

# §1. Introduction

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## ON TORRES-TYPE RELATIONS FOR THE ALEXANDER POLYNOMIALS OF LINKS

by V. G. TURAEV

### § 1. INTRODUCTION

The classical formula of Torres [5] relates the (first) Alexander polynomial of a link  $K$  in  $S^3$  with that of the sublink of  $K$  obtained by deleting a component. The aim of the present paper is to establish a Torres-type formula for Alexander polynomials of higher-dimensional links. We also discuss analogous formulas for higher Alexander polynomials of links in  $S^3$ .

An  $n$ -component link in the sphere  $S^m$  is an ordered collection of  $n$  disjoint smooth imbedded oriented  $(m-2)$ -dimensional spheres in  $S^m$ . With each odd-dimensional link  $K \subset S^{2r+1}$  one associates a  $\Lambda_n$ -module  $H_r(\tilde{X})$ , where  $\Lambda_n$  is the Laurent polynomial ring  $\mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ ,  $X$  is the exterior of  $K$  and  $\tilde{X}$  is the maximal abelian covering of  $X$ . The module  $H_r(\tilde{X})$  algebraically gives rise to a sequence of Fitting (or determinantal) invariants  $\Delta_1(K), \Delta_2(K), \dots$ , which are elements of  $\Lambda_n$  defined up to multiplication by monomials  $\pm t_1^{s_1} \dots t_n^{s_n}$  (see [1] or § 3). The polynomial  $\Delta_i(K)$  is called the  $i$ -th Alexander polynomial of  $K$ . The first Alexander polynomial  $\Delta_1(K)$  is also denoted by  $\Delta(K)$  and called "the Alexander polynomial of  $K$ ".

**THEOREM (Torres [5]).** *Let  $K$  be an  $n$ -component link in  $S^3$  with  $n \geq 2$  and let  $L$  be the sublink of  $K$  obtained by deleting the  $n$ -th component. Then*

$$\Delta(K)(t_1, \dots, t_{n-1}, 1) = \begin{cases} (t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1)\Delta(L) & \text{if } n > 2 \\ \frac{t_1^{l_1} - 1}{t_1 - 1} \Delta(L) & \text{if } n = 2 \end{cases}$$

where  $l_i$  denotes the linking number of the  $i$ -th and  $n$ -th components of  $K$ .

The following theorem can be considered as a high-dimensional variant of the Torres theorem.

THEOREM 1. Let  $K$  be an  $n$ -component link in  $S^m$  with odd  $m \geq 5$ . Let  $L$  be the sublink of  $K$  obtained by deleting the  $n$ -th component. Then there exists an element  $\lambda$  of  $\Lambda_{n-1}$  such that

$$(1) \quad \Delta(L) = \Delta(K)(t_1, \dots, t_{n-1}, 1) \cdot \lambda \bar{\lambda}.$$

Here the overbar denotes the involution of the Laurent polynomial ring  $\Lambda_{n-1}$  which sends each polynomial  $f(t_1, \dots, t_{n-1})$  into  $f(t_1^{-1}, \dots, t_{n-1}^{-1})$ .

It is well known that for any link  $K \subset S^m$  with odd  $m \geq 5$  the Alexander polynomial  $\Delta(K)$  is non-zero. Moreover,

$$\text{aug}(\Delta(K)) = \Delta(K)(1, 1, \dots, 1) = \pm 1$$

(see [1]). This implies that  $\text{aug}(\lambda) = \pm 1$  for any  $\lambda$  satisfying (1). It seems that there are no other restrictions on  $\lambda$ ; one may even guess that for any  $\Delta \in \Lambda_n$ ,  $\lambda \in \Lambda_{n-1}$  with  $\text{aug}(\Delta) = \text{aug}(\lambda) = \pm 1$  and  $\bar{\Delta} \doteq \Delta$  there exists a pair  $K, L$  as in Theorem 1 such that  $\Delta(K) \doteq \Delta$  and  $\Delta(L) \doteq \Delta(t_1, \dots, t_{n-1}, 1)\lambda\bar{\lambda}$ . Here and below the symbol  $\doteq$  denotes the equality of Laurent polynomials up to multiplication by a monomial  $\pm t_1^{s_1} \dots t_n^{s_n}$ .

Let us call two Laurent polynomials  $\Delta, \Delta' \in \Lambda_n$  algebraically cobordant if there exist polynomials  $\lambda, \lambda' \in \Lambda_n$  such that  $\Delta\lambda\bar{\lambda} \doteq \Delta'\lambda'\bar{\lambda}'$  and  $\text{aug}(\lambda) = \text{aug}(\lambda') = \pm 1$ . This terminology is suggested by the fact that Alexander polynomials of (smoothly) cobordant links are algebraically cobordant (see [4]). The formula (1) enables us to calculate Alexander polynomials of all sublinks of a given link, up to algebraic cobordism. It is curious to note that if  $K, K'$  are  $n$ -component links in  $S^m$  with odd  $m \geq 5$  and if polynomials  $\Delta(K), \Delta(K')$  are algebraically cobordant then Theorem 1 implies that Alexander polynomials of corresponding sublinks of  $K, K'$  are algebraically cobordant to each other. This fact reflects the evident property of geometric cobordisms: corresponding sublinks of cobordant links are cobordant.

I do not know if it is possible to associate with a link  $K$  some preferred  $\lambda = \lambda(K)$  satisfying (1).

The remaining part of the Introduction is concerned with the classical links. The symbols  $K, L, n, l_1, \dots, l_{n-1}$  denote the same objects as in the Torres theorem formulated above. It may well happen that some of the Alexander polynomials  $\Delta_1(K), \Delta_2(K), \dots$  are equal to zero. Denote by  $u = u(K)$  the minimal integer  $u \geq 1$  such that  $\Delta_u(K) \neq 0$ . Since  $\Delta_{i+1}(K)$  divides  $\Delta_i(K)$  for all  $i$ ,  $\Delta_i(K) = 0$  for  $i < u$  and  $\Delta_i(K) \neq 0$  for  $i \geq u(K)$ .

In view of the Torres theorem it is natural to look for a relationship between  $\Delta_{u(K)}(K)$  and a corresponding invariant of  $L$ . In the case  $u(K) = 1$  we have the Torres formula, so we shall restrict ourselves to the case  $u(K) \geq 2$  (i.e. the case  $\Delta(K) = 0$ ).

The integers  $u(K), u(L)$  are related by the inequality  $u(L) \geq u(K) - 1$  (see [1] or § 4). If  $l_i \neq 0$  at least for one  $i = 1, \dots, n - 1$  then the stronger inequality holds:  $u(L) \geq u(K)$ . These inequalities suggest to relate  $\Delta_u(K)$  (where we put  $u = u(K)$ ) with  $\Delta_{u-1}(L)$  and  $\Delta_u(L)$ . The following relationship between  $\Delta_u(K)$  and  $\Delta_u(L)$  was established in [4].

**THEOREM ([4, Theorem 5.5.1]).** *If  $u = u(K) \geq 2$  then there exist an element  $\lambda$  of  $\Lambda_{n-1}$  and a subset  $\beta$  of the set  $\{1, 2, \dots, n-1\}$  such that*

$$(2) \quad (t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1) \Delta_u(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_u(K) (t_1, \dots, t_{n-1}, 1).$$

Several remarks are in order. a) The non-trivial case of the Theorem is the case where at least one of the integers  $l_1, \dots, l_{n-1}$  is non-zero: otherwise  $t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1 = 0$  and we may put  $\lambda = 0$ . b) Formula (2) is proved in [4] under the additional condition  $u(L) = u(K)$ . However if  $u(L) < u(K)$  then we have the trivial case  $l_1 = l_2 = \dots = l_{n-1} = 0$ ; if  $u(L) > u(K)$  then  $\Delta_{u(K)}(L) = 0$  and we may put  $\lambda = 0$ . c) Formula (2) combines the factors from the Torres formula, formula (1) and a new factor  $\prod (t_i - 1)$ . All these factors may be non-trivial (see [4]). d) An explicit construction of the set  $\beta = \beta(K)$  is given in [4, § 5]. I do not know if there exists a preferred  $\lambda = \lambda(K)$  which satisfies (2).

The relationships between the polynomials  $\Delta_u(K)$  and  $\Delta_{u-1}(L)$  were first considered by Levine [2] in the case  $u = 2$ .

**THEOREM (Levine [2]).** *If  $u(K) \geq 2$  then there exist an element  $\lambda \in \Lambda_{n-1}$  and a set  $\beta \subset \{1, 2, \dots, n-1\}$  such that*

$$\Delta(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_2(K) (t_1, \dots, t_{n-1}, 1).$$

Note that in the case  $u(K) > 2$  the Levine's theorem is evident: if  $u(K) > 2$  then  $u(L) \geq u(K) - 1 > 1$  so that  $\Delta(L) = \Delta_2(K) = 0$ .

The following theorem generalizes the Levine's result.

**THEOREM 2.** *If  $u = u(K) \geq 2$  then there exist an element  $\lambda$  of  $\Lambda_{n-1}$  and a set  $\beta \subset \{1, 2, \dots, n-1\}$  such that*

$$\Delta_{u-1}(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda \bar{\lambda} \cdot \Delta_u(K) (t_1, \dots, t_{n-1}, 1).$$

The non-trivial case of Theorem 2 is the case  $l_1 = l_2 = \dots = l_{n-1} = 0$ : otherwise  $u(L) \geq u$  so that  $\Delta_{u-1}(L) = 0$  and we may put  $\lambda = 0$ .

The proof of Theorems 1, 2 goes along the same lines as the proof of the formula (2) given in [4]. These proofs are based on a relationship between the Alexander polynomials and Reidemeister-type torsions, established in [4]. This relationship is recalled in § 2. In § 3 several easy algebraic lemmas are proved. Theorems 1, 2 are proved in § 4.

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## § 2. TORSIONS OF CHAIN COMPLEXES AND MANIFOLDS

2.1. THE TORSION OF A CHAIN COMPLEX (see [3]). Let  $Q$  be a field. If  $a = (a_1, \dots, a_n)$  and  $b = (b_1, \dots, b_n)$  are two bases of a  $Q$ -module then  $a_i = \sum_{j=1}^n c_{i,j} b_j$  where  $(c_{i,j})$  is a non-singular  $n \times n$ -matrix over  $Q$ ; the determinant  $\det(c_{i,j}) \in Q \setminus 0$  is denoted by  $[a/b]$ .

Let  $C = (C_m \rightarrow \dots \rightarrow C_0)$  be a chain  $Q$ -complex. Suppose that each  $Q$ -module  $C_i$  is finite dimensional with a preferred basis  $c_i$  and each  $Q$ -module  $H_i(C)$  also has a preferred basis  $h_i$ . (The case  $C_i = 0$  or  $H_i(C) = 0$  is not excluded; by definition the zero module has the empty basis.) In this setting one defines the torsion  $\tau(C) \in Q$  as follows. For each  $i = 1, 2, \dots, m$  choose a sequence  $b_i = (b_1^i, \dots, b_{r_i}^i)$  of elements of  $C_i$  such that  $\partial_{i-1}(b_i) = (\partial_{i-1}(b_1^i), \dots, \partial_{i-1}(b_{r_i}^i))$  is a basis in  $\text{Im}(\partial_{i-1}: C_i \rightarrow C_{i-1})$ . For each  $i = 0, 1, \dots, m$  choose a lifting  $\tilde{h}_i$  of the basis  $h_i$  to  $\text{Ker } \partial_{i-1}$ . The combined sequence  $\partial_i(b_{i+1})\tilde{h}_i b_i$  is a basis in  $C_i$ . (It is understood that  $b_0 = \emptyset$  and  $b_{m+1} = \emptyset$ ). Put

$$(3) \quad \tau(C) = \prod_{i=0}^m [\partial_i(b_{i+1})\tilde{h}_i b_i / c_i]^{\varepsilon(i)}$$

where  $\varepsilon(i) = (-1)^{i+1}$ . Clearly,  $\tau(C) \in Q \setminus 0$ . It is easy to verify that  $\tau(C)$  does not depend on the choice of  $b_i$  and  $\tilde{h}_i$ .

(Note that the torsion of  $C$  defined in Milnor's survey article [3] equals  $\pm \tau(C)^{-1} \in Q / \pm 1$  and that Milnor uses the additive notation for the multiplication in  $Q \setminus 0 = K_1(Q)$ .)

2.1.1. LEMMA (multiplicativity of torsion). *Let  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$  be a short exact sequence of  $m$ -dimensional chain complexes over a field  $Q$ .*