

3. Applications

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

$$(*) \quad \mathcal{O}(\tilde{X}) \subset \bigcap_{x \in X} S_x^{-1} A.$$

For $f \in \mathcal{O}(\tilde{X})$, $a \in \tilde{X}$ and $b \in p^{-1}(p(a))$, it is now possible to conclude that $f(a) = f(b)$ is true. Let $x := \pi(a)$. Due to (*), functions $g \in S_x$ and $h \in A$ exist with $f = h/g \circ \pi$. Since a and b are equivalent with respect to the equivalence relation R , $f(a) = f(b)$ follows, and a continuous function $F: Y \rightarrow \mathbb{C}$ exists with $F \circ p = f$.

Since the Stein complex structure on Y is not in general the canonical ringed quotient structure, it is still necessary to verify that F is holomorphic in order to prove the density of A in $\mathcal{O}(\tilde{X})$. To that end, let $H \in \mathcal{O}(Y)$ and $G \in \mathcal{O}(Y)$ have the property that $H \circ p = h$ and $G \circ p = g \circ \pi$. Such functions exist because $p^*(\mathcal{O}(Y)) = \bar{A}$ holds. Then $F = H/G$ follows, and the germ $F_{p(a)}$ is the germ of a holomorphic function at $p(a)$, since the germ $G_{p(a)}$ of G at $p(a)$ is a unit. The surjectivity of p implies that F is holomorphic on Y , completing the proof of the theorem.

Note that the topology induced by $\mathcal{O}(\tilde{X})$ on any subalgebra A of $\mathcal{O}(\tilde{X})$ is the metrizable topology of uniform convergence on compact subsets of X . Because the closure $\bar{A}$ of A in $\mathcal{O}(\tilde{X})$ is its completion, $\bar{A}$ can be obtained without referring directly to $\mathcal{O}(\tilde{X})$. Thus the Main Theorem can be stated as follows:

If $\tilde{X}$ denotes the normalization of an irreducible Stein space X , then $\mathcal{O}(\tilde{X})$ is the completion of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$.

3. APPLICATIONS

In this section X will denote an irreducible Stein space with normalization $\pi: \tilde{X} \rightarrow X$, $\widetilde{\mathcal{O}(X)}$ will be the integral closure of the holomorphic functions $\mathcal{O}(X)$ on X , $\mathcal{O}(\tilde{X})$ the Fréchet algebra of weakly holomorphic functions on X (or equivalently, the Fréchet algebra of holomorphic functions $\mathcal{O}(\tilde{X})$ on $\tilde{X}$), and

$$S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\} \quad \text{for } x \in X.$$

Although the example given in the first section shows that the algebras $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ are not always equal, the inclusion (*) in the proof of the Main Theorem implies that they are locally equal in the following sense.

THEOREM 2. For every $x \in X$, the localizations of $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ with respect to S_x coincide.

The next theorem implies an algebraic description of the topological closure of $\widetilde{\mathcal{O}(X)}$ in $\tilde{\mathcal{O}}(X)$.

THEOREM 3. $\mathcal{O}(\tilde{X}) = \bigcap_{x \in X} S_x^{-1} \widetilde{\mathcal{O}(X)}.$

Proof. Let $f \in M(\tilde{X}) = M(X)$ be such that for every $x \in X$ there is a $g \in S_x$ and an $h \in \widetilde{\mathcal{O}(X)}$, with $f = h/g \circ \pi$. Then the germ f_a of f at an arbitrary point $a \in \tilde{X}$ is holomorphic, because the germ of $g \circ \pi$ at a is a unit. Hence $f \in \mathcal{O}(\tilde{X})$, and the assertion is proved.

COROLLARY 2. The topological closure of $\widetilde{\mathcal{O}(X)}$ in $\tilde{\mathcal{O}}(X)$ is the intersection of the localizations of $\widetilde{\mathcal{O}(X)}$ with respect to S_x for all $x \in X$.

The next result characterizes the weakly holomorphic functions on X as being exactly those meromorphic functions on X which are almost integral over $\mathcal{O}(X)$.

COROLLARY 3. $\mathcal{O}(\tilde{X})$ is completely normal.

Proof. Let $f \in M(\tilde{X})$ be almost integral over $\mathcal{O}(\tilde{X})$. Then f is almost integral over $\mathcal{O}(X)$ and therefore over $S_x^{-1} \widetilde{\mathcal{O}(X)}$ for every $x \in X$ which has been shown to be completely normal in the proof of the Main Theorem. An application of Theorem 3 yields $f \in \mathcal{O}(\tilde{X})$ and hence the assertion.

Using the classical Oka-Weil-Cartan Theorem [1, Anhang zu VI, § 4], an immediate consequence of the Main Theorem is

THEOREM 4. $\tilde{X}$ is $\widetilde{\mathcal{O}(X)}$ -convex, $\widetilde{\mathcal{O}(X)}$ -separable and has local coordinates by functions in $\widetilde{\mathcal{O}(X)}$.

A property which ensures that the holomorphic functions on $\tilde{X}$ are integral over the holomorphic functions on X is that $\mathcal{O}(\tilde{X})$ is a finite $\mathcal{O}(X)$ -module.

THEOREM 5. Let $u \in \mathcal{O}(X)$ be any global universal denominator for X . Then $\mathcal{O}(\tilde{X})$ is isomorphic to the closed ideal $u\mathcal{O}(\tilde{X})$ in $\mathcal{O}(X)$, and $\mathcal{O}(\tilde{X})$ is a finite $\mathcal{O}(X)$ -module if and only if this ideal is finitely generated.

Proof. Recall that a global universal denominator u for X always exists [10, E.73a]. The multiplication map

$$\mathcal{O}(\tilde{X}) \rightarrow \mathcal{O}(X), \quad f \mapsto uf,$$

defines an injective $\mathcal{O}(X)$ -module homomorphism. Thus, $\mathcal{O}(\tilde{X})$ is isomorphic to the ideal $u\mathcal{O}(\tilde{X})$ in $\mathcal{O}(X)$ which will now be denoted by I . Consider the transporter ideal $J := \tilde{\mathcal{O}} : \frac{1}{u} \mathcal{O}$ of $\frac{1}{u} \mathcal{O}$ into $\tilde{\mathcal{O}}$ which is a coherent sheaf of ideals in $\tilde{\mathcal{O}}$. The global sections $J(X)$ form a closed ideal of $\mathcal{O}(X)$ by a theorem of Cartan [4, 5], due again to the fact that X is Stein. Because $J(X) = I$ holds, the assertion follows.

COROLLARY 4. *If $\mathcal{O}(\tilde{X})$ does not coincide with $\widetilde{\mathcal{O}(X)}$, the closed ideal $u\mathcal{O}(\tilde{X})$ in $\mathcal{O}(X)$ is not finitely generated.*

In a Stein algebra $\mathcal{O}(X)$, every finitely generated ideal is closed, as Cartan [4, 5] showed. If X is at least two-dimensional, Forster [6] gave examples of closed ideals in $\mathcal{O}(X)$ which are not finitely generated. According to Corollary 4, the space constructed in § 1 gives a one-dimensional example.

REFERENCES

- [1] BEHNKE, H. und P. THULLEN. *Theorie der Funktionen mehrerer komplexen Veränderlichen*, 2. edition. Springer-Verlag, Berlin, Heidelberg, New York, 1970.
- [2] BINGENER, J. und U. STORCH. Resträume zu analytischen Mengen in Steinschen Räumen. *Math. Ann.* 210 (1974), 33-53.
- [3] BOURBAKI, N. *Algèbre commutative*. Hermann, Paris, 1969.
- [4] CARTAN, H. *Séminaire*. E.N.S. 1951/1952.
- [5] ———. Idéaux et modules de fonctions analytiques de variables complexes. *Bull. Soc. Math. France* 78 (1950), 28-64.
- [6] FORSTER, O. Zur Theorie der Steinschen algebren und Moduln. *Math. Zeitschr.* 97 (1967), 376-405.
- [7] FOSSUM, R. *The divisor class group of a Krull domain*. Ergebnisse der Math. und ihrer Grenzgebiete 74, Springer-Verlag, Berlin, Heidelberg, New York, 1973.
- [8] GRAUERT, H. and R. REMMERT. *Coherent analytic sheaves*. Grundle. Math. Wiss. 265. Springer-Verlag, Berlin, Heidelberg, New York, 1984.
- [9] ———. *Theorie der Steinschen Räume*. Grundle. Math. Wiss. 227. Springer-Verlag, Berlin, Heidelberg, New York, 1977.