

3. Construction of Gröbner Bases

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **27.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

(2) $\Rightarrow$ (3): trivial.

(3) $\Rightarrow$ (1): By (3) we have $\text{in}(Q) \in \langle \text{in}(F) \rangle$ for every $Q \in J - \{0\}$. Hence $\langle \text{in}(J) \rangle = \langle \text{in}(F) \rangle$.

2.6. COROLLARY. Let F be a Gröbner basis of an ideal $J \leq R[X]$.

1) F generates J .

2) Let $Q \in R[X]$. Then $Q \in J$ iff a rest of Q after dividing by F is zero.

Proof. Obvious.

2.7. Another characterisation of Gröbner bases can be given as follows:

We shall say that a set $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$ of admissible combinations of F (with pairwise different degrees) is an " F -admissible set", if for all α we have $\deg(L_\alpha) = \alpha$ and $\text{lc}(L_\alpha)$ generates the ideal

$${}_R \langle \text{lc}(P) \mid P \in \langle \text{in}(F) \rangle, \deg(P) = \alpha \rangle.$$

Any F -admissible set is R -linearly independent.

If R is a field the condition on $\text{lc}(L_\alpha)$ is superfluous.

PROPOSITION. Let J be an ideal in $R[X]$ containing F . Then the following conditions are equivalent:

- (1) F is a Gröbner basis of J .
- (2) There is an F -admissible set which is a R -basis of J .
- (3) Every F -admissible set is a R -basis of J .

Proof. Let $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$ be a F -admissible set.

(1) $\Rightarrow$ (3): Let Q be an element of $J - \{0\}$. Division of Q by $\{L_{\deg(Q)}\}$, of its rest $\bar{Q}$ by $\{L_{\deg(\bar{Q})}\}$, ... yields in a finite number of steps an expression of Q as R -linear combination of L_α 's.

(3) $\Rightarrow$ (2): trivial.

(2) $\Rightarrow$ (1): Suppose that $\{L_\alpha \mid \alpha \in \mathcal{D}(F)\}$ is a R -basis of J . For every $Q \in J - \{0\}$ the initial term of $L_{\deg(Q)}$ divides $\text{in}(Q)$, hence $\text{in}(Q) \in \langle \text{in}(F) \rangle$.

3. CONSTRUCTION OF GRÖBNER BASES

3.1. Definition. Let P, Q be elements of $R[X]$, let $\alpha, \beta \in \mathbb{N}^n$ and let $a, b \in R$. Then the polynomial

$$S(P, Q) := aX^\alpha P - bX^\beta Q$$

is called a "S(ubtraction)-polynomial of P, Q " iff

$$\alpha + \deg(P) = \beta + \deg(Q) = \min(\mathcal{D}(\{P\}) \cap \mathcal{D}(\{Q\}))$$

and $\text{lc}(P) \cdot a = \text{lc}(Q) \cdot b =$ a least common multiple of $\text{lc}(P)$ and $\text{lc}(Q)$.

3.2. *Example.* Consider the graded lexicographic ordering on $\mathbb{N}^2$ and

$$P := 6X_1^3X_2 + 1, \quad Q := 8X_1X_2^2 + 3X_1X_2 + X_2 \in \mathbb{Z}[X_1, X_2].$$

Then

$$4X_2P - 3X_1^2Q = -9X_1^3X_2 - 3X_1^2X_2 + 4X_2 \quad \text{and} \quad -4X_2P + 3X_1^2Q$$

are S-polynomials of P, Q .

See figure 5.

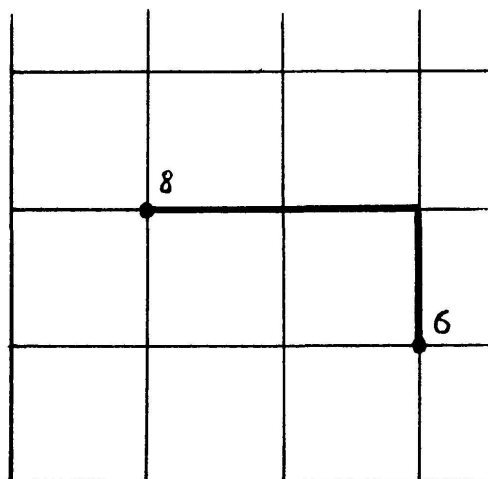


FIGURE 5.

3.3 *Remark.* For $P, Q \in R[X]$, $S(P, Q)$ as defined above is unique up to multiplication by an invertible element of R . Therefore we shall call it "the" S-polynomial of P, Q .

3.4. **LEMMA.** Let $P_1, \dots, P_k \in R[X]$, $c_1, \dots, c_k \in R$ such that $\deg(P_1) = \dots = \deg(P_k) =: \delta$ but $\deg(\sum_{i=1}^k c_i P_i) \neq \delta$.

Then $\sum_{i=1}^k c_i P_i$ is a R -linear combination of the S-polynomials $S(P_i, P_j)$, $1 \leq i, j \leq k$.

Proof. By induction on k .

Let $l_i := \text{lc}(P_i)$, $1 \leq i \leq k$. Then $\sum_{i=1}^k c_i l_i = 0$.

It is sufficient to prove the existence of $a_{ij}, b_{ij} \in R$ such that

$$\sum_{i=1}^k c_i P_i = \sum_{1 \leq i, j \leq k} (a_{ij} P_i - b_{ij} P_j) \quad \text{and} \quad a_{ij} l_i = b_{ij} l_j, \quad 1 \leq i, j \leq n.$$

For $k = 2$ we have $c_1 P_1 + c_2 P_2 = c_1 P_1 - (-c_2) P_2$ and $c_1 l_1 = (-c_2) l_2$.
 $k = 3$: Let l be a greatest common divisor of l_1, l_2, l_3 . Since $c_2 l_2 = -c_1 l_1 - c_3 l_3$, a greatest common divisor of l_1 and l_3 divides $c_2 l$. Hence there are elements $x_2, x_3 \in R$ such that $c_2 l = x_1 l_1 + x_3 l_3$.

Then $d_1 := (-x_1 l_2 - c_1 l)/l$, $d_2 := (-x_1 l_1)/l$, $d_3 := (x_3 l_2)/l$ are elements of R . Furthermore, we have

$$\begin{aligned} (c_1 + d_1) l_1 &= d_2 l_2 \\ (c_2 + d_2) l_2 &= d_3 l_3 \\ (c_3 + d_3) l_3 &= d_1 l_1 \quad \text{and} \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^3 c_i P_i &= [(c_1 + d_1) P_1 - d_2 P_2] + [(c_2 + d_2) P_2 - d_3 P_3] \\ &\quad + [(c_3 + d_3) P_3 - d_1 P_1]. \end{aligned}$$

$$k > 3: \text{ Let } Q := \sum_{i=3}^k c_i P_i \quad \text{and} \quad m := \sum_{i=3}^k c_i l_i.$$

If $m = 0$, we can apply the induction hypothesis to Q .

If $m \neq 0$, by the $k = 3$ case there are $d_1, d_2, d_3 \in R$ such that

$$\begin{aligned} c_1 P_1 + c_2 P_2 + Q &= [(c_1 + d_1) P_1 - d_2 P_2] + [(c_2 + d_2) P_2 - d_3 Q] \\ &\quad + [(1 + d_3) Q - d_1 P_1] \end{aligned}$$

$$\text{and} \quad (c_1 + d_1) l_1 = d_2 l_2, \quad (c_2 + d_2) l_2 = d_3 m, \quad (1 + d_3) m = d_1 l_1.$$

Therefore, we can apply the induction hypothesis to $(c_2 + d_2) P_2 - \sum_{i=3}^k d_3 c_i P_i$

and to $-d_1 P_1 + \sum_{i=3}^k (1 + d_3) c_i P_i$ and thus terminate the proof.

Remark. If R is a field, the proof is trivial: Let $l_i := \text{lc}(P_i)$ and

$$\begin{aligned} P'_i &:= (P_i / l_i), \quad 1 \leq i \leq k, \quad \text{then} \quad \sum_{i=1}^k c_i P_i = c_1 l_1 (P'_1 - P'_2) \\ &\quad + (c_1 l_1 + c_2 l_2) (P'_2 - P'_3) + \dots + \left(\sum_{i=1}^{k-1} c_i l_i \right) (P'_{k-1} - P'_k). \end{aligned}$$

3.5. THEOREM. Let J be an ideal of $R[X]$ generated by a finite subset $F \subseteq R[X] - \{0\}$.

Then the following assertions are equivalent:

- (1) F is a Gröbner basis of J .
- (2) For all $P, Q \in F$ a rest of $S(P, Q)$ after division by F is zero.

Proof.

(1) $\Rightarrow$ (2): Let $P, Q \in F$. Then $S(P, Q)$ and its rest after division by F are elements of J . Therefore, this implication is a special case of proposition 2.5., (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Let $A \in J - \{0\}$. We have to show that $\text{in}(A) \in \langle \text{in}(F) \rangle$. Since J is generated by F , there are elements $c(\gamma, P) \in R$ such that $A = \sum_{P \in F, \gamma \in \mathbb{N}^n} c(\gamma, P) X^\gamma P$.

Let $\delta := \max \{ \gamma + \deg(P) \mid c(\gamma, P) \neq 0 \}$ and $L := \sum_{\substack{\gamma, P \\ \gamma + \deg(P) = \delta}} c(\gamma, P) X^\gamma P$.

By lemma 1.3. we may assume that δ is minimal, i.e.:

if $A = \sum_{P \in F, \gamma \in \mathbb{N}^n} d(\gamma, P) X^\gamma P$ then $\delta \leq \max \{ \gamma + \deg(P) \mid d(\gamma, P) \neq 0 \}$.

Suppose that $\deg(L) < \delta$. Then the lemma above yields

$$L = \sum_{P, Q \in F, \alpha \in \mathbb{N}^n} a(\alpha, P, Q) X^\alpha S(P, Q), \quad a(\alpha, P, Q) \in R$$

(note that for $\beta, \gamma \in \mathbb{N}^n$ there is an $\alpha \in \mathbb{N}^n$ such that $S(X^\beta P, X^\gamma Q) = X^\alpha S(P, Q)$).

But according to (2) the S -polynomials are admissible combinations of F and clearly the same holds for the $X^\alpha S(P, Q)$'s. Since their degree is smaller than δ , this is a contradiction to the minimality of δ . Hence $\deg(L) = \delta$. But then $\text{in}(A) = \text{in}(L) \in \langle \text{in}(F) \rangle$.

3.6. THEOREM. Let J be the ideal generated by F . Then a Gröbner basis of J can be constructed (in a finite number of steps) by the following algorithm:

$$F_0 := F$$

$$F_{i+1} := F_i \cup (\overline{\{S(P, Q) \mid P, Q \in F_i\}} - \{0\})$$

($\overline{S(P, Q)}$ is a rest of $S(P, Q)$ after division by F_i). If $F_i = F_{i+1}$, then F_i is a Gröbner basis of J .

Proof. By the preceding theorem we only have to show that there is a $k \in \mathbb{N}$ such that $F_k = F_{k+1}$.

If $F_i \subset F_{i+1}$ then $\langle \text{in}(F_i) \rangle \subset \langle \text{in}(F_{i+1}) \rangle$. Since the strictly ascending sequence $\langle \text{in}(F_0) \rangle \subset \langle \text{in}(F_1) \rangle \subset \dots$ must be finite, there is a $k \in \mathbb{N}$ with $F_k = F_{k+1}$.

3.7. *Example.* Consider the graded lexicographic ordering on $\mathbb{N}^2$ and

$$F := \{P_1 := 2X_1X_2^2 - X_1, P_2 := 3X_1^2X_2 - X_2\} \subseteq \mathbb{Z}[X_1, X_2].$$

Then

$$\begin{aligned} F_0 = F \quad \text{and} \quad S(P_1, P_2) &= 3X_1P_1 - 2X_2P_2 = -3X_1^2 + 2X_2^2 \\ &= \overline{S(P_1, P_2)} = : P_3. \end{aligned}$$

So

$$F_1 = \{P_1, P_2, P_3\} \quad \text{and} \quad \overline{S(P_1, P_2)}^{F_1} = 0,$$

$$\overline{S(P_1, P_3)}^{F_1} = 4X_2^4 - 3X_1^2 = : P_4, \quad \overline{S(P_2, P_3)}^{F_1} = 2X_2^3 - X_2 = : P_5.$$

Therefore $F_2 = \{P_1, P_2, P_3, P_4, P_5\}$ and all rests after division by F_2 of S -polynomials are 0. Hence F_2 is a Gröbner basis of the ideal generated by F .

See figure 6.

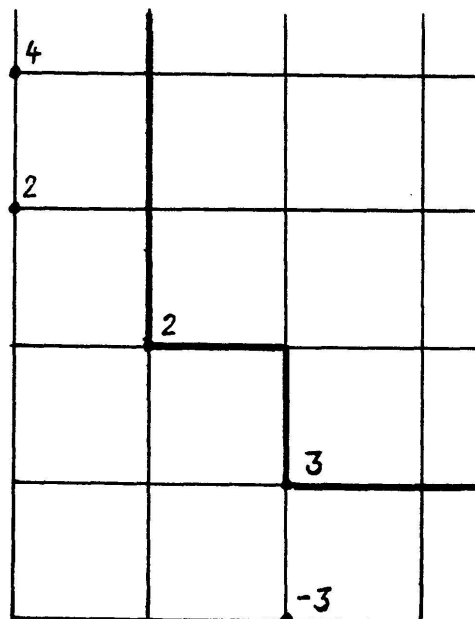


FIGURE 6.

3.8. *Remark.* Let G be a Gröbner basis of an ideal J . We shall say that G is "simplified" if all $P \in G$ fulfill the following two conditions:

$$\text{lc}(P) \text{ generates the ideal } {}_R \langle \text{lc}(Q) \mid Q \in J, \deg(Q) = \deg(P) \rangle$$

and

$$\text{in}(P) \notin \langle \text{in}(G - \{P\}) \rangle.$$

It is easy to see that the elements of a simplified Gröbner basis have pairwise different degrees.

If R is a field then G is simplified iff the elements of G have pairwise different degrees and $\deg(G)$ is the set of minimal elements (with respect to the natural partial ordering on $\mathbb{N}^n$) in $\deg(J)$.

If G is not simplified, then in the following way we can construct (in a finite number of steps) a simplified Gröbner basis of J :

For every $P \in G$ choose an admissible combination P' of G such that $\deg(P) = \deg(P')$ and $\text{lc}(P')$ generates the ideal

$${}_R \langle \text{lc}(Q) \mid Q \in J, \deg(Q) = \deg(P) \rangle.$$

Then $G' := \{P' \mid P \in G\}$ is a Gröbner basis of J , since $\langle \text{in}(J) \rangle = \langle \text{in}(G) \rangle \subseteq \langle \text{in}(G') \rangle \subseteq \langle \text{in}(J) \rangle$.

If there is a $P' \in G'$ with $\text{in}(P') \in \langle \text{in}(G' - \{P'\}) \rangle$, then $G' - \{P'\}$ is a Gröbner basis, since then $\langle \text{in}(G' - \{P'\}) \rangle = \langle \text{in}(G') \rangle = \langle \text{in}(J) \rangle$.

Replace G' by $G' - \{P'\}$. After finitely many eliminations of this kind we obtain a simplified Gröbner basis.

In example 3.7. the Gröbner basis F_2 is not simplified, since $\text{in}(P_2) = -X_2 \text{in}(P_3)$ and $\text{in}(P_4) = 2X_2 \text{in}(P_5)$. $\{P_1, P_3, P_5\}$ is a simplified Gröbner basis of the ideal generated by F_2 .

4. APPLICATION TO SYSTEMS OF ALGEBRAIC EQUATIONS

Let J be an ideal in $R[X]$, generated by a subset $F \neq \{0\}$.

4.1. We may consider F as a system of algebraic equations in n variables. We denote by K an algebraic closure of the quotient field of R .

Let $Z(F)$ (resp. $Z_K(F)$) be the set $\{z \in R^n$ (resp. K^n) $\mid P(z) = 0$ for all $P \in F\}$ of common zeros in R^n (resp. K^n) of the elements of F . Clearly $Z(F) = Z(J)$ and $Z_K(F) = Z_K(J)$.