

## 2. Construction of $O(\tilde{X})$ from $O(X)$ for Stein spaces $X$

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2. CONSTRUCTION OF  $\mathcal{O}(\tilde{X})$  FROM  $\mathcal{O}(X)$  FOR STEIN SPACES  $X$ 

According to a theorem of Oka [12], the normalization sheaf  $\tilde{\mathcal{O}}$  of weakly holomorphic functions on a complex space  $(X, \mathcal{O})$  is coherent. Consequently, there is a canonical topology making  $\tilde{\mathcal{O}}$  a Fréchet sheaf; the global weakly holomorphic functions  $\tilde{\mathcal{O}}(X)$  will always carry this topology. Since the holomorphic functions  $\mathcal{O}(\tilde{X})$  on the normalization  $\tilde{X}$  of  $X$  are topologically isomorphic to  $\tilde{\mathcal{O}}(X)$  [8, 8.3], the question posed in the introduction can now be answered.

**MAIN THEOREM.** *For an irreducible Stein space  $X$ , the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$  is dense in  $\tilde{\mathcal{O}}(X)$ .*

*Proof.* Let  $\pi: \tilde{X} \rightarrow X$  be the normalization of  $X$  and put  $A := \widetilde{\mathcal{O}(X)}$ . Since  $\pi$  is proper,  $\tilde{X}$  is  $\mathcal{O}(X)$ -convex and therefore  $\bar{A}$ -convex. Note that Corollary 1 implies  $A \subset \tilde{\mathcal{O}}(X)$  and that  $\bar{A}$  is the closure of  $A$  with respect to the canonical topology in  $\tilde{\mathcal{O}}(X)$ .

Consider the equivalence relation  $R$  on  $\tilde{X}$  defined by  $\bar{A}$ , i.e.  $(x, y) \in R$  iff for every  $f \in \bar{A}$ ,  $f(x) = f(y)$ . Rossi's theorem [13] ensures that the topological quotient  $Y := \tilde{X}/R$  can be given the complex structure of a Stein space such that the projection  $p: \tilde{X} \rightarrow Y$  is holomorphic and proper and the map  $p^*: \mathcal{O}(Y) \rightarrow \mathcal{O}(\tilde{X})$ ,  $f \mapsto f \circ p$ , induces an isomorphism  $\mathcal{O}(Y) \cong \bar{A}$ .

It suffices to show that every  $f \in \mathcal{O}(\tilde{X})$  can be factorized through a holomorphic function on  $Y$ , meaning that an  $F \in \mathcal{O}(Y)$  exists with  $F \circ p = f$ . This will be accomplished by first factorizing  $f \in \mathcal{O}(\tilde{X})$  through a continuous function  $F$  on  $Y$  and then proving that  $F$  is actually holomorphic. The existence of such a continuous factor  $F$  for  $f$  is equivalent to demonstrating that every  $f \in \mathcal{O}(\tilde{X})$  is constant on the fibers of  $p$ . The validity of this geometric statement will be shown now using commutative algebra.

$\mathcal{O}(\tilde{X})$  is almost integral over  $\mathcal{O}(X)$  (see § 1), and hence over the localization  $S_x^{-1}A$  of  $A$  with respect to  $S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\}$  for every  $x \in X$ . Moreover,

$$S_x^{-1}A = \widetilde{S_x^{-1}\mathcal{O}(X)}$$

holds [3, V, 1.5, Corollary 1]. The localization  $S_x^{-1}\mathcal{O}(X) = \mathcal{O}(X)_{m(x)}$  of the Stein algebra  $\mathcal{O}(X)$  at the maximal ideal  $m(x) := \{f \in \mathcal{O}(X) : f(x) = 0\}$  is noetherian — even more, it's excellent [2, p. 35]. According to a theorem of Mori-Nagata, the integral closure of a noetherian integral domain is completely normal [7, 4.3, 3.6], implying

$$(*) \quad \mathcal{O}(\tilde{X}) \subset \bigcap_{x \in X} S_x^{-1} A.$$

For  $f \in \mathcal{O}(\tilde{X})$ ,  $a \in \tilde{X}$  and  $b \in p^{-1}(p(a))$ , it is now possible to conclude that  $f(a) = f(b)$  is true. Let  $x := \pi(a)$ . Due to (\*), functions  $g \in S_x$  and  $h \in A$  exist with  $f = h/g \circ \pi$ . Since  $a$  and  $b$  are equivalent with respect to the equivalence relation  $R$ ,  $f(a) = f(b)$  follows, and a continuous function  $F: Y \rightarrow \mathbf{C}$  exists with  $F \circ p = f$ .

Since the Stein complex structure on  $Y$  is not in general the canonical ringed quotient structure, it is still necessary to verify that  $F$  is holomorphic in order to prove the density of  $A$  in  $\mathcal{O}(\tilde{X})$ . To that end, let  $H \in \mathcal{O}(Y)$  and  $G \in \mathcal{O}(Y)$  have the property that  $H \circ p = h$  and  $G \circ p = g \circ \pi$ . Such functions exist because  $p^*(\mathcal{O}(Y)) = \bar{A}$  holds. Then  $F = H/G$  follows, and the germ  $F_{p(a)}$  is the germ of a holomorphic function at  $p(a)$ , since the germ  $G_{p(a)}$  of  $G$  at  $p(a)$  is a unit. The surjectivity of  $p$  implies that  $F$  is holomorphic on  $Y$ , completing the proof of the theorem.

Note that the topology induced by  $\mathcal{O}(\tilde{X})$  on any subalgebra  $A$  of  $\mathcal{O}(\tilde{X})$  is the metrizable topology of uniform convergence on compact subsets of  $X$ . Because the closure  $\bar{A}$  of  $A$  in  $\mathcal{O}(\tilde{X})$  is its completion,  $\bar{A}$  can be obtained without referring directly to  $\mathcal{O}(\tilde{X})$ . Thus the Main Theorem can be stated as follows:

*If  $\tilde{X}$  denotes the normalization of an irreducible Stein space  $X$ , then  $\mathcal{O}(\tilde{X})$  is the completion of the integral closure  $\widetilde{\mathcal{O}(X)}$  of  $\mathcal{O}(X)$ .*

### 3. APPLICATIONS

In this section  $X$  will denote an irreducible Stein space with normalization  $\pi: \tilde{X} \rightarrow X$ ,  $\widetilde{\mathcal{O}(X)}$  will be the integral closure of the holomorphic functions  $\mathcal{O}(X)$  on  $X$ ,  $\tilde{\mathcal{O}}(X)$  the Fréchet algebra of weakly holomorphic functions on  $X$  (or equivalently, the Fréchet algebra of holomorphic functions  $\mathcal{O}(\tilde{X})$  on  $\tilde{X}$ ), and

$$S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\} \quad \text{for } x \in X.$$

Although the example given in the first section shows that the algebras  $\widetilde{\mathcal{O}(X)}$  and  $\mathcal{O}(\tilde{X})$  are not always equal, the inclusion (\*) in the proof of the Main Theorem implies that they are locally equal in the following sense.

**THEOREM 2.** *For every  $x \in X$ , the localizations of  $\widetilde{\mathcal{O}(X)}$  and  $\mathcal{O}(\tilde{X})$  with respect to  $S_x$  coincide.*