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THE EULER CLASS OF ORTHOGONAL RATIONAL REPRESENTATIONS OF FINITE GROUPS

by Jean-Marc Piveteau

If $\rho: G \to SO_n(\mathbf{R})$ is an orthogonal representation of the group G, then the Euler class $e(\rho)$ is defined as Euler class of the flat real vector bundle over BG associated with ρ. For representations of finite groups over a number field $\mathbf{K}$ there is a uniform bound, depending on $\mathbf{K}$ and on the degree of the representation only, for the order of the Euler class. This bound has been extensively studied by Eckmann and Mislin ([1], [2], [3]). In this note we discuss analogous bounds for orthogonal representations over the field $\mathbf{Q}$ of rational numbers. Since the best upper bound for odd dimensional representations is equal to two (cf. [3]), we consider the case of even dimensional $\mathbf{Q}$-representations. We will write $F_{\mathbf{Q}}(m)$ for the best upper bound for the order of the Euler Class $e(\rho)$, where ρ ranges over all $2m$-dimensional representations of finite groups over $\mathbf{Q}$. Thus, for every representation $\rho: G \to SO_{2m}(\mathbf{Q})$ of any finite group G, it follows that $F_{\mathbf{Q}}(m) \cdot e(\rho) = 0 \in H^{2m}(G; \mathbf{Z})$, and $F_{\mathbf{Q}}(m)$ is the best possible. The prime factorisation of the numbers $F_{\mathbf{Q}}(m)$ is given as follows:

Main Theorem. *For odd* m *we have* $F_{\mathbf{Q}}(m) = 4$. *For even* m, *if we write* $F_{\mathbf{Q}}(m, p)$ *for the* p-*primary part of* $F_{\mathbf{Q}}(m)$ $(p$: *prime*$)$, *we have*:

$$F_{\mathbf{Q}}(m, p) = \begin{cases} 1, \text{ if } n \neq 0 \bmod (p-1) \text{ or if } n = Np^k(p-1) \text{ with} \\ \qquad \text{g.c.d. } (p, N) = 1, \; N \text{ odd and } p = 7 \bmod 8, \\ p\text{-primary part of } \operatorname{den}(B_m/m) \text{ otherwise,} \end{cases}$$

where B_m *is the* m-*th Bernoulli-number and* $\operatorname{den}(B_m/m)$ *is the denominator of* B_m/m *written in its lowest terms.*

Note that $F_{\mathbf{Q}}(m)$ is a lower bound for the order of the universal profinite Euler class $\hat{e}_{2m}(\mathbf{Q})$ considered by Eckmann and Mislin in [3].

The two first sections contain preliminary results about bilinear forms and orthogonal representations. In the last section, we prove the main theorem.

This paper is a summary of some results of the thesis [8] I have written under the direction of Guido Mislin. I want to express him on this occasion my gratefulness for his stimulating advices and the interest he constantly showed for this work.

1. Invariant Bilinear Forms

Let $\mathbf{K}$ be a field of characteristic 0, V a finite dimensional vector space over $\mathbf{K}$ and $\rho: G \to GL(V)$ a $\mathbf{K}$-representation of the group G. A $\mathbf{K}$-bilinearform $\alpha: V \times V \to \mathbf{K}$ is called ρ-*invariant* if

$$\alpha(\rho(g)x, \rho(g)y) = \alpha(x, y) \qquad \forall x, y \in V, \quad \forall g \in G.$$

If G is finite, then for any bilinear form γ the form $\bar{\gamma}$ defined by

$$\bar{\gamma}(x, y) := \sum_{g \in G} \gamma(\rho(g)x, \rho(g)y)$$

is ρ-invariant.

(1.1) *Remark.* If α is definit (i.e. $\alpha(x, x) = 0 \Rightarrow x = 0$) and if ρ splits in a direct sum $\rho = \rho_1 \oplus \rho_2$, the restriction ρ' of ρ to the orthogonal complement of the invariant space corresponding to ρ_1 is equivalent to ρ_2. Since we always can substitute a representation or a bilinear form by an equivalent one, we can assume that the representation space of a sum is an orthogonal sum of corresponding invariant subspaces.

We call *standard bilinear form* (of dimension m) the map $\beta_m: \mathbf{K}^m \times \mathbf{K}^m \to \mathbf{K}$ given by

$$\beta_m(x, y) := \sum_{i=1}^{m} x_i y_i \quad \text{with} \quad x = (x_1, \ldots, x_m) \quad \text{and} \quad y = (y_1, \ldots, y_m).$$

The group $O_m(\mathbf{K})$ is the subgroup of $GL_m(\mathbf{K})$ of matrices (a_{ij}) such that $\sum_k a_{ik}a_{jk} = \delta_{ij}$ for all i, j. The group $SO_m(\mathbf{K})$ is the subgroup of $O_m(\mathbf{K})$ of matrices (a_{ij}) with $\det(a_{ij}) = 1$. It is therefore evident that a representation $\rho: G \to GL_m(\mathbf{K})$ is realizable over $O_m(\mathbf{K})$ if and only if there is a ρ-invariant symmetric bilinear form which is equivalent to the standard bilinear form.

Let p be a prime number. Up to equivalence, there is a unique irreducible faithful $\mathbf{Q}$-representation σ of $\mathbf{Z}/p$; it is given by

$$\sigma: \mathbf{Z}/p \to GL_{p-1}(\mathbf{Q})$$

$$1 \mapsto A := \begin{bmatrix} 0 & . & . & . & -1 \\ 1 & . & . & . & -1 \\ & & . & & \\ . & . & . & 1 & -1 \end{bmatrix}$$

We can identify the irreducible faithful $\mathbf{Q}[\mathbf{Z}/p]$-Module $\mathbf{Q}^{p-1}$ with $\mathbf{Q}(\zeta_p)$ (ζ_p: primitive p-th root of unity, $1 \in \mathbf{Z}/p$ acts on $\mathbf{Q}(\zeta_p)$ by multiplication with ζ_p). Any symmetric σ-invariant bilinear form is given by $tr_{\mathbf{Q}(\zeta_p)/\mathbf{Q}}(ax\bar{y}$ with $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ (cf. [4] or [6]). We write γ_a for the σ-invariant bilinear form corresponding to $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$.

(1.2) LEMMA. *The discriminant of* γ_a *in* $\mathbf{Q}/\mathbf{Q}^{*2}$ *is equal to* p *mod* $\mathbf{Q}^{*2}$.

Proof. Since $a \in \mathbf{L} := \mathbf{Q}(\zeta_p + \zeta_p^{-1})$ we have: $\gamma_a = tr_{\mathbf{L}/\mathbf{Q}}(tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}} ax\bar{y})$. An easy computation shows that $tr_{\mathbf{Q}(\zeta_p)/\mathbf{L}}(ax\bar{y})$ is a 2-dimensional symmetric $\mathbf{L}$-bilinearform with discriminant $4 - (\zeta_p + \zeta_p^{-1})^2 \bmod \mathbf{L}^{*2} \in \mathbf{L}/\mathbf{L}^{*2}$. Applying [7, Lemma 2.2] we conclude that the discriminant of γ_a is independant of $a \in \mathbf{L}$. Consider now the matrix representation of σ given before (σ: irreducible faithful $\mathbf{Q}$-representation of $\mathbf{Z}/p$). Let C be the $(p-1) \times (p-1)$-matrix given by:

$$C := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \cdot & \cdot \\ & & & -1 & 2 \end{bmatrix}$$

It is easy to check that C is the matrix of a σ-invariant symmetric bilinear form. The Lemma follows since the determinant of C is equal to p.

2. ORTHOGONAL REPRESENTATIONS OF p-GROUPS

Let $p > 2$ be an odd prime. The integer $l_{\mathbf{Q}}(p)$ is defined by

$$l_{\mathbf{Q}}(p) := \text{g.c.d.} \left\{ m > 1 \;\middle|\; \begin{array}{l} \text{the } m\text{-fold direct sum } \sigma \oplus \ldots \oplus \sigma \text{ of the irre-} \\ \text{ducible faithful } \mathbf{Q}\text{-representation } \sigma \text{ of } \mathbf{Z}/p \text{ is} \\ \text{equivalent to an orthogonal representation} \end{array} \right\}$$

The importance played by cyclic groups in the investigation of representations of p-groups is given by the following result (cf. [1, Theorem (1.10)]):

(2.1) PROPOSITION. *Let* G *be a finite p-group* ($p>2$) *and let* ρ *be an irreducible* $\mathbf{Q}$*-representation of* G. *Then either* ρ *is induced from a representation* θ *of a normal subgroup of index* p, *or* ρ *factors through a* $\mathbf{Q}$*-representation of* $\mathbf{Z}/p$.

The degree of an irreducible non trivial **Q**-representation of a finite p-group is therefore of the form $p^k(p-1)$ $(k=0, 1, 2, \ldots)$, cf. [1, Corollary (1.11)].

(2.2) PROPOSITION. *Let G be a p-group $(p>2)$ and $\rho: G \to SO_{2m}(\mathbf{Q})$ a representation of G with $2m \neq 0 \bmod (l_{\mathbf{Q}}(p) \cdot (p-1))$. Then ρ has a fixed point (i.e. $\rho = 1 \oplus \tau$ where 1 is the unique 1-dimensional* **Q***-representation of G).*

We will need the following lemma for the proof of (2.2):

(2.3) LEMMA. *Let $\rho: G \to GL_m(\mathbf{Q})$ be an irreducible non trivial representation of the p-group $G(p>2)$ and let ψ be a ρ-invariant symmetric bilinear form. If we write σ for the irreducible faithful representation of $\mathbf{Z}/p$, then there exist σ-invariant bilinear forms $\Gamma_1, \ldots, \Gamma_s$ such that ψ is equivalent to the orthogonal sum $\Gamma_1 \perp \ldots \perp \Gamma_s$.*

Proof. Let $p^k(p-1)$ be the degree of ρ. We prove the lemma by induction on k. For $k = 0$, ρ factors through the irreducible faithful representation σ of $\mathbf{Z}/p$. Every ρ-invariant symmetric bilinearform ψ is therefore σ-invariant. For $k > 0$, ρ is induced by a representation θ of a normal subgroup H of index p. The restriction ρ_H of ρ to H splits in a direct sum: $\rho = \theta_1 \oplus \ldots \oplus \theta_p$ with $\theta = \theta_1$ and θ_i is irreducible for $i = 1, \ldots, p$. By (1.1) we can assume that $\mathbf{Q}^m$ is the orthogonal sum of the corresponding irreducible invariant subspaces. The assertion follows by induction.

Proof of (2.2). If $G = \mathbf{Z}/p$, we split ρ in a direct sum: $\rho = n_0 1 \oplus n_1\sigma$ (1: one dimensional representation of $\mathbf{Z}/p$; σ: irreducible faithful representation of $\mathbf{Z}/p$). If $n_0 = 0$ then n_1 must be a multiple of $l_{\mathbf{Q}}(p)$, i.e. we have $2m = 0 \bmod (p-1)l_{\mathbf{Q}}(p)$. Contradiction.

If G is not $\mathbf{Z}/p$, we split ρ in a direct sum of irreducible representations: $\rho = \rho_1 \oplus \ldots \oplus \rho_t$, chosen in such a way that $\mathbf{Q}^{2m}$ is the orthogonal sum of the corresponding invariant subspaces. Suppose now that ρ has no fixed points. Then all ρ_i are non trivial and it follows from (2.3) that any ρ-invariant symmetric bilinear form is equivalent to an orthogonal sum of σ-invariant symmetric bilinear forms. We can therefore construct a representation $\mathbf{Z}/p \to SO_{2m}(\mathbf{Q})$ without fixed points, what contradicts the first part of the proof.

The rest of the section is devoted to the computation of $l_{\mathbf{Q}}(p)$, p odd prime.

(2.4) PROPOSITION. $l_{\mathbf{Q}}(p) = \begin{cases} 2 & \textit{if } p \neq 7 \bmod 8 \\ 4 & \textit{otherwise.} \end{cases}$

Proof. For each $a \in \mathbf{Q}(\zeta_p + \zeta_p^{-1})$, the discriminant of γ_a is not a square in $\mathbf{Q}$ (cf. lemma (1.2)). Therefore $l_{\mathbf{Q}}(p)$ must be even. The 4-fold orthogonal sum of a $\mathbf{Q}$-bilinear form is equivalent to the standard bilinear form, since every integer is sum of four squares. Let C be the matrix considered in the proof of lemma (1.2). If it is possible to find two rational numbers u and v such that the matrix $X_{u,v}$

$$X_{u,v} := \begin{bmatrix} uC & 0 \\ 0 & vC \end{bmatrix}$$

represents a bilinear form $\xi_{u,v}$ which is equivalent to the standard one, then the representation $\sigma \oplus \sigma$ is equivalent to an orthogonal representation. This sufficient condition is also necessary if $p = 3 \bmod 4$ (cf. [5]). For a prime p, let $\mathbf{Q}_p$ be the field of p-adic numbers and write $\mathbf{Q}_\infty$ for $\mathbf{R}$ as usual. For $a, b \in \mathbf{Q}$ and for $v = 2, 3, 5, 7, \ldots, \infty$ we write $(a, b)_v$ for the Hilbert symbol of a and b relatively to $\mathbf{Q}_v$. For a bilinearform α given in an orthogonal base by the diagonal matrix

$$\begin{bmatrix} a_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & a_n \end{bmatrix}$$

we write $H_v(\alpha)$ $(v = 2, 3, \ldots, \infty)$ for the Hasse invariant, which is defined by

$$H_v(\alpha) = \prod_{i<j} (a_i, a_j)_v$$

Using the formulas given for example by [9] to compute the Hilbert symbol, one check that:

$$\begin{array}{lllll} H_v(\xi_{1,1}) & = & 1 & \text{if } p \neq 3 \bmod 4 & \text{for } v = 2, 3, 5, 7, \ldots, \infty\,, \\ H_2(\xi_{u,v}) & = & -1 & \text{if } p = 7 \bmod 8 & \text{for any } u \text{ and any } v\,, \\ H_v(\xi_{2p,1}) & = & 1 & \text{if } p = 1 \bmod 8 & \text{for } v = 2, 3, 5, 7, \ldots \infty\,. \end{array}$$

Since the discriminant of $\xi_{u,v}$ is $1 \in \mathbf{Q}/\mathbf{Q}^{*2}$ and since $\xi_{u,v}$ is positive definit for any u and any v, it follows that $\sigma \oplus \sigma$ is equivalent to an orthogonal representation if and only if $p \neq 7 \bmod 8$. It remains to show that, for $p = 7 \bmod 8$, the $2n$-fold orthogonal sum μ given by the matrix H:

$$H := \begin{bmatrix} u_1 C & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & u_{2n} C \end{bmatrix}$$

is isomorphic to the standard bilinear form if and only if n is even. Let u_{odd} and u_{even} defined by:

$$u_{\text{even}} := \prod_{k=1}^{n} u_{2k} \qquad u_{\text{odd}} := \prod_{k=1}^{n} u_{2k-1};$$

an easy computation shows that $H_v(\xi_{u_{\text{even}}, u_{\text{odd}}}) = H_v(\mu)$ if n is odd. The proposition follows.

3. Proof of the main theorem

(3.1) Lemma. *Let p be a prime number $(p>2)$. For every integer m satisfying $2m \neq 0 \bmod (p-1) \cdot l_{\mathbf{Q}}(p)$ we have $F_{\mathbf{Q}}(m, p) = 1$.*

Proof. Let G be a p-group, $p > 2$. It follows from (2.2) that any representation ρ of G splits: $\rho = 1 \oplus \tau$ (1 is the 1-dimensional representation of G). Then we have $e(\rho) = e(1)e(\tau) = 0$.

We are now able to prove the main theorem. It has been showed in [3] that $F_{\mathbf{Q}}(n) = 4$ if n is odd. If n is even, four cases have to be distinguished. If $p = 2$ then the $n/2^{N-2}$-fold sum of the irreducible faithful representation of $\mathbf{Z}/2^N$, where 2^N is the 2-primary part of $\operatorname{den}(B_n/n)$, is an orthogonal representation with Euler class of order 2^N (cf. [1]). Let now p be an odd prime. Since the irreducible faithful representation ν of $\mathbf{Z}/p^r (r \geqslant 1)$ is induced by the irreducible faithful representation of $\mathbf{Z}/p \subset \mathbf{Z}/p^r$, the M-fold sum of ν is equivalent to an orthogonal representation if and only if $l_{\mathbf{Q}}(p)$ divides M. Write $n = Np^k(p-1)$ with g.c.d. $(N, p) = 1$. If N is even, the $2N$-fold sum of the irreducible faithful representation of $\mathbf{Z}/p^{k+1}$ is orthogonal and has Euler class of order p^{k+1} (cf. [1]); if N is odd and $p \neq 7 \bmod 8$ then the $2N$-fold sum of the irreducible faithful representation of $\mathbf{Z}/p^{k+1}$ is orthogonal and has Euler class of order p^{k+1} (cf. [1]). In the three cases, the statement follows from the well known characterization of $\operatorname{den}(B_n/n)$ (cf. [1] for example). Eventually, applying (3.1) we see that $F_{\mathbf{Q}}(n, p) = 1$ if N is odd and $p = 7 \bmod 8$.

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Jean-Marc Piveteau

Department of Mathematics
ETH — Zürich
CH - 8092 Zürich (Switzerland)