Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **29.03.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

L'Enseignement Mathématique, t. 34 (1988), p. 357-364

GLOBAL CONSTRUCTION OF THE NORMALIZATION OF STEIN SPACES

by Sandra HAYES and Geneviève POURCIN

INTRODUCTION

A fundamental tool in the theory of complex manifolds X is Riemann's Theorem on Removable Singularities of holomorphic functions which ensures that all functions holomorphic outside of a rare analytic subset of X and locally bounded on X can be extended to functions holomorphic on all of X. In other words, all weakly holomorphic functions on X are actually holomorphic. Although this theorem does not hold for arbitrary complex spaces, Oka [12] showed in 1951 that every complex space X can be modified to a complex space $\tilde{X}$ for which Riemann's continuation theorem is valid, the so-called normalization of X.

Stein spaces X are complex spaces which can be completely described by the algebra $\mathcal{O}(X)$ of global holomorphic functions. Since a complex space is Stein if and only if its normalization is Stein [11], it is natural to ask if the normalization $\tilde{X}$ of a Stein space X can be constructed just from the holomorphic functions on X. Phrased differently, the question is whether the algebra $\mathcal{O}(\tilde{X})$ of all holomorphic functions on $\tilde{X}$ or equivalently, the algebra $\tilde{\mathcal{O}}(X)$ of all weakly holomorphic functions on X, can be derived from the algebra $\mathcal{O}(X)$ of holomorphic functions on X.

The purpose of this paper is to demonstrate that this is possible when X is irreducible: $\tilde{\mathcal{O}}(X)$ is the topological closure of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$. An example given in § 1 shows that $\widetilde{\mathcal{O}(X)}$ is not in general topologically closed even if X is locally irreducible. $\tilde{\mathcal{O}}(X)$ can also be obtained by taking the intersection of the localizations $S_x^{-1}\,\widetilde{\mathcal{O}(X)}$ of the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ with respect to $S_x := \{g \in \mathcal{O}(X) : g(x) \neq 0\}$ for every $x \in X$ (see § 3).

The proof relies on an analytic and an algebraic theorem, namely Rossi's theorem [13] generalizing the Remmert quotient and the integral closure theorem of Mori-Nagata [7].

An analytic consequence of the construction presented here is that the normalization $\tilde{X}$ of an irreducible Stein space X is $\widetilde{\mathcal{O}(X)}$-convex, $\widetilde{\mathcal{O}(X)}$-separable and has local coordinates by functions in $\widetilde{\mathcal{O}(X)}$. Some algebraic results are that $\mathcal{O}(\tilde{X})$ is completely normal and that the two algebras $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ are always locally equal, i.e. their localizations at all maximal ideals in $\mathcal{O}(X)$ are equal.

In this paper, a complex space refers to a reduced complex space with countable topology.

1. Example of a Stein space X with $\widetilde{\mathcal{O}(X)} \neq \mathcal{O}(\tilde{X})$

Let $(X, \mathcal{O})$ be a complex space with normalization $\pi: \tilde{X} \to X$. Since π is surjective, the map $\pi^*: \mathcal{O}(X) \to \mathcal{O}(\tilde{X})$, $f \mapsto f \circ \pi$, is injective and the holomorphic functions $\mathcal{O}(X)$ on X can be considered to be a subring of the holomorphic functions $\mathcal{O}(\tilde{X})$ on the normalization $\tilde{X}$ of X; this will be indicated by $\mathcal{O}(X) \subset \mathcal{O}(\tilde{X})$. If X is irreducible and Stein, then $\mathcal{O}(\tilde{X})$ contains the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ but does not always coincide with it, as will be shown in this section.

For an irreducible complex space $(X, \mathcal{O})$, the integral domain $\mathcal{O}(X)$ is said to be *normal*, if it is integrally closed in its field of fractions $Q(\mathcal{O}(X))$, i.e. $\widetilde{\mathcal{O}(X)} = \mathcal{O}(X)$. Recall that $Q(\mathcal{O}(X))$ is the field of meromorphic functions $M(X)$ on X when X is irreducible and Stein due to Theorem B [10, 53.1, 52.17], and that the algebras $M(X)$ and $M(\tilde{X})$ are isomorphic for every complex space X [8, p. 161].

The following characterization of normal irreducible Stein spaces X by their global function algebra $\mathcal{O}(X)$ is essentially contained in [2, § 1, p. 35].

Theorem 1. *An irreducible Stein space X is normal if and only if the integral domain $\mathcal{O}(X)$ is normal.*

An analysis of the proof shows that even when X is just irreducible and normal, $\mathcal{O}(X)$ is also normal. Theorem 1 implies

Corollary 1. *For an irreducible Stein space X with normalization $\tilde{X}$, the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ is contained in $\mathcal{O}(\tilde{X})$.*

The following example shows that there are functions $f \in \mathcal{O}(\tilde{X})$ which are not integral over $\mathcal{O}(X)$. In this example, $X := (\mathbf{C}, \mathcal{O}')$ is an irreducible