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ON TORRES-TYPE RELATIONS FOR THE ALEXANDER POLYNOMIALS OF LINKS

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§ 1. INTRODUCTION

The classical formula of Torres [5] relates the (first) Alexander polynomial of a link K in S^3 with that of the sublink of K obtained by deleting a component. The aim of the present paper is to establish a Torres-type formula for Alexander polynomials of higher-dimensional links. We also discuss analogous formulas for higher Alexander polynomials of links in S^3.

An n-component link in the sphere S^m is an ordered collection of n disjoint smooth imbedded oriented $(m-2)$-dimensional spheres in S^m. With each odd-dimensional link $K \subset S^{2r+1}$ one associates a Λ_n-module $H_r(\tilde{X})$, where Λ_n is the Laurent polynomial ring $\mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$, X is the exterior of K and $\tilde{X}$ is the maximal abelian covering of X. The module $H_r(\tilde{X})$ algebraically gives rise to a sequence of Fitting (or determinantal) invariants $\Delta_1(K), \Delta_2(K), \dots$, which are elements of Λ_n defined up to multiplication by monomials $\pm t_1^{s_1} \dots t_n^{s_n}$ (see [1] or § 3). The polynomial $\Delta_i(K)$ is called the i-th Alexander polynomial of K. The first Alexander polynomial $\Delta_1(K)$ is also denoted by $\Delta(K)$ and called "the Alexander polynomial of K".

THEOREM (Torres [5]). *Let K be an n-component link in S^3 with $n \geqslant 2$ and let L be the sublink of K obtained by deleting the n-th component. Then*

$$\Delta(K)(t_1, \dots, t_{n-1}, 1) = \begin{cases} (t_1^{l_1} \dots t_{n-1}^{l_{n-1}} - 1)\Delta(L) & \text{if} \quad n > 2 \\ \dfrac{t_1^{l_1} - 1}{t_1 - 1} \Delta(L) & \text{if} \quad n = 2 \end{cases}$$

where l_i denotes the linking number of the i-th and n-th components of K.

The following theorem can be considered as a high-dimensional variant of the Torres theorem.

THEOREM 1. *Let* K *be an n-component link in* S^m *with odd* $m \geqslant 5$. *Let* L *be the sublink of* K *obtained by deleting the n-th component. Then there exists an element* λ *of* Λ_{n-1} *such that*

$$(1) \qquad \Delta(L) = \Delta(K)(t_1, \ldots, t_{n-1}, 1) \cdot \lambda\bar{\lambda}.$$

Here the overbar denotes the involution of the Laurent polynomial ring Λ_{n-1} which sends each polynomial $f(t_1, \ldots, t_{n-1})$ into $f(t_1^{-1}, \ldots, t_{n-1}^{-1})$.

It is well known that for any link $K \subset S^m$ with odd $m \geqslant 5$ the Alexander polynomial $\Delta(K)$ is non-zero. Moreover,

$$\text{aug}(\Delta(K)) = \Delta(K)(1, 1, \ldots, 1) = \pm 1$$

(see [1]). This implies that $\text{aug}(\lambda) = \pm 1$ for any λ satisfying (1). It seems that there are no other restrictions on λ; one may even guess that for any $\Delta \in \Lambda_n$, $\lambda \in \Lambda_{n-1}$ with $\text{aug}(\Delta) = \text{aug}(\lambda) = \pm 1$ and $\bar{\Delta} \doteq \Delta$ there exists a pair K, L as in Theorem 1 such that $\Delta(K) \doteq \Delta$ and $\Delta(L) \doteq \Delta(t_1, \ldots, t_{n-1}, 1)\lambda\bar{\lambda}$. Here and below the symbol $\doteq$ denotes the equality of Laurent polynomials up to multiplication by a monomial $\pm t_1^{s_1} \ldots t_n^{s_n}$.

Let us call two Laurent polynomials $\Delta, \Delta' \in \Lambda_n$ algebraically cobordant if there exist polynomials $\lambda, \lambda' \in \Lambda_n$ such that $\Delta\lambda\bar{\lambda} \doteq \Delta'\lambda'\bar{\lambda'}$ and $\text{aug}(\lambda) = \text{aug}(\lambda') = \pm 1$. This terminology is suggested by the fact that Alexander polynomials of (smoothly) cobordant links are algebraically cobordant (see [4]). The formula (1) enables us to calculate Alexander polynomials of all sublinks of a given link, up to algebraic cobordism. It is curious to note that if K, K' are n-component links in S^m with odd $m \geqslant 5$ and if polynomials $\Delta(K)$, $\Delta(K')$ are algebraically cobordant then Theorem 1 implies that Alexander polynomials of corresponding sublinks of K, K' are algebraically cobordant to each other. This fact reflects the evident property of geometric cobordisms: corresponding sublinks of cobordant links are cobordant.

I do not know if it is possible to associate with a link K some preferred $\lambda = \lambda(K)$ satisfying (1).

The remaining part of the Introduction is concerned with the classical links. The symbols $K, L, n, l_1, \ldots, l_{n-1}$ denote the same objects as in the Torres theorem formulated above. It may well happen that some of the Alexander polynomials $\Delta_1(K), \Delta_2(K), \ldots$ are equal to zero. Denote by $u = u(K)$ the minimal integer $u \geqslant 1$ such that $\Delta_u(K) \neq 0$. Since $\Delta_{i+1}(K)$ divides $\Delta_i(K)$ for all i, $\Delta_i(K) = 0$ for $i < u$ and $\Delta_i(K) \neq 0$ for $i \geqslant u(K)$.

In view of the Torres theorem it is natural to look for a relationship between $\Delta_{u(K)}(K)$ and a corresponding invariant of L. In the case $u(K) = 1$ we have the Torres formula, so we shall restrict ourselves to the case $u(K) \geqslant 2$ (i.e. the case $\Delta(K) = 0$).

The integers $u(K), u(L)$ are related by the inequality $u(L) \geqslant u(K) - 1$ (see [1] or § 4). If $l_i \neq 0$ at least for one $i = 1, ..., n-1$ then the stronger inequality holds: $u(L) \geqslant u(K)$. These inequalities suggest to relate $\Delta_u(K)$ (where we put $u = u(K)$) with $\Delta_{u-1}(L)$ and $\Delta_u(L)$. The following relationship between $\Delta_u(K)$ and $\Delta_u(L)$ was established in [4].

THEOREM ([4, Theorem 5.5.1]). *If* $u = u(K) \geqslant 2$ *then there exist an element* λ *of* Λ_{n-1} *and a subset* β *of the set* $\{1, 2, ..., n-1\}$ *such that*

(2) $$(t_1^{l_1} \ldots t_{n-1}^{l_{n-1}} - 1)\Delta_u(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda\bar{\lambda} \cdot \Delta_u(K)(t_1, ..., t_{n-1}, 1) .$$

Several remarks are in order. a) The non-trivial case of the Theorem is the case where at least one of the integers $l_1, ..., l_{n-1}$ is non-zero: otherwise $t_1^{l_1} \ldots t_{n-1}^{l_{n-1}} - 1 = 0$ and we may put $\lambda = 0$. b) Formula (2) is proved in [4] under the additional condition $u(L) = u(K)$. However if $u(L) < u(K)$ then we have the trivial case $l_1 = l_2 = ... = l_{n-1} = 0$; if $u(L) > u(K)$ then $\Delta_{u(K)}(L) = 0$ and we may put $\lambda = 0$. c) Formula (2) combines the factors from the Torres formula, formula (1) and a new factor $\prod (t_i - 1)$. All these factors may be non-trivial (see [4]). d) An explicit construction of the set $\beta = \beta(K)$ is given in [4, § 5]. I do not know if there exists a preferred $\lambda = \lambda(K)$ which satisfies (2).

The relationships between the polynomials $\Delta_u(K)$ and $\Delta_{u-1}(L)$ were first considered by Levine [2] in the case $u = 2$.

THEOREM (Levine [2]). *If* $u(K) \geqslant 2$ *then there exist an element* $\lambda \in \Lambda_{n-1}$ *and a set* $\beta \subset \{1, 2, ..., n-1\}$ *such that*

$$\Delta(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda\bar{\lambda} \cdot \Delta_2(K)(t_1, ..., t_{n-1}, 1) .$$

Note that in the case $u(K) > 2$ the Levine's theorem is evident: if $u(K) > 2$ then $u(L) \geqslant u(K) - 1 > 1$ so that $\Delta(L) = \Delta_2(K) = 0$.

The following theorem generalizes the Levine's result.

THEOREM 2. *If* $u = u(K) \geqslant 2$ *then there exist an element* λ *of* Λ_{n-1} *and a set* $\beta \subset \{1, 2, ..., n-1\}$ *such that*

$$\Delta_{u-1}(L) = \prod_{i \in \beta} (t_i - 1) \cdot \lambda\bar{\lambda} \cdot \Delta_u(K)(t_1, ..., t_{n-1}, 1) .$$

The non-trivial case of Theorem 2 is the case $l_1 = l_2 = \dots = l_{n-1} = 0$: otherwise $u(L) \geqslant u$ so that $\Delta_{u-1}(L) = 0$ and we may put $\lambda = 0$.

The proof of Theorems 1, 2 goes along the same lines as the proof of the formula (2) given in [4]. These proofs are based on a relationship between the Alexander polynomials and Reidemeister-type torsions, established in [4]. This relationship is recalled in § 2. In § 3 several easy algebraic lemmas are proved. Theorems 1, 2 are proved in § 4.

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§ 2. Torsions of chain complexes and manifolds

2.1. The torsion of a chain complex (see [3]). Let Q be a field. If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are two bases of a Q-module then $a_i = \sum_{j=1}^{n} c_{i,j} b_j$ where $(c_{i,j})$ is a non-singular $n \times n$-matrix over Q; the determinant $\det(c_{i,j}) \in Q \backslash 0$ is denoted by $[a/b]$.

Let $C = (C_m \to \cdots \to C_0)$ be a chain Q-complex. Suppose that each Q-module C_i is finite dimensional with a preferred basis c_i and each Q-module $H_i(C)$ also has a preferred basis h_i. (The case $C_i = 0$ or $H_i(C) = 0$ is not excluded; by definition the zero module has the empty basis.) In this setting one defines the torsion $\tau(C) \in Q$ as follows. For each $i = 1, 2, \dots, m$ choose a sequence $b_i = (b^i_1, \dots, b^i_{r_i})$ of elements of C_i such that $\partial_{i-1}(b_i) = (\partial_{i-1}(b^i_1), \dots, \partial_{i-1}(b^i_{r_i}))$ is a basis in $\operatorname{Im}(\partial_{i-1}\colon C_i \to C_{i-1})$. For each $i = 0, 1, \dots, m$ choose a lifting $\tilde{h}_i$ of the basis h_i to $\operatorname{Ker} \partial_{i-1}$. The combined sequence $\partial_i(b_{i+1})\tilde{h}_i b_i$ is a basis in C_i. (It is understood that $b_0 = \emptyset$ and $b_{m+1} = \emptyset$). Put

$$\tau(C) = \prod_{i=0}^{m} [\partial_i(b_{i+1})\tilde{h}_i b_i / c_i]^{\varepsilon(i)} \tag{3}$$

where $\varepsilon(i) = (-1)^{i+1}$. Clearly, $\tau(C) \in Q \backslash 0$. It is easy to verify that $\tau(C)$ does not depend on the choice of b_i and $\tilde{h}_i$.

(Note that the torsion of C defined in Milnor's survey article [3] equals $\pm\, \tau(C)^{-1} \in Q/\pm 1$ and that Milnor uses the additive notation for the multiplication in $Q \backslash 0 = K_1(Q)$.)

2.1.1. Lemma (multiplicativity of torsion). *Let* $0 \to C' \to C \to C'' \to 0$ *be a short exact sequence of m-dimensional chain complexes over a field* Q.

Suppose that for all $i = 0, 1, \ldots, m$ *the modules* C_i, C'_i, C''_i *are provided with preferred bases* c'_i, c_i, c''_i *which are compatible, in the sense that* $[c'_i c''_i / c_i] = \pm 1$. *Suppose that for all* $i = 0, 1, \ldots, m$ *the homology modules* $H_i(C), H_i(C'), H_i(C'')$ *are provided with preferred bases. Let* $\mathscr{H}$ *be the homology sequence of the sequence* $0 \to C' \to C \to C'' \to 0$:

$$\mathscr{H} = (H_m(C') \to H_m(C) \to \cdots \to H_0(C) \to H_0(C'')) .$$

Consider $\mathscr{H}$ *as an acyclic based chain complex over* Q. *Then* $\tau(C) = \pm \tau(C')\tau(C'')\tau(\mathscr{H})$.

For a proof see [3].

2.2. The torsion ω. Let M be an orientable compact smooth manifold of odd dimension m with $\operatorname{rg} H_1(M) \geqslant 1$. Denote the free abelian group $H_1(M)/\operatorname{Tors} H_1(M)$ by G. Denote the fraction field of the group ring $\mathbf{Z}[G]$ by Q. Provide Q with the involution $q \mapsto \bar{q}$ which sends $g \in G$ to g^{-1}. The field Q defines via the natural homomorphism $\mathbf{Z}[\pi_1(M)] \to Q$ a system of local coefficients on M. We shall denote this system by the same symbol Q. Assume that $H_*(\partial M ; Q) = 0$. In this setting one can consider a torsion-type invariant $\omega(M)$ of M which is "an element of $Q \backslash 0$ defined up to multiplication by $\pm g q \bar{q}$ with $g \in G$ and $q \in Q \backslash 0$" (see [4]).

Recall the definition of $\omega(M)$ given in [4, § 5]. Let $\tilde{M} \to M$ be the regular covering of M corresponding to the kernel of the natural homomorphism $\pi_1(M) \to G$. Fix a C^1-triangulation of M and the induced G-equivariant triangulation of $\tilde{M}$. Choose over each simplex of the (fixed) triangulation of M a simplex of the triangulation of $\tilde{M}$. These simplices in $\tilde{M}$ being arbitrarily oriented and ordered determine "natural" bases of the modules of the simplicial chain $\mathbf{Z}[G]$-complex $C_*(\tilde{M} ; \mathbf{Z})$. These bases induce "natural" Q-bases in the chain Q-complex

$$C = Q \otimes_{\mathbf{Z}[G]} C_*(\tilde{M} ; \mathbf{Z}) .$$

For all $i = 0, 1, \ldots, m$ choose an arbitrary Q-basis h_i in $H_i(M ; Q) = H_i(C)$. Denote by $\tau(C, h_0, \ldots, h_m)$ the torsion of C with respect to the bases in chain modules constructed above and the bases $h_0, h_1, \ldots, h_m$ in homology. Since $H_*(\partial M ; Q) = 0$ the semi-linear intersection form $H_i(M ; Q) \times H_{m-i}(M ; Q) \to Q$ is non-singular. Let v_i be the matrix of this form regarding the bases h_i and h_{m-i}. Put

$$d = \tau(C, h_0, h_1, \ldots, h_m) \prod_{i=0}^{r} (\det v_i)^{-\varepsilon(i)} \in Q \backslash 0$$

where $r = (m-1)/2$ and $\varepsilon(i) = (-1)^{i+1}$. It is easy to show that under a different choice of natural bases and bases $h_0, h_1, \ldots, h_m$ the element d is replaced by $\pm g q \bar{q} d$ with $g \in G$, $q \in Q \backslash 0$. Thus the set $\{\pm g q \bar{q} d \mid g \in Q \backslash 0\} \subset Q$ does not depend on the choice of bases. It also does not depend on the choice of triangulation in M. It is this set which is $\omega(M)$.

An explicit formula established in [4] enables us to calculate $\omega(M)$ in terms of the orders of $\mathbf{Z}[G]$-modules $H_*(\partial \tilde{M}) = H_*(\partial \tilde{M}; \mathbf{Z})$, $H_*(\tilde{M}) = H_*(\tilde{M}; \mathbf{Z})$ and related modules. (The notion of the order of a module is recalled in Sec. 3.1.) Denote by J the image of the inclusion homomorphism $H_r(\partial \tilde{M}) \to H_r(\tilde{M})$ where $r = (m-1)/2$. Then up to multiples of type $q\bar{q}$ with $q \in Q \backslash 0$

$$\omega(M) = \operatorname{ord}\left(\operatorname{Tors}_{\mathbf{Z}[G]} H_r(M, \partial M)\right) (\operatorname{ord} J)^{\varepsilon(r)} \prod_{i=0}^{r-1} [\operatorname{ord} H_i(\partial M)]^{\varepsilon(i)} \tag{4}$$

(see [4, Theorem 5.1.1]). Note that the equalities $Q \otimes_{\mathbf{Z}[G]} H_*(\partial \tilde{M}) = H_*(\partial \tilde{M}; Q) = 0$ imply that $H_*(\partial \tilde{M})$ and J are torsion $\mathbf{Z}[G]$-modules. Therefore $\operatorname{ord} H_i(\partial \tilde{M})$ and $\operatorname{ord} J$ are non-zero elements of $\mathbf{Z}[G]$.

We shall apply formula (4) in the case where M is the exterior of an n-component link $K \subset S^m$ with odd m. The condition $H_*(\partial M; Q) = 0$ is always fulfilled in this case. Here the field Q is canonically identified with the field of rational functions of n variables $Q_n = Q(t_1, \ldots, t_n)$. Thus $\omega(M) \subset Q_n$. If $m \geqslant 5$ then (4) implies that

$$\Delta(K)(t_1, \ldots, t_n) \cdot \prod_{i=1}^{n} (t_i - 1) \subset \omega(M)\,.$$

If $m = 3$ then there exists a unique subset $\alpha = \alpha(K)$ of the set $\{1, 2, \ldots, n\}$ such that

$$\Delta_{u(K)}(K)(t_1, \ldots, t_n) \cdot \prod_{i \in \alpha} (t_i - 1) \subset \omega(M)\,.$$

For proofs and details consult [4, § 5].

§ 3. Algebraic lemmas

3.1. Preliminary definitions. For a finitely generated module H over a (commutative) domain R we denote by $\operatorname{rk}_R H$ or, briefly, by $\operatorname{rk} H$ the integer $\dim_Q(Q \otimes_R H)$ where $Q = Q(R)$ denotes the field of fractions of R. For a R-linear homomorphism $f: H \to H'$ we put $\operatorname{rk} f = \operatorname{rk}_R f(H)$. Note that if $\bar{R}$ is the localization of R at some multiplicative system then $Q(\bar{R}) = Q(R)$ and therefore the (exact) functor $(H \mapsto \bar{R} \otimes_R H,\ f \mapsto \operatorname{id}_{\bar{R}} \otimes f)$

preserves the ranks of modules and homomorphisms. If H, H' are finitely generated free R-modules and if A is the matrix of a R-homomorphism $H \to H'$ with respect to some bases then $\operatorname{rk} f = \operatorname{rk} A$ where $\operatorname{rk} A$ is the maximal integer r such that some $r \times r$-minor of A is non-zero.

If R is a unique factorization domain with 1 and if A is a matrix with $n < \infty$ columns and possibly infinite number of rows then $\Delta_i(A)$ denotes the greatest common divisor of the $(n-i+1) \times (n-i+1)$-minors of A. Here $i = 1, 2, \ldots$ and $\Delta_i(A)$ is an element of R defined up to a unit multiple. If H is a finitely generated module over R and A is a presentation matrix of H then $\Delta_i(A)$ depends only on H and i; one defines $\Delta_i(H) = \Delta_i(A)$. Clearly $\Delta_i(H) = 0$ for $i \leqslant \operatorname{rg} H = n - \operatorname{rg} A$ and $\Delta_i(H) \neq 0$ for $i > \operatorname{rg} H$. The invariant $\Delta_1(H)$ is denoted also by $\operatorname{ord} H$; it is called the order of H. It is clear that $\operatorname{ord} H \neq 0$ iff $H = \operatorname{Tors}_R H$. For proofs and further information see [1].

Recall, finally, that a local ring is a domain K which has a unique maximal (proper) ideal. The quotient of K by this ideal is a field which we shall call "the field associated to K".

3.2. LEMMA. *Let* R, R' *be (commutative) domains with* 1 *and let* $\varphi: R \to R'$ *be a ring homomorphism. Let* $C = (\cdots \to C_{i+1} \to C_i \to \cdots)$ *be a finitely generated free chain complex over* R *and let* C' *be the chain* R'*-complex* $R' \otimes_R C$. *Then:* (i) $\operatorname{rk}_{R'} H_i(C') \geqslant \operatorname{rk}_R H_i(C)$ *and* $\operatorname{rk} \partial'_i \leqslant \operatorname{rk} \partial_i$ *for all* i *where* ∂_i, ∂'_i *are the boundary homomorphisms* $C_{i+1} \to C_i, C'_{i+1} \to C'_i$; (ii) *if* $\operatorname{rk} H_i(C') = \operatorname{rk} H_i(C)$ *for some* i *then* $\operatorname{rk} \partial'_j = \operatorname{rk} \partial_j$ *for* $j = i, i+1$; (iii) *if* R, R' *are unique factorization Noetherian domains and if* $\operatorname{rk} H_i(C') = \operatorname{rk} H_i(C)$ *then* $\varphi(\operatorname{ord}(\operatorname{Tors}_R H_i(C)))$ *divides* $\operatorname{ord}(\operatorname{Tors}_{R'} H_i(C'))$.

Proof. Let $n = \operatorname{rk} C_i$. Let $A = (a_{p,q})$, $1 \leqslant q \leqslant n$, $1 \leqslant p$, be the matrix of ∂_i with respect to some bases in C_i, C_{i+1}. Then $A' = (\varphi(a_{p,q}))$ is the matrix of ∂'_i with respect to the induced bases in C'_i, C'_{i+1}. It is evident that $\operatorname{rk} \partial'_i = \operatorname{rk} A' \leqslant \operatorname{rk} A = \operatorname{rk} \partial_i$. Therefore

$$\operatorname{rk} H_i(C') = n - \operatorname{rk} \partial'_i - \operatorname{rk} \partial'_{i+1} \geqslant n - \operatorname{rk} \partial_i - \operatorname{rk} \partial_{i+1} = \operatorname{rk} H_i(C).$$

These inequalities imply (i) and (ii).

Put $r = n - \operatorname{rk} A + 1$ and denote the R-module $C_i/\operatorname{Im}\partial_i$ by J. Since A is a presentation matrix of J we have $\operatorname{ord}(\operatorname{Tors}_R J) = \Delta_r(A)$ (see [1, p. 31]). From the exact sequence $0 \to H_i(C) \to J \to C_{i-1}$ we obtain that $\operatorname{Tors} J = \operatorname{Tors} H_i(C)$. Thus $\operatorname{ord}(\operatorname{Tors} H_i(C)) = \Delta_r(A)$. Analogously $\operatorname{ord}(\operatorname{Tors} H_i(C')) = \Delta_{r'}(A')$ where $r' = n - \operatorname{rk} A' + 1$. If $\operatorname{rk} H_i(C) = \operatorname{rk} H_i(C')$ then $\operatorname{rk} A = \operatorname{rk} A'$ and therefore $r = r'$. It is evident that $\varphi(\Delta_j(A))$ divides $\Delta_j(A')$ for all j. This implies (iii).

3.3. Lemma. *Let* R *be a local ring and* F *be the associated field. Let* $f: C_1 \to C_0$ *be a R-homomorphism of finitely generated free R-modules and let* $\bar{f}: F \otimes_R C_1 \to F \otimes_R C_0$ *be the induced F-homomorphism. If* $\operatorname{rk} f = \operatorname{rk} \bar{f}$ *then with respect to some bases in* C_1, C_0 *the homomorphism* f *is presented by the matrix* $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ *where* E *is the unit matrix of order* $\operatorname{rk} f$.

Proof. Since F is a field we can choose bases d_0, d_1 respectively in $F \otimes_k C_0$, $F \otimes_K C_1$ so that the matrix of $\bar{f}$ regarding these bases has the form $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$. Let $\mathscr{D}_i$ be a lifting of d_i to $C_i, i = 1, 2$. Here $\mathscr{D}_i$ is a sequence of $\operatorname{rg} C_i$ elements of C_i. In view of Nakayama's lemma $\mathscr{D}_i$ generate C_i. This implies that $\mathscr{D}_i$ generates the $(\operatorname{rg} C_i)$-dimensional vector space $Q(R) \otimes_R C_i$ over the field $Q(R)$. Therefore, the elements of the sequence $\mathscr{D}_i$ are linearly independent over $Q(R)$ and, hence, over R. Thus $\mathscr{D}_i$ is a basis of C_i for $i = 0, 1$. The matrix of f with respect to bases $\mathscr{D}_0, \mathscr{D}_1$ has the form $\begin{bmatrix} E+U & Z \\ X & Y \end{bmatrix}$ where U, X, Y, Z are matrices over the maximal ideal u of R. Note that $\det(E+U) = 1 \pmod{u}$. Since all elements of $R \backslash U$ are invertible in R the square matrix $E + U$ is invertible over R. Therefore we can choose bases in C_0, C_1 so that the corresponding matrix of f equals $\begin{bmatrix} E & 0 \\ 0 & Y' \end{bmatrix}$. Since $\operatorname{rk} f = \operatorname{rk} \bar{f} = \operatorname{rk} E$, $Y' = 0$.

3.4. Lemma. *Let* R *be a local ring and* F *be the associated field. Let* $C = (\cdots \to C_{i+1} \to C_i \to \cdots)$ *be a finitely generated free chain complex over* R. *Let* C' *be the chain F-complex* $F \otimes_R C$. *Let* ∂_i, ∂'_i *be the boundary homomorphisms* $C_{i+1} \to C_i, C'_{i+1} \to C'_i$. *If* $\operatorname{rk}_R H_i(C) = \operatorname{rk}_F H_i(C')$ *for some* i *then:* $H_i(C), \operatorname{Im} \partial_{i+1}, \operatorname{Im} \partial_i$ *are free R-modules and* $C_i = \operatorname{Im} \partial_{i+1} \oplus H_i(C) \oplus \operatorname{Im} \partial_i$; *the projection* $C \to C'$ *induces F-isomorphisms* $F \otimes_R H_i(C) \to H_i(C'), F \otimes_R \operatorname{Im} \partial_j \to \operatorname{Im} \partial'_j$ *with* $j = i, i+1$.

This Lemma directly follows from Lemmas 3.2 (ii) and 3.3.

§ 4. Proof of Theorems 1 and 2

4.1. Proof of Theorem 1. Denote by Q_n the fraction field of the ring $\Lambda_n = \mathbf{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. Denote by Q_n^0 the subring of Q_n which consists of rational functions fg^{-1} with $f, g \in \Lambda_n$ and $g \notin (t_n - 1)\Lambda_n$ (so that

$g(t_1, ..., t_{n-1}, 1) \neq 0$). The homomorphism $f \mapsto f(t_1, ..., t_{n-1}, 1): \Lambda_n \to \Lambda_{n-1}$ uniquely extends to a ring homomorphism $Q_n^0 \to Q_{n-1}$ which is denoted by φ.

Denote by X the exterior of K and by Y the exterior of L.

We shall prove the following two statements.

(4.1.1). $\varphi(\Delta(K)) = \Delta(K)(t_1, ..., t_{n-1}, 1)$ divides $\Delta(L)$ in Λ_{n-1}.

(4.1.2). There exists a representative ω of the torsion $\omega(X) \subset Q_n$ such that $(t_n - 1)\omega \in Q_n^0$ and $\varphi((t_n - 1)\omega)$ represents $\omega(Y) \subset Q_{n-1}$.

Let us show first that these two statements imply the Theorem. Let ω be the element of Q_n produced by (4.1.2). Put $\pi = \prod_{i=1}^{n-1} (t_i - 1)$. According to the results formulated in Sec. 2.2 the product $(t_n - 1)\pi \cdot \Delta(K)$ represents $\omega(X)$. Thus

$$\omega \doteq \frac{f\bar{f}}{g\bar{g}}(t_n - 1)\pi\Delta(K)$$

where $f, g \in \Lambda_n \backslash 0$. We may assume that $f\bar{f}$ and $g\bar{g}$ are relatively prime. If $t_n - 1$ does not divide g then $\omega \in Q_n^0$ and $\varphi((t_n - 1)\omega) = 0$ which contradicts to the inclusion $\varphi((t_n - 1)\omega) \in \omega(Y)$. Thus $g = (t_n - 1)h$ with $h \in \Lambda_n$. In view of (4.1.1), $\varphi(\Delta(K)) \neq 0$, i.e. $t_n - 1$ does not divide $\Delta(K)$. If $\varphi(h) = 0$ then $(t_n - 1)^2$ divides g which obviously contradicts the inclusion $(t_n - 1)\omega \in Q_n^0$. Thus $\varphi(h) \neq 0$. We have

$$h\bar{h}(t_n - 1)\omega \doteq f\bar{f}\,\pi\Delta(K).$$

Since $\varphi(h\bar{h}(t_n - 1)\omega) \neq 0$ we have $\varphi(f) \neq 0$. This implies that $\pi \cdot \varphi(\Delta(K)) \doteq q\bar{q}\,\varphi((t_n - 1)\omega)$ where $q = \varphi(h)/\varphi(f)$. Thus $\pi\varphi(\Delta(K))$ represents $\omega(Y)$. Since $\pi\Delta(L) \in \omega(Y)$ we have

$$\varphi(\Delta(K))\lambda\bar{\lambda} = \Delta(L)\mu\bar{\mu}$$

with non-zero $\lambda, \mu \in \Lambda_{n-1}$. We may assume that $\lambda\bar{\lambda}$ and $\mu\bar{\mu}$ are relatively prime. Since $\varphi(\Delta(K))$ divides $\Delta(L)$ we immediately obtain $\mu\bar{\mu} = 1$. Thus, $\Delta(L) = \varphi(\Delta(K))\lambda\bar{\lambda}$.

Let us prove (4.1.1) and (4.1.2). We may assume that $X \subset Y$ and that $Y \backslash X$ is the interior of the regular neighborhood $U \subset Y$ of the n-th component of K in Y. Let $p: \tilde{X} \to X$ and $q: \tilde{Y} \to Y$ be the maximal abelian coverings with the groups of covering transformations respectively $H_1(X) \approx \mathbf{Z}^n$ (generators $t_1, ..., t_n$) and $H_1(Y) \approx \mathbf{Z}^{n-1}$ (generators $t_1, ..., t_{n-1}$). It is clear that p is the composition of an infinite cyclic covering $\tilde{X} \to q^{-1}(X)$ and the covering $q: q^{-1}(X) \to X$.

Fix a C^1-triangulation of Y so that X and U are simplicial subcomplexes of Y. Fix also the induced equivariant triangulations in $\tilde{X}$ and $\tilde{Y}$.

The ring Λ_{n-1} determines via the natural homomorphism $\mathbf{Z}[\pi_1(Y)] \to \mathbf{Z}[H_1 Y] = \Lambda_{n-1}$ a system of local coefficients on Y which we denote by the same symbol Λ_{n-1}. According to definitions, for any simplicial subsets $A \supset B$ of Y the Λ_{n-1}-module $H_*(A, B; \Lambda_{n-1})$ equals $H_*(C(q^{-1}(A), q^{-1}(B); \mathbf{Z}))$. Here the simplicial chain complex $C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$ is a finitely generated free Λ_{n-1}-complex. Analogously Λ_n defines a system of local coefficients on X and for simplicial subsets $A \supset B$ of X the Λ_n-module $H_*(A, B; \Lambda_n)$ equals $H_*(C(p^{-1}(A), p^{-1}(B); \mathbf{Z}))$. Note that

$$\Lambda_{n-1} \otimes_{\Lambda_n} C_*(p^{-1}(A), p^{-1}(B); \mathbf{Z}) = C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$$

where Λ_n acts on Λ_{n-1} via φ.

Claim 1. For $i \neq 1, m-1$,

$$\mathrm{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \mathrm{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = \mathrm{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = 0 .$$

For $i = 1, m-1$,

$$\mathrm{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \mathrm{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = n-1 ; \ \mathrm{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = n-2 .$$

Proof of Claim 1. We shall compute the rank of $H_i(X; \Lambda_n)$; modules $H_i(X; \Lambda_{n-1})$ and $H_i(Y; \Lambda_{n-1})$ can be treated similarly.

Denote by V a wedge of n circles in X such that the inclusion homomorphism $H_1(V; \mathbf{Z}) \to H_1(X; \mathbf{Z}) = \mathbf{Z}^n$ is bijective. Then $H_i(X, V, \mathbf{Z}) = 0$ for $i \leqslant m-2$. Therefore an application of Lemma 3.2(i) to complexes $C_*(\tilde{X}, p^{-1}(V); \mathbf{Z})$ and $C_*(X, V; \mathbf{Z})$ gives that $\mathrm{rk}_{\Lambda_n} H_i(X, V; \Lambda_n) = 0$ for $i \leqslant m-2$. This implies that $\mathrm{rk}\, H_i(X; \Lambda_n) = \mathrm{rk}\, H_i(V; \Lambda_n)$ for $i \leqslant m-3$ and that $\mathrm{rk}\, H_{m-2}(X; \Lambda_n) \leqslant \mathrm{rk}\, H_{m-2}(V; \Lambda_n)$. The rank of $H_i(V; \Lambda_n)$ can be computed directly: It is equal to 0 if $i \neq 1$ and to $n-1$ if $i = 1$. Thus the rank of $H_i(X; \Lambda_n)$ equals 0 if $i \neq 1$, $m-1$ and equals $n-1$ if $i = 1$. The equality $\mathrm{rk}\, H_{m-1}(X; \Lambda_n) = n-1$ follows from duality or from the equalities

$$\sum_{i=0}^{m} (-1)^i \,\mathrm{rk}\, H_i(X; \Lambda_n) = \chi(X) = 0 .$$

Claim 2. The exact homology sequence of (Y, X) with coefficients in Λ_{n-1} splits into short exact sequences

$$0 \to H_m(Y, X; \Lambda_{n-1}) \to H_{m-1}(X; \Lambda_{n-1}) \to H_{m-1}(Y; \Lambda_{n-1}) \to 0,$$
$$0 \to H_i(X; \Lambda_{n-1}) \overset{\approx}{\to} H_i(Y; \Lambda_{n-1}) \to 0, \quad (i \neq 1, m-1)$$
$$0 \to H_2(Y, X; \Lambda_{n-1}) \overset{\partial_1}{\to} H_1(X; \Lambda_{n-1}) \to H_1(Y; \Lambda_{n-1}) \to 0.$$

Proof of Claim 2. Clearly, $H_i(Y, X; \Lambda_{n-1}) = H_i(U, \partial U; \Lambda_{n-1}) = 0$ for $i \neq 2, m$. Therefore the only thing to prove is the injectivity of ∂_1. According to Claim 1 rk $H_1(X; \Lambda_{n-1}) = n - 1$ and rk $H_1(Y; \Lambda_{n-1}) = n - 2$. Since $H_2(Y, X; \Lambda_{n-1}) = \Lambda_{n-1}$ we see that ∂_1 is injective.

Proof of (4.1.1). In view of the equalities rg $H_i(X; \Lambda_n) =$ rg $H_i(X; \Lambda_{n-1})$, $i = 0, 1, \ldots$ we may apply Lemma 3.2 (iii) to the chain complexes $C_*(\tilde{X}; \mathbf{Z})$ and $C_*(q^{-1}(X); \mathbf{Z})$ respectively over Λ_n and Λ_{n-1}. Since $m - 1 > r > 1$ Claims 1, 2 show that $H_r(X; \Lambda_n)$ and $H_r(X; \Lambda_{n-1})$ are torsion modules respectively over Λ_n and Λ_{n-1} and $H_r(X, \Lambda_{n-1}) = H_r(Y; \Lambda_{n-1})$. By definition $\Delta(K) = \text{ord } H_r(X; \Lambda_n)$ and $\Delta(L) = \text{ord } H_r(Y; \Lambda_{n-1}) = \text{ord } H_r(X; \Lambda_{n-1})$. Lemma 3.2 (iii) directly implies that $\varphi(\Delta(K))$ divides $\Delta(L)$.

It remains to prove Statement (4.1.2) which is, of course, the core of Theorem 1. For simplicial subsets $A \supset B$ of Y we shall denote by $C(A, B)$ the (simplicial) chain Q_{n-1}-complex $Q_{n-1} \otimes_{\Lambda_{n-1}} C_*(q^{-1}(A), q^{-1}(B); \mathbf{Z})$. Clearly

$$H_i(A, B; Q_{n-1}) = H_i(C(A, B)) = Q_{n-1} \otimes_{\Lambda_{n-1}} H_i(A, B; \Lambda_{n-1}).$$

Consider the short exact sequence of chain Q_{n-1}-complexes

(5) $$0 \to C(X) \to C(Y) \to C(Y, X) \to 0.$$

Provide the homology modules of complexes $C(X)$, $C(Y)$, $C(Y, X)$ with bases as follows. It is evident that $H_i(C(Y, X)) = 0$ for $i \neq 2, m$ and

$$H_i(C(Y, X)) = H_i(C(U, \partial U)) = H_i(U, \partial U; Q_{n-1}) = Q_{n-1}$$

for $i = 2, m$. Fix a lifting $\tilde{U} \subset \tilde{Y}$ of $U \approx S^{m-2} \times D^2$. Fix in $H_m(C(Y, X))$ the generator $[\tilde{U}, \partial\tilde{U}]$. Fix in $H_2(C(Y, X))$ the generator $[\Delta, \partial\Delta]$ where Δ is the meridional disk of $\tilde{U}$.

It follows from Claim 1 that $H_i(C(X)) = H_i(C(Y)) = 0$ for $i \neq 1, m - 1$. Fix an arbitrary basis f in the $(n-2)$-dimensional vector Q_{n-1}-space $H_1(Y; Q_{n-1})$. Fix the dual basis g in $H_{m-1}(Y; Q_{n-1})$. It follows from Claim 2 that inclusion homomorphisms $H_i(C(X)) \to H_i(C(Y))$ are surjective for all i. Let F and G be sequences of $n - 2$ vectors in $H_1(C(X))$ and in $H_{m-1}(C(X))$ whose images under these inclusion homomorphisms are equal respectively to f and g. Claim 2 implies that $[\partial\tilde{U}]$, G is a basis in $H_{m-1}(C(X))$ and

$[\partial\Delta]$, F is a basis in $H_1(C(X))$. Now all homology modules of complexes $C(X)$, $C(Y)$, $C(Y, X)$ are provided with bases.

Provide the modules of $C(X)$, $C(Y)$, $C(Y, X)$ with natural bases (see Sec. 2.2). We may choose these bases to be compatible in the sense of Lemma 2.1.1. According to this Lemma

$$\tau(C(Y)) = \pm\, \tau(C(X))\tau(C(Y, X))\tau(\mathscr{H})$$

where $\mathscr{H}$ is the homology sequence associated with the exact sequence (5). It is evident that $\tau(\mathscr{H}) = \pm 1$. It is easy to verify that $\tau(C(Y, X)) = \tau(C(U, \partial U)) = \pm 1$. (Indeed, the pair $(U, \partial U)$ has a cell structure such that Int U contains 2 open cells; the meridional disc and its complement; for such cell structure the equality $\tau(C(U, \partial U)) = \pm 1$ is evident. The case of an arbitrary cell structure (or triangulation) follows from the invariance of torsion under cell subdivision). Thus $\tau(C(Y)) = \pm\, \tau(C(X))$. Note that $\tau(C(Y))$ represents $\omega(Y)$. Therefore $\tau(C(X))$ also represents $\omega(Y)$.

Consider the chain complex

$$C = Q_n^0 \otimes_{\Lambda_n} C_*(\tilde{X}; \mathbf{Z}).$$

Note that Q_n^0 is a local ring with the maximal ideal $(t_n - 1)Q_n^0$ and associated field Q_{n-1}. Clearly, $Q_{n-1} \otimes_{Q_n^0} C = C(X)$. The natural bases in chain modules of $C(X)$ lift to natural bases in chain modules of C. Claim 1 implies that for all $i \geqslant 0$

$$\mathrm{rk}_{Q_n^0} H_i(C) = \mathrm{rk}_{\Lambda_n} H_i(X; \Lambda_n) = \mathrm{rk}_{Q_{n-1}} H_i(C(X)).$$

Therefore we may apply Lemma 3.4 to complexes C, $C(X)$. This lemma shows that: $H_i(C) = H_i(C(X)) = 0$ for $i \neq 1, m-1$; the basis $[\partial\Delta]$, F in $H_1(C(X))$ lifts to a basis, say, $f_0, f_1, \ldots, f_{n-2}$ in $H_1(C)$; the basis $[\partial\tilde{U}]$, G in $H_{m-1}(C(X))$ lifts to a basis, say, $g_0, g_1, \ldots, g_{n-2}$ in $H_{m-1}(C)$; the submodules of cycles and boundaries of C are free in all dimensions. Thus we may apply the constructions of Sec. 2.1 to C which gives rise to a torsion $\tau(C) \in Q_n^0$. It follows directly from the formula (3) that $\varphi(\tau(C)) = \tau(C(X))$. Thus $\varphi(\tau(C))$ represents $\omega(Y)$.

Let v be the matrix of the semi-linear intersection pairing

$$< \,,\, > : H_1(X; Q_n^0) \times H_{m-1}(X; Q_n^0) \to Q_n^0$$

with respect to bases $f_0, f_1, \ldots, f_{n-2}$ and $g_0, g_1, \ldots, g_{n-2}$. (Here $H_i(X; Q_n^0) = H_i(C)$). It is clear that $\tau(C)\,(\det v)^{-1}$ represents $\omega(X)$. Put $\omega = \tau(C)\,(\det v)^{-1}$. We shall prove that

(6) $$\det v = \pm (t_n - 1) + (t_n - 1)^2 a$$

where $a \in Q_n^0$. Then $(t_n - 1)\omega \in Q_n^0$ and

$$\varphi((t_n - 1)\omega) = \varphi(\tau(C)[\pm 1 + (t_n - 1)a]^{-1}) = \pm\, \varphi(\tau(C)) \in \omega(Y).$$

This would complete the proof of (4.1.2).

It is obvious that

$$v = \begin{bmatrix} <f_0, g_0> & (t_n - 1)\alpha \\ (t_n - 1)\beta & E + (t_n - 1)\gamma \end{bmatrix}$$

where α, β, γ are respectively a $(n-2)$-row, $(n-2)$-column and $(n-2) \times (n-2)$-matrix over Q_n^0. It turns out that

$$<f_0, g_0> = \pm\,(t_n - 1) + (t_n - 1)^2 b \tag{7}$$

with $b \in Q_n^0$. This immediately implies (6).

I shall prove (7) for a special choice of f_0 which is sufficient for our aims. Let $\theta\colon [0, 1] \to \partial\tilde{X}$ be a path whose projection to $\tilde{Y}$ is a loop parametrizing $\partial\Delta \subset \partial\tilde{U}$. Let $\eta\colon [0, 1] \to \tilde{X}$ be a path such that $\eta(0) = \theta(0)$ and $\eta(1) = t_1 \cdot \theta(0)$. Consider the singular chain $\vartheta = \theta - t_1\theta + t_n\eta - \eta$. It is easy to check up that ϑ is a cycle in $\tilde{X}$ and that its homology class $[\vartheta] \in H_1(C)$ projects to $(1 - t_1)\,[\partial\Delta] \in H_1(C(X))$. Put $f_0 = (1 - t_1)^{-1}[\vartheta]$. Then $<f_0, g_0> = (1 - t_1)^{-1} <[\vartheta], g_0> = (1 - t_1)^{-1}(t_n - 1) <\eta, g_0>$ where in the right part the brackets $<\ ,\ >$ denote the intersection pairing

$$H_1(X, \partial X; Q_n^0) \times H_{m-1}(X; Q_n^0) \to Q_n^0.$$

The image of $<\eta, g_0>$ under $\varphi\colon Q_n^0 \to Q_{n-1}$ can be computed using the analogous pairing

$$H_1(X, \partial X; Q_{n-1}) \times H_{m-1}(X; Q_{n-1}) \to Q_{n-1}.$$

Namely, $\varphi(<\eta, g_0>) = \pm\,(t_1 - 1)$. Thus $<\eta, g_0> = \pm\,(t_1 - 1) + (t_n - 1)c$ with $c \in Q_n^0$. Therefore $<f_0, g_0> = \pm\,(t_n - 1) + (t_n - 1)^2 b$ where $b = (1 - t_1)^{-1}c$. This implies (7).

4.2. *Proof of Theorem 2.* We may assume that $\Delta_{u-1}(L) \neq 0$ and $l_1 = l_2 = \cdots = l_{n-1} = 0$. Then the n-th component of K lifts to the maximal abelian covering of the exterior Y of L. The remaining part of the proof is analogous to the proof of Theorem 1. Note, however, the necessary changes. In Claim 1 for $i = 1, 2$

$$\mathrm{rk}_{\Lambda_n} H_i(X; \Delta_n) = \mathrm{rk}_{\Lambda_{n-1}} H_i(X; \Lambda_{n-1}) = u - 1;\ \mathrm{rk}_{\Lambda_{n-1}} H_i(Y; \Lambda_{n-1}) = u - 2.$$

In the proof of (4.1.1) one should take into account that $\mathrm{Tors}_{\Lambda_{n-1}} H_1(X;\Lambda_{n-1})$ injects into $\mathrm{Tors}_{\Lambda_{n-1}} H_1(Y;\Lambda_{n-1})$ and thus the order of the first of these 2 modules divides the order of the second one.

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