1. Example of a Stein space X with $\widetilde{O(X)} \neq O(\tilde{X})$

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An analytic consequence of the construction presented here is that the normalization $\tilde{X}$ of an irreducible Stein space X is $\widetilde{\mathcal{O}(X)}$-convex, $\widetilde{\mathcal{O}(X)}$-separable and has local coordinates by functions in $\widetilde{\mathcal{O}(X)}$. Some algebraic results are that $\mathcal{O}(\tilde{X})$ is completely normal and that the two algebras $\widetilde{\mathcal{O}(X)}$ and $\mathcal{O}(\tilde{X})$ are always locally equal, i.e. their localizations at all maximal ideals in $\mathcal{O}(X)$ are equal.

In this paper, a complex space refers to a reduced complex space with countable topology.

1. Example of a Stein space X with $\widetilde{\mathcal{O}(X)} \neq \mathcal{O}(\tilde{X})$

Let $(X, \mathcal{O})$ be a complex space with normalization $\pi : \tilde{X} \to X$. Since π is surjective, the map $\pi^* : \mathcal{O}(X) \to \mathcal{O}(\tilde{X})$, $f \mapsto f \circ \pi$, is injective and the holomorphic functions $\mathcal{O}(X)$ on X can be considered to be a subring of the holomorphic functions $\mathcal{O}(\tilde{X})$ on the normalization $\tilde{X}$ of X; this will be indicated by $\mathcal{O}(X) \subset \mathcal{O}(\tilde{X})$. If X is irreducible and Stein, then $\mathcal{O}(\tilde{X})$ contains the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ but does not always coincide with it, as will be shown in this section.

For an irreducible complex space $(X, \mathcal{O})$, the integral domain $\mathcal{O}(X)$ is said to be *normal*, if it is integrally closed in its field of fractions $Q(\mathcal{O}(X))$, i.e. $\widetilde{\mathcal{O}(X)} = \mathcal{O}(X)$. Recall that $Q(\mathcal{O}(X))$ is the field of meromorphic functions $M(X)$ on X when X is irreducible and Stein due to Theorem B [10, 53.1, 52.17], and that the algebras $M(X)$ and $M(\tilde{X})$ are isomorphic for every complex space X [8, p. 161].

The following characterization of normal irreducible Stein spaces X by their global function algebra $\mathcal{O}(X)$ is essentially contained in [2, § 1, p. 35].

Theorem 1. *An irreducible Stein space X is normal if and only if the integral domain $\mathcal{O}(X)$ is normal.*

An analysis of the proof shows that even when X is just irreducible and normal, $\mathcal{O}(X)$ is also normal. Theorem 1 implies

Corollary 1. *For an irreducible Stein space X with normalization $\tilde{X}$, the integral closure $\widetilde{\mathcal{O}(X)}$ of $\mathcal{O}(X)$ is contained in $\mathcal{O}(\tilde{X})$.*

The following example shows that there are functions $f \in \mathcal{O}(\tilde{X})$ which are not integral over $\mathcal{O}(X)$. In this example, $X := (\mathbf{C}, \mathcal{O}')$ is an irreducible and locally irreducible Stein space given by a substructure of the canonical complex plane $(\mathbf{C}, \mathcal{O})$, which is then the normalization $\tilde{X}$ of X. The substructure is defined by a "Strukturausdünnung" (see [10]) which results by replacing the stalks $\mathcal{O}_n$, $n \in \mathbf{N}$, with the stalks of generalized Neil parabolas becoming steeper as n increases. More precisely, let $(p_n)_{n \in \mathbf{N}}$ be a strictly increasing sequence of prime numbers. For every $n \in \mathbf{N}$,

$$X_n := \{(x, y) \in \mathbf{C}^2 : x^{p_n} = y^{p_n + 1}\}$$

is an irreducible, locally irreducible analytic subset of $\mathbf{C}^2$ with the origin as the only singularity and with normalization

$$\pi_n : \mathbf{C} \to X_n, \ t \mapsto (t^{p_n + 1}, t^{p_n}).$$

Let $f \in \mathcal{O}(\mathbf{C})$ be the identity and denote by $\mathcal{O}_{X_n}$ the canonical complex structure on X_n. The germ $f_0 \in \mathcal{O}_0$ of f at the origin is integral over $\mathcal{O}_{X_n, 0}$ with respect to a polynomial of degree p_n, and p_n is the minimal degree of all such polynomials.

Now define $X := (\mathbf{C}, \mathcal{O}')$ as a substructure of the canonical plane $(\mathbf{C}, \mathcal{O})$ with stalks

$$\mathcal{O}'_x \cong \begin{cases} \mathcal{O}_x, & x \notin \mathbf{N} \\ \mathcal{O}_{X_n, 0}, & x = n \in \mathbf{N} \end{cases}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}'_n & \to & \mathcal{O}_n \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{O}_{X_n, 0} & \xrightarrow{\pi_n^*} & \mathcal{O}_0, \end{array}$$

where $\mathcal{O}'_n \to \mathcal{O}_n$ is the map induced by the identity $(\mathbf{C}, \mathcal{O}) \to (\mathbf{C}, \mathcal{O}')$ and $\mathcal{O}_n \cong \mathcal{O}_0$ is determined by the translation $\mathbf{C} \to \mathbf{C}$, $z \mapsto z - n$.

The identity $f \in \mathcal{O}(\mathbf{C})$ is not integral over $\mathcal{O}'(\mathbf{C})$, because otherwise every polynomial of integral dependence would have degree at least p_n for all $n \in \mathbf{N}$.

In conclusion it should be mentioned that $\mathcal{O}(\tilde{X})$ is almost integral over $\mathcal{O}(X)$ [7, § 3] for every irreducible Stein space X, since X has a global universal denominator [10, E.73a].