

# §5. Monopoles and Instantons

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*Remarks.* 1) It would be interesting to see what kind of harmonic representatives for classes in  $H^1(M; \mathbf{R})$  can be found.

2) Theorem 4.2 generalizes to identify elements of  $H^j(M, \delta M; \mathbf{R})$  with  $L^2$  harmonic forms for any oriented  $n$ -dimensional Riemannian manifold  $M$  for which a conformal compactification of  $M \times S^k$  exists, for all  $k$ , provided  $j < n/2$ .

§ 5. MONOPOLES AND INSTANTONS

Our goal is now to exploit the compactification  $X$  of  $M \times S^1$  (see § 2) to get monopoles on  $M$  from  $S^1$ -invariant instantons on  $X$ . We shall also relate the instanton number on  $X$  to various topological invariants of the monopoles on  $M$ . General background for this section can be found in Freed-Uhlenbeck [12] and Jaffe-Taubes [22]. More specifically our approach here is very similar to the one taken in Atiyah [2].

Let  $P$  be a principal  $SU(2)$ -bundle over  $X$ , with  $c_2(P) = k \geq 0$ . Recall that  $X$  comes naturally with a conformal structure. This enables us to talk about *instantons or anti-self-dual connections*  $A$  on  $P$ . These are defined to be the solutions of the *anti-self-duality equation*:

$$5.1 \quad F^A = - *_4 F^A \quad (*_4 \text{ the Hodge star on } \Lambda^2(X)).$$

Here  $F^A$  is the curvature of  $A$ , a section of  $\Lambda^2(X) \otimes g_P$  with  $g_P = P \times_{Ad} su(2)$ . The instantons are the absolute minima of the *Yang-Mills functional*:

$$5.2 \quad YM(A) = (16\pi^2)^{-1} \int_X \langle F^A \wedge *F^A \rangle$$

where  $\langle \alpha, \beta \rangle = -2 \cdot tr(\alpha\beta)$  is an invariant inner product on  $su(2)$ . For an instanton  $YM(A) = k$ .

Next assume that the double cover  $\tilde{S}^1$  of  $S^1$  acts on  $P$  by bundle automorphisms, covering the action on  $X$ ; the double cover will be needed in order to include the spin bundles of  $X$ . Our interest will now lie in  $\tilde{S}^1$ -invariant instantons on  $P$ . To relate these to objects on  $M$  introduce the map:

$$j: M \rightarrow X : m \rightarrow i'(m, 1) \quad (\text{compare 2.2}),$$

which is a diffeomorphism onto its image. Let  $v$  be the vectorfield on  $P$  induced by the  $\tilde{S}^1$ -action. If we interpret an  $\tilde{S}^1$ -invariant connection  $A$  as a 1-form on  $P$ , then define the Higgs-field  $\Phi$  to be the  $su(2)$ -valued function  $j^*A(\frac{1}{2}v)$  on  $j^*P$ . It is easy to see that  $\Phi$  is a section of  $j^*g_P$ .

Further  $A_3 = j^*A$  defines a connection on the bundle  $j^*P$  over  $M$ . A little computation shows that the  $\tilde{S}^1$ -invariant connection  $A$  is anti-self-dual iff  $(A_3, \Phi)$  satisfy the so called *Bogomol'nyi equation* on  $M$ :

$$5.3 \quad d^{A_3}\Phi = - *_3 F^{A_3}.$$

As 5.3 is the standard equation describing *magnetic monopoles* on three dimensional manifolds, this leads to the definition.

*Definition 5.1.* A monopole on  $P$  is an  $\tilde{S}^1$ -invariant instanton on  $P$ .

Normally one defines a monopole by imposing certain asymptotic conditions rather than requiring it to extend over a compact manifold. In Braam [10] it is explained that results of the Sibners imply that this amounts to the same. We shall see below that the boundary data are the same.

If  $GA(P)$  denotes the group of  $\tilde{S}^1$ -invariant gauge transformations on  $P$ , then  $GA(P)$  leaves the set of monopoles invariant. Just as for instantons one can therefore define a *monopole moduli space*, equal to:

$$5.4 \quad \{\text{solutions of 5.3}\}/GA(P)$$

In Braam [10] is shown that under some assumptions these moduli spaces are non-empty finite dimensional manifolds.

We shall now return to our  $\tilde{S}^1$ -equivariant bundle  $P$  and relate topological invariants of the action to asymptotic invariants of  $(A_3, \Phi)$  on  $M$ . Restricted to one of the fixed surfaces  $S_j$ ,  $\tilde{S}^1$  acts by gauge transformations on  $P$ . The fibres of  $E = P \times_{SU(2)} \mathbf{C}^2$  over  $S_j$  decompose into eigenspaces for the  $\tilde{S}^1$  action. Denote by  $m_j \in \mathbf{Z}_{\geq 0}$  the  $\tilde{S}^1$ -weight which is non-negative.

If  $m_j > 0$  then:

$$5.5 \quad E|_{S_j} \cong L_j \oplus L_j^*$$

where  $L_j$  is the complex line bundle in  $E$  of weight  $m_j$  and  $L_j^*$  that of weight  $-m_j$ ; because  $c_1(E|_{S_j}) = 0$ ,  $L_j^*$  is also the dual of  $L_j$ . In order to define the first Chern classes of  $L_j$  it is convenient to have an orientation of  $S_j$ . Recall that  $X$  is oriented and that a neighbourhood of  $S_j$  in  $X$  looks like  $S_j \times \mathbf{R}^2$ . The  $\mathbf{R}^2$  is oriented by the  $S^1$ -action, and this induces an orientation of  $S_j$ . Now write  $c_1(L_j) = -k_j \cdot x_j$  with  $k_j \in \mathbf{Z}$  and  $x_j$  the positive generator of  $H^2(S_j; \mathbf{Z})$ . If  $m_j = 0$  then  $E|_{S_j}$  is trivial as an  $\tilde{S}^1$ -equivariant vector bundle. We shall leave  $k_j$  undefined in this case.

There is one important constraint on the  $m_j$ . This becomes clear by remarking that  $-1 \in \tilde{S}^1$  acts as a gauge transformation on all of  $E$ , i.e. as

+ 1 or as - 1. This implies that either all  $m_j$  are even or they are all odd. In Braam [10] we have shown that any set of invariants  $(m_j, k_j)$  satisfying this constraint arises from a suitable  $\tilde{S}^1$ -equivariant bundle, and that the  $\tilde{S}^1$ -isomorphism class is determined by  $(m_j, k_j)$ .

*Definition 5.2.* The moduli space of monopoles on a principal  $SU(2)$ -bundle  $P$  with invariants  $(m_j, k_j)$  will be denoted by  $\mathcal{M}(m_j, k_j)$ .

Having defined the relevant invariants of  $P$ , the question now arises what they amount to in terms of asymptotic conditions for a pair  $(A_3, \Phi)$  on  $M$ . The vector field  $v$  on  $P$  turns vertical over  $S_j$ . This shows that:

$$5.6 \quad |\Phi(y)| \rightarrow m_j \quad \text{if} \quad y \rightarrow S_j \subset \delta M .$$

This is the Prasad-Sommerfeld boundary condition used in physics and the numbers  $m_j$  are called the *masses* of the monopole.

The solutions of the Bogomol'nyi equation 5.3 are minima of the *energy functional*:

$$5.7 \quad E(A_3, \Phi) = (8\pi)^{-1} \int_M |F^{A_3}|^2 + |d_{A_3}\Phi|^2 dV_3 .$$

If the pair  $(A_3, \Phi)$  arises from an invariant connection  $A$  on  $P$  then  $E(A_3, \Phi) = YM(A)$ . If we assume that  $(A_3, \Phi)$  satisfies 5.4, then:

$$|d_{A_3}\Phi|^2 dV_3 = |F^{A_3}|^2 dV_3 = \langle F^{A_3} \wedge d_{A_3}\Phi \rangle = d \langle F^{A_3} \cdot \Phi \rangle ,$$

by the Bianchi identity. It follows that:

$$E(A_3, \Phi) = - 2 \sum_j (8\pi)^{-1} \cdot \int_{S_j} \langle F^{A_3} \cdot \Phi \rangle .$$

The minus sign appears because the boundary orientation of  $S_j$  does not agree with orientation we have given it above. A moments reflection shows that  $2 \cdot (8\pi)^{-1} \cdot \int_{S_j} \langle F^{A_3} \cdot \Phi \rangle = - m_j \cdot k_j$ . Putting things together we get:

$$5.7 \quad \sum m_j \cdot k_j = E(A_3, \Phi) = YM(A) = k .$$

This is essentially the localization formula in equivariant cohomology applied to the equivariant  $c_2(P)$ , see Atiyah [2].

Exactly what the physical symmetry breaking would lead one to expect does indeed happen: far away in  $M$ , that is near an  $S_j$  with  $m_j \neq 0$ , the connection almost becomes a  $U(1)$ -connection on  $L_j$ , the bundle of eigenvectors of  $\Phi$  of eigenvalue  $\frac{1}{2} \cdot m_j$ . The *charges*  $k_j$  appear as first Chern classes of these line bundles on the boundary surfaces. This is of course nothing but the quantized charge of a  $U(1)$ -monopole, a so called Dirac monopole, on  $L_j$ . Dirac monopoles have singularities, but the genuine non-

Abelian character of  $SU(2)$ -monopoles in the core of  $M$  allows for non-singular solutions.

From 5.7 we see that  $\sum m_j \cdot k_j \geq 0$  is necessary for the existence of monopoles, however this is by no means sufficient as we shall see below (also compare Braam [10]).

We shall end this section by giving some simple examples of monopoles.

*Examples 5.3.* 1) Monopoles with all  $m_j = 0$ . For these monopoles  $YM(A) = 0$ , so we are dealing with flat connections. The Higgs field  $\Phi$  vanishes, this follows from the Bogomol'nyi equation. It is not hard to see that the moduli space  $\mathcal{M}(0, 0)$  equals the space of all representations  $\pi_1(X) \rightarrow SU(2)$  modulo conjugacy: one assigns to a flat connection its holonomy representation. This space can be very non-trivial; e.g. if  $M = H^3/\text{Fuchsian group} \cong S \times \mathbf{R}$ , with  $S$  a surface, then  $\mathcal{M}(0, 0)$  is the space of representations of  $\pi_1(S) \rightarrow SU(2)$  modulo conjugacy. By the theorem of Narasimham-Seshadri this is the same as the moduli space of semi-stable  $SL(2, \mathbf{C})$ -bundles on  $S$ , for any complex structure on  $S$ . The topology of this  $\mathcal{M}(0, 0)$  was investigated by Atiyah-Bott [4].

2) Next keep  $k_j = 0$  but take at least one  $m_j$  to be nonzero. The connections are still flat so  $\Phi$  is covariantly constant. This shows that  $\mathcal{M}(m_j, 0) = \emptyset$  unless all  $m_j$  are equal. Further

$$\begin{aligned} \mathcal{M}(m, 0) &\cong \text{Repr}(\pi_1(M), S^1) \cong \text{Repr}(H_1(M; \mathbf{Z}), S^1) \\ &\cong H_1(X; \mathbf{Z})_{\text{tor}} \times \{H_1(X; \mathbf{R})/H_1(X; \mathbf{Z})\}. \end{aligned}$$

3) For  $M \cong H^3$  all monopoles were determined by Atiyah [2]. The moduli space  $\mathcal{M}(m, k)$  equals  $\{\phi: S^2 \rightarrow S^2; \phi \text{ rational, degree } \phi = k, \phi(\infty) = 0\}$ , modulo multiplication by complex scalars of length 1. The monopole associated to the rational function  $\sum_j \exp(i\alpha_j) \cdot \lambda_j / (z - a_j)$  with  $\lambda_j \in \mathbf{R}_{>0}$ ,  $a_j \in \mathbf{C}$ , represents  $k$  lumps, centered at approximately  $(a_j, \lambda_j) \in \mathbf{R}_+^3 \cong H^3$ , with relative phase factors  $\exp(i(\alpha_{j_1} - \alpha_{j_2}))$ .

4) Monopoles arising from Riemannian curvature. If  $X$  is a oriented Riemannian 4-manifold then one can write the curvature tensor  $R: \Lambda^2 \rightarrow \Lambda^2$  as  $\begin{bmatrix} W_+ + (R_{sc}/3) & B \\ B^* & W_- + (R_{sc}/3) \end{bmatrix}$  relative to the decomposition  $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$ , in which  $B$  equals the Ricci curvature and  $W_{\pm}$  the Weyl tensor. If  $X$  is a conformally flat spin manifold with a metric of zero scalar curvature then this curvature tensor equals  $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$ . It follows that the connection

on the spin bundle  $S_+$  is anti-self-dual. Recall (see § 3) that for  $\Gamma$  Fuchsian, extended Fuchsian or a suitable Schottky group  $X_\Gamma$  admits such a metric. The connection on  $S_+$  is a monopole because the metrics are  $S^1$ -invariant. The mass(es) is (are) 1 by proposition 2.2, and the charges  $k_j$  equal  $g - 1$ , where  $g$  is the genus of the fixed surface(s). Choosing a different spin structure amounts to tensoring the bundle with a 2-torsion element in  $\text{Repr}(\pi_1(M), S^1)$ , compare 2).

In section 7 we shall come to grips with explicit formulae for nontrivial monopoles on certain handlebodies. In Braam-Hurtubise [11] the moduli spaces of monopoles on a solid torus are investigated in considerable detail. A general existence theory for monopoles on hyperbolic manifolds has been developed in Braam [10].

§ 6. TWISTOR SPACES

To a conformally flat oriented 4-manifold  $X$  there are naturally associated two complex manifolds  $Z_+$  and  $Z_-$ , the *twistor spaces* of  $X$ . Applying our construction of § 2 we thus get twistor spaces for hyperbolic 3-manifolds. It will be shown here that these carry a lot of geometric information associated to the 3-manifold  $M$ , such as the complete geodesic flow. Also they allow for a description of monopoles through holomorphic geometry. For the rest of this section let  $X$  be the conformal compactification of  $M \times S^1$ , with  $M$  a hyperbolic 3-manifold  $H^3/\Gamma$  as in § 2. We shall state those properties of  $Z_\pm$  that we will need, and refer to Atiyah [1] and Atiyah-Hitchin-Singer [5] for proofs and more details. The general line of thought in this section is very similar to that of Hitchin [20] and Atiyah [2].

If  $S_+(S_-)$  is the spin bundle of positive (negative) chirality on  $X$ , then  $Z_+(Z_-)$  can be realised as the  $\mathbf{CP}^1$ -bundles over  $X$ :

$$P(S_+) \rightarrow X \quad (P(S_-) \rightarrow X),$$

where  $P( )$  denotes projectivization of vectorbundles. A remarkable fact is that  $Z_+$  and  $Z_-$  are *complex manifolds* with a complex structure encoded in the conformal structure of  $X$ . However, the twistor spaces are only Kähler if  $X \cong S^4$  or  $X \cong \mathbf{CP}^2$ , which in our case results in  $\Gamma = \{e\}$  (see Hitchin [19]). There is an orientation reversing isometry of  $X$  arising from conjugation of the circles. This interchanges the two spin bundles and makes