3. Helices in S^3

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3. HELICES IN S^3

A spherical helix in S^3 is a curve $p(t)$ of constant geodesic curvature and torsion. As in R^3, two spherical helices of the same curvature and torsion are congruent.

If the curvature is nonzero, then we can define a Frenet frame $T(t)$, $N(t)$, $B(t)$ along $p(t)$ in the usual way, and get the Frenet equations:

$$T' = \kappa N\,, \quad N' = -\kappa T - \tau B\,, \quad B' = \tau N\,.$$

Here we assume that t is an arc length parameter along $p(t)$, and use primes $'$ to denote covariant differentiation of vector fields along this path.

A model helix in S^3 is given by

$$p(t) = (\cos\alpha\cos at,\ \cos\alpha\sin at,\ \sin\alpha\cos bt,\ \sin\alpha\sin bt)\,.$$

Here α ranges between 0 and $\pi/2$ and determines the shape of the flat torus

$$x_1^2 + x_2^2 = \cos^2\alpha\,, \quad x_3^2 + x_4^2 = \sin^2\alpha\,,$$

on which the helix $p(t)$ lies. We take the numbers a and b to be $\geqslant 0$, and require that

$$a^2\cos^2\alpha + b^2\sin^2\alpha = 1\,,$$

so that the helix will be traversed at unit speed. Every spherical helix in S^3 is congruent to one of these models.

Next, we give formulas for the curvature κ, torsion τ, and writhe $\rho = \sqrt{\kappa^2 + \tau^2}$ of the model helix $p(t)$ in terms of the descriptive parameters α, a and b. These formulas are given as general information only, and will not be used here.

We first record two simple inequalities which follow from the equality $a^2\cos^2\alpha + b^2\sin^2\alpha = 1$.

Note that $a = 1$ and $b = 1$ satisfies this equation. So if one of these quantities increases above 1, the other must decrease below 1. Arranging matters so that a is the larger of the two, we will then have

$$(a^2-1)(1-b^2) \geqslant 0\,.$$

In addition,

$$a^2 + b^2 \geqslant a^2\cos^2\alpha + b^2\sin^2\alpha = 1\,,$$

so we have

$$a^2 + b^2 - 1 \geqslant 0\,.$$

The formulas for curvature, torsion and writhe are as follows.

$$\begin{aligned}
\text{Curvature} &= \kappa = \sqrt{(a^2-1)(1-b^2)} \\
\text{Torsion} &= \tau = ab \\
\text{Writhe} &= \rho = \sqrt{a^2+b^2-1}\,.
\end{aligned}$$

Consider the 3-dimensional linear space of vector fields

$$aT(t) + bN(t) + cB(t)$$

which can be written as constant coefficient combinations of the Frenet vectors along the helix $p(t)$. Covariant differentiation along the helix maps this linear space to itself according to the Frenet formulas.

We've already noted in the introduction that the instantaneous axis vector $U = \tau T - \kappa B$ satisfies $U' = 0$.

Consider the vectors N and $V = (\kappa/\rho)T + (\tau/\rho)B$, which form an orthonormal basis for the orthogonal complement of U. Note that

$$\begin{aligned}
N' &= -\kappa T - \tau B = -\rho V\,, \quad \text{and} \\
V' &= (\kappa/\rho)T' + (\tau/\rho)B' = (\kappa/\rho)(\kappa N) + (\tau/\rho)(\tau N) = \rho N\,.
\end{aligned}$$

Thus, covariant differentiation along the helix kills the instantaneous axis vector and takes the orthogonal 2-plane to itself by a 90° rotation, followed by multiplication by the writhe.

4. Sasaki's equations

Let M be any Riemannian manifold, and UM its unit tangent bundle with the Riemannian metric described in section 1.

THEOREM (Sasaki [Sa], 1958). *The curve $(p(t), v(t))$ in UM is a constant speed geodesic there if and only if both of the following equations hold:*

1) $v'' = -\langle v', v'\rangle\, v$

2) $p'' = R(v', v)p'$.

Here, primes denote ordinary derivatives with respect to t when applied to functions, and covariant derivatives along the path $p(t)$ when applied to vector fields. For example, the first prime in p'' represents ordinary differentiation, the second, covariant differentiation. The symbol R denotes the Riemann curvature transformation

$$R\colon TM_p \times TM_p \to \mathrm{Hom}(TM_p, TM_p)\,.$$