

Appendix 1. The Cayley numbers

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correspond to the coordinate transformations $t \rightarrow tB(\lambda)/N(\lambda)^{1/2}$ in $\mathcal{I}_4$, where $B(\lambda)$ are the matrices given in (1.7) in Theorem 1.6. By Theorem 2.5, the elements $B(\lambda)/N(\lambda)^{1/2}$ of $SO(4)$ form a subgroup isomorphic with S^3 . Therefore, the bundle group $O(4)$ in $\mathcal{H}\mathcal{S}_4$ can be replaced by S^3 . Similarly, the bundle group $O(2)$ in $\mathcal{H}\mathcal{S}_2$ can be replaced by S^1 . With these observations, we can now prove the following theorem by proceeding as in the proof of Theorem 5.3.

THEOREM 5.4. *The representative coordinate bundles constructed in § 4 for the sphere bundles $\mathcal{H}\mathcal{S}_2$ and $\mathcal{H}\mathcal{S}_4$, with bundle groups S^1 and S^3 respectively, are topologically the same as the representative coordinate bundles constructed in § 3 for the sphere bundles $\mathcal{I}_2$ and $\mathcal{I}_4$, respectively.*

Finally, we remark that representative coordinate bundles of the bundles $\mathcal{S}\mathcal{L}_n$ in Theorem 4.2 are topologically essentially the same as the representative coordinate bundles of the bundles $\mathcal{I}\mathcal{L}_n$ in Theorem 3.2.

APPENDIX 1. THE CAYLEY NUMBERS

The Cayley numbers, denoted by X, Y, Z, W , etc. are ordered pairs (q_1, q_2) of quaternions subject to the rules and having the properties listed below. The set of all Cayley numbers, therefore, forms a (non-commutative and non-associative) real division algebra. No proof of the properties will be given as they can all be checked by direct computations.

(1) The *addition* is defined by

$$(q_1, q_2) + (q'_1, q'_2) = (q_1 + q'_1, q_2 + q'_2).$$

The *zero* is $O = (O, O)$.

(2) The *multiplication* is defined by

$$(q_1, q_2)(q'_1, q'_2) = (q_1q'_1 - q'_2{}^*q_2, q'_2q_1 + q_2q'_1{}^*),$$

where $q'_1{}^*, q'_2{}^*$ are respectively the conjugates of (the quaternions) q'_1, q'_2 . The (two-sided) *unit* is $1 \equiv (1, 0)$.

(3) Multiplication is

(i) distributive with respect to addition, i.e.,

$$(X + Y)W = XW + YW, \quad W(X + Y) = WX + WY;$$

- (ii) not commutative, i.e., generally, $XY \neq YX$ (but see (4) (iv) below);
 - (iii) not associative, i.e., generally, $(XY)W \neq X(YW)$ (but see (7) below).
- (4) The *real part* of $X \equiv (q_1, q_2)$ is $\text{Re } X = (\text{Re } q_1, 0) \equiv \text{Re } q_1$. X is said to be *real* if $X = \text{Re } X$; i.e., (q_1, q_2) is real iff q_1 is real and $q_2 = 0$.
- (i) $\text{Re}(X + Y) = \text{Re}(X) + \text{Re}(Y)$.
 - (ii) $\text{Re}(XY) = \text{Re}(YX)$.
 - (iii) $\text{Re}(CX) = 0$ for all X implies that $C = 0$.
 - (iv) $CX = XC$ for all X iff C is real. In this case, $C = (c_1, 0)$, where $c_1 = \text{real}$, and $CX = (c_1q_1, c_1q_2) = XC$.
- (5) The *conjugate* of $X \equiv (q_1, q_2)$ is $X^* = (q_1^*, -q_2)$.
- (i) $(X + Y)^* = X^* + Y^*$,
 - (ii) $(XY)^* = Y^*X^*$.
 - (iii) $X^* = X$ iff X is real.
- (6) The *norm* of X is the non-negative real number $N(X) \equiv XX^*$, which is also equal to X^*X . The *length* of X is the non-negative real number $|X| \equiv N(X)^{1/2} = (XX^*)^{1/2}$.
- (i) $N(X) = 0$ iff $X = 0$.
 - (ii) If $X \neq 0$, then $X^{-1} \equiv X^*/N(X)$ is a right and left inverse of X .
 - (iii) $N(XY) = N(X)N(Y)$. It follows from this that $XY = 0$ iff $X = 0$ or $Y = 0$.
- (7) Though multiplication is generally non-associative,
- (i) $(XY)Y^* = X(Y Y^*)$.
 - (ii) If $Y \neq 0$, then $(XY)Y^{-1} = X = Y^{-1}(YX)$.
 - (iii) $\text{Re}((XY)W) = \text{Re}(X(YW))$.

APPENDIX 2. THE HOPF FIBERING AND MUTUALLY ISOCLINIC PLANES

At the beginning of § 4, we described how H. Hopf obtained his fibering of S^{2n-1} by S^{n-1} over S^n , $n = 2, 4$, or 8 , by intersecting the unit sphere S^{2n-1} in $R^{2n} = Q_n \times Q_n$ with the Q_n -lines $Y = CX$ and $X = 0$. In Theorem 5.2, we proved that the Hopf fibering and maximal set of mutually isoclinic n -planes in R^{2n} are equivalent concepts. Here we prove, directly, the