

§5. Monopoles and Instantons

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Remarks. 1) It would be interesting to see what kind of harmonic representatives for classes in $H^1(M; \mathbf{R})$ can be found.

2) Theorem 4.2 generalizes to identify elements of $H^j(M, \delta M; \mathbf{R})$ with L^2 harmonic forms for any oriented n -dimensional Riemannian manifold M for which a conformal compactification of $M \times S^k$ exists, for all k , provided $j < n/2$.

§ 5. MONOPOLES AND INSTANTONS

Our goal is now to exploit the compactification X of $M \times S^1$ (see § 2) to get monopoles on M from S^1 -invariant instantons on X . We shall also relate the instanton number on X to various topological invariants of the monopoles on M . General background for this section can be found in Freed-Uhlenbeck [12] and Jaffe-Taubes [22]. More specifically our approach here is very similar to the one taken in Atiyah [2].

Let P be a principal $SU(2)$ -bundle over X , with $c_2(P) = k \geq 0$. Recall that X comes naturally with a conformal structure. This enables us to talk about *instantons or anti-self-dual connections* A on P . These are defined to be the solutions of the *anti-self-duality equation*:

$$5.1 \quad F^A = - *_4 F^A \quad (*_4 \text{ the Hodge star on } \Lambda^2(X)).$$

Here F^A is the curvature of A , a section of $\Lambda^2(X) \otimes g_P$ with $g_P = P \times_{Ad} su(2)$. The instantons are the absolute minima of the *Yang-Mills functional*:

$$5.2 \quad YM(A) = (16\pi^2)^{-1} \int_X \langle F^A \wedge *F^A \rangle$$

where $\langle \alpha, \beta \rangle = -2 \cdot \text{tr}(\alpha\beta)$ is an invariant inner product on $su(2)$. For an instanton $YM(A) = k$.

Next assume that the double cover \tilde{S}^1 of S^1 acts on P by bundle automorphisms, covering the action on X ; the double cover will be needed in order to include the spin bundles of X . Our interest will now lie in \tilde{S}^1 -invariant instantons on P . To relate these to objects on M introduce the map:

$$j: M \rightarrow X: m \rightarrow i'(m, 1) \quad (\text{compare 2.2}),$$

which is a diffeomorphism onto its image. Let v be the vectorfield on P induced by the \tilde{S}^1 -action. If we interpret an \tilde{S}^1 -invariant connection A as a 1-form on P , then define the Higgs-field Φ to be the $su(2)$ -valued function $j^*A(\frac{1}{2}v)$ on j^*P . It is easy to see that Φ is a section of j^*g_P .

Further $A_3 = j^*A$ defines a connection on the bundle j^*P over M . A little computation shows that the \tilde{S}^1 -invariant connection A is anti-self-dual iff (A_3, Φ) satisfy the so called *Bogomol'nyi equation* on M :

$$5.3 \quad d^{A_3}\Phi = - *_3 F^{A_3}.$$

As 5.3 is the standard equation describing *magnetic monopoles* on three dimensional manifolds, this leads to the definition.

Definition 5.1. A monopole on P is an \tilde{S}^1 -invariant instanton on P .

Normally one defines a monopole by imposing certain asymptotic conditions rather than requiring it to extend over a compact manifold. In Braam [10] it is explained that results of the Sibners imply that this amounts to the same. We shall see below that the boundary data are the same.

If $GA(P)$ denotes the group of \tilde{S}^1 -invariant gauge transformations on P , then $GA(P)$ leaves the set of monopoles invariant. Just as for instantons one can therefore define a *monopole moduli space*, equal to:

$$5.4 \quad \{\text{solutions of 5.3}\}/GA(P)$$

In Braam [10] is shown that under some assumptions these moduli spaces are non-empty finite dimensional manifolds.

We shall now return to our \tilde{S}^1 -equivariant bundle P and relate topological invariants of the action to asymptotic invariants of (A_3, Φ) on M . Restricted to one of the fixed surfaces S_j , \tilde{S}^1 acts by gauge transformations on P . The fibres of $E = P \times_{SU(2)} \mathbb{C}^2$ over S_j decompose into eigenspaces for the \tilde{S}^1 action. Denote by $m_j \in \mathbb{Z}_{\geq 0}$ the \tilde{S}^1 -weight which is non-negative.

If $m_j > 0$ then:

$$5.5 \quad E|_{S_j} \cong L_j \oplus L_j^*$$

where L_j is the complex line bundle in E of weight m_j and L_j^* that of weight $-m_j$; because $c_1(E|_{S_j}) = 0$, L_j^* is also the dual of L_j . In order to define the first Chern classes of L_j it is convenient to have an orientation of S_j . Recall that X is oriented and that a neighbourhood of S_j in X looks like $S_j \times \mathbb{R}^2$. The \mathbb{R}^2 is oriented by the S^1 -action, and this induces an orientation of S_j . Now write $c_1(L_j) = -k_j \cdot x_j$ with $k_j \in \mathbb{Z}$ and x_j the positive generator of $H^2(S_j; \mathbb{Z})$. If $m_j = 0$ then $E|_{S_j}$ is trivial as an \tilde{S}^1 -equivariant vector bundle. We shall leave k_j undefined in this case.

There is one important constraint on the m_j . This becomes clear by remarking that $-1 \in \tilde{S}^1$ acts as a gauge transformation on all of E , i.e. as

+ 1 or as - 1. This implies that either all m_j are even or they are all odd. In Braam [10] we have shown that any set of invariants (m_j, k_j) satisfying this constraint arises from a suitable \tilde{S}^1 -equivariant bundle, and that the \tilde{S}^1 -isomorphism class is determined by (m_j, k_j) .

Definition 5.2. The moduli space of monopoles on a principal $SU(2)$ -bundle P with invariants (m_j, k_j) will be denoted by $\mathcal{M}(m_j, k_j)$.

Having defined the relevant invariants of P , the question now arises what they amount to in terms of asymptotic conditions for a pair (A_3, Φ) on M . The vector field v on P turns vertical over S_j . This shows that:

$$5.6 \quad |\Phi(y)| \rightarrow m_j \quad \text{if} \quad y \rightarrow S_j \subset \delta M.$$

This is the Prasad-Sommerfeld boundary condition used in physics and the numbers m_j are called the *masses* of the monopole.

The solutions of the Bogomol'nyi equation 5.3 are minima of the *energy functional*:

$$5.7 \quad E(A_3, \Phi) = (8\pi)^{-1} \int_M |F^{A_3}|^2 + |d_{A_3}\Phi|^2 dV_3.$$

If the pair (A_3, Φ) arises from an invariant connection A on P then $E(A_3, \Phi) = YM(A)$. If we assume that (A_3, Φ) satisfies 5.4, then:

$$|d_{A_3}\Phi|^2 dV_3 = |F^{A_3}|^2 dV_3 = \langle F^{A_3} \wedge d_{A_3}\Phi \rangle = d\langle F^{A_3} \cdot \Phi \rangle,$$

by the Bianchi identity. It follows that:

$$E(A_3, \Phi) = -2 \sum_j (8\pi)^{-1} \cdot \int_{S_j} \langle F^{A_3} \cdot \Phi \rangle.$$

The minus sign appears because the boundary orientation of S_j does not agree with orientation we have given it above. A moments reflection shows that $2 \cdot (8\pi)^{-1} \cdot \int_{S_j} \langle F^{A_3} \cdot \Phi \rangle = -m_j \cdot k_j$. Putting things together we get:

$$5.7 \quad \sum m_j \cdot k_j = E(A_3, \Phi) = YM(A) = k.$$

This is essentially the localization formula in equivariant cohomology applied to the equivariant $c_2(P)$, see Atiyah [2].

Exactly what the physical symmetry breaking would lead one to expect does indeed happen: far away in M , that is near an S_j with $m_j \neq 0$, the connection almost becomes a $U(1)$ -connection on L_j , the bundle of eigenvectors of Φ of eigenvalue $\frac{1}{2} \cdot m_j$. The *charges* k_j appear as first Chern classes of these line bundles on the boundary surfaces. This is of course nothing but the quantized charge of a $U(1)$ -monopole, a so called Dirac monopole, on L_j . Dirac monopoles have singularities, but the genuine non-

Abelian character of $SU(2)$ -monopoles in the core of M allows for non-singular solutions.

From 5.7 we see that $\sum m_j \cdot k_j \geq 0$ is necessary for the existence of monopoles, however this is by no means sufficient as we shall see below (also compare Braam [10]).

We shall end this section by giving some simple examples of monopoles.

Examples 5.3. 1) Monopoles with all $m_j = 0$. For these monopoles $YM(A) = 0$, so we are dealing with flat connections. The Higgs field Φ vanishes, this follows from the Bogomol'nyi equation. It is not hard to see that the moduli space $\mathcal{M}(0, 0)$ equals the space of all representations $\pi_1(X) \rightarrow SU(2)$ modulo conjugacy: one assigns to a flat connection its holonomy representation. This space can be very non-trivial; e.g. if $M = H^3/\text{Fuchsian group} \cong S \times \mathbf{R}$, with S a surface, then $\mathcal{M}(0, 0)$ is the space of representations of $\pi_1(S) \rightarrow SU(2)$ modulo conjugacy. By the theorem of Narasimham-Seshadri this is the same as the moduli space of semi-stable $SL(2, \mathbf{C})$ -bundles on S , for any complex structure on S . The topology of this $\mathcal{M}(0, 0)$ was investigated by Atiyah-Bott [4].

2) Next keep $k_j = 0$ but take at least one m_j to be nonzero. The connections are still flat so Φ is covariantly constant. This shows that $\mathcal{M}(m_j, 0) = \emptyset$ unless all m_j are equal. Further

$$\begin{aligned} \mathcal{M}(m, 0) &\cong \text{Repr}(\pi_1(M), S^1) \cong \text{Repr}(H_1(M; \mathbf{Z}), S^1) \\ &\cong H_1(X; \mathbf{Z})_{\text{tor}} \times \{H_1(X; \mathbf{R})/H_1(X; \mathbf{Z})\}. \end{aligned}$$

3) For $M \cong H^3$ all monopoles were determined by Atiyah [2]. The moduli space $\mathcal{M}(m, k)$ equals $\{\phi: S^2 \rightarrow S^2; \phi \text{ rational, degree } \phi = k, \phi(\infty) = 0\}$, modulo multiplication by complex scalars of length 1. The monopole associated to the rational function $\sum_j \exp(i\alpha_j) \cdot \lambda_j/(z - a_j)$ with $\lambda_j \in \mathbf{R}_{>0}$, $a_j \in \mathbf{C}$, represents k lumps, centered at approximately $(a_j, \lambda_j) \in \mathbf{R}_+^3 \cong H^3$, with relative phase factors $\exp(i(\alpha_{j_1} - \alpha_{j_2}))$.

4) Monopoles arising from Riemannian curvature. If X is a oriented Riemannian 4-manifold then one can write the curvature tensor $R: \Lambda^2 \rightarrow \Lambda^2$ as $\begin{bmatrix} W_+ + (R_{sc}/3) & B \\ B^* & W_- + (R_{sc}/3) \end{bmatrix}$ relative to the decomposition $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$, in which B equals the Ricci curvature and W_{\pm} the Weyl tensor. If X is a conformally flat spin manifold with a metric of zero scalar curvature then this curvature tensor equals $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$. It follows that the connection

on the spin bundle S_+ is anti-self-dual. Recall (see § 3) that for Γ Fuchsian, extended Fuchsian or a suitable Schottky group X_Γ admits such a metric. The connection on S_+ is a monopole because the metrics are S^1 -invariant. The mass(es) is (are) 1 by proposition 2.2, and the charges k_j equal $g - 1$, where g is the genus of the fixed surface(s). Choosing a different spin structure amounts to tensoring the bundle with a 2-torsion element in $\text{Repr}(\pi_1(M), S^1)$, compare 2).

In section 7 we shall come to grips with explicit formulae for nontrivial monopoles on certain handlebodies. In Braam-Hurtubise [11] the moduli spaces of monopoles on a solid torus are investigated in considerable detail. A general existence theory for monopoles on hyperbolic manifolds has been developed in Braam [10].

§ 6. TWISTOR SPACES

To a conformally flat oriented 4-manifold X there are naturally associated two complex manifolds Z_+ and Z_- , the *twistor spaces* of X . Applying our construction of § 2 we thus get twistor spaces for hyperbolic 3-manifolds. It will be shown here that these carry a lot of geometric information associated to the 3-manifold M , such as the complete geodesic flow. Also they allow for a description of monopoles through holomorphic geometry. For the rest of this section let X be the conformal compactification of $M \times S^1$, with M a hyperbolic 3-manifold H^3/Γ as in § 2. We shall state those properties of Z_\pm that we will need, and refer to Atiyah [1] and Atiyah-Hitchin-Singer [5] for proofs and more details. The general line of thought in this section is very similar to that of Hitchin [20] and Atiyah [2].

If $S_+(S_-)$ is the spin bundle of positive (negative) chirality on X , then $Z_+(Z_-)$ can be realised as the \mathbf{CP}^1 -bundles over X :

$$P(S_+) \rightarrow X \quad (P(S_-) \rightarrow X),$$

where $P(\)$ denotes projectivization of vectorbundles. A remarkable fact is that Z_+ and Z_- are *complex manifolds* with a complex structure encoded in the conformal structure of X . However, the twistor spaces are only Kähler if $X \cong S^4$ or $X \cong \mathbf{CP}^2$, which in our case results in $\Gamma = \{e\}$ (see Hitchin [19]). There is an orientation reversing isometry of X arising from conjugation of the circles. This interchanges the two spin bundles and makes