

# 4. Properness

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LEMMA 2.1. Let  $A, B$  be metric spaces, with  $A \neq \emptyset$  and  $B$  connected. Let  $P: A \rightarrow B$  be a continuous map. Assume:

- (i)  $P$  is open,
- (ii)  $P$  is proper, that is, for any compact subset  $K$  in  $B$ ,  $P^{-1}(K)$  is compact. Then  $P$  is surjective.

*Proof.* We only need to prove that  $P(A)$  is closed. Let  $b_0$  be a point in  $\overline{P(A)}$ . Since  $B$  is a metric space, there exists a sequence  $(b_i)_{i>0}$  in  $P(A)$  converging to  $b_0$ . The subset  $K = \{b_0, b_1, b_2, \dots\}$  is compact, hence so is  $PP^{-1}(K)$ . The latter contains  $b_1, \dots, b_i, \dots$ , hence  $b_0$ , and it is obviously contained in  $P(A)$ . Q.E.D.

In order to make use of this lemma, we shall need some inverse function theorem for (i), and some *a priori* estimates for (ii).

### 3. LOCAL INVERSION

THEOREM 3.1. Let  $X$  be a smooth compact manifold,  $V$  and  $W$  smooth vector bundles on  $X$ ,  $U$  an open set in  $C^\infty(X, V)$ , and  $P: U \rightarrow C^\infty(X, W)$ , a smooth nonlinear elliptic partial differential operator. Let  $A$  and  $B$  be LCFC submanifolds of  $U$  and of  $C^\infty(X, W)$  respectively, such that the restriction  $P_A$  of  $P$  to  $A$ , sends  $A$  into  $B$ . Then the Jacobian criterion holds for  $P_A$ , namely, if the derivative of  $P_A: A \rightarrow B$  is invertible at  $\varphi_0 \in A$ , then  $P_A$  is a local diffeomorphism near  $\varphi_0$ .

This is a convenient variant of the Nash-Moser theorem (e.g. [14]) regarding suitable restrictions of elliptic operators. It is established in a separate paper [11] (see also [22]). It relies only on the classical (Banach) inverse function theorem combined with *elliptic regularity*.

*Remark 3.2.* The Nash-Moser theorem has been studied by many authors, see the bibliography below and further references in [14] [15] [25].

### 4. PROPERNESS

In view of (2), theorem 3.1 implies that  $P_\lambda$  is open. We want to apply lemma 2.1 in order to prove that  $P_\lambda$  is surjective from  $A_\lambda$  to  $B_\lambda$ . Since  $P_\lambda(A_\lambda) \neq \emptyset$  (it contains 0), and since  $B_\lambda$  is connected, this amounts to proving that  $P_\lambda$  is *proper*. Let us explain why *a priori* estimates imply properness.

Concerning subsets in  $A_\lambda$  we have

PROPOSITION 4.1. *A subset  $S$  in  $A_\lambda$  is relatively compact in  $A_\lambda$  iff its closure  $\bar{S}$  in  $C^\infty(X)$  lies inside  $A_\lambda$  and  $S$  is bounded in  $C^\infty(X)$ .*

This readily follows from Ascoli theorem which implies the well-known fact [12] (p. 231) that in  $C^\infty(X)$  (and in any closed LCFC submanifold of  $C^\infty(X)$ , such as  $B_\lambda$ , as well) bounded subsets are relatively compact and vice-versa; hence, compact subset of  $A_\lambda$  are nothing but bounded closed strictly interior subsets of  $A_\lambda$ . Explicitely, let us state the

COROLLARY 4.2. *A closed subset  $S$  in  $A_\lambda$  is compact if and only if there exists a sequence  $(C_i), i \in \mathbf{N}$ , of positive numbers, such that for any  $\varphi$  in  $S$  the following estimates hold:*

$$\begin{aligned} \|(g')^{-1}\| &=:\sup_X |(g')^{-1}| \leq C_0, \\ \forall i \in \mathbf{N}, \quad \|D^i\varphi\| &=:\sup_X |D^i\varphi| \leq C_i, \end{aligned}$$

where  $|\cdot|$  denotes some natural norms of tensors in the original metric  $g$ , and  $D =: (\nabla, \bar{\nabla})$  is the total covariant differentiation with respect to the metric  $g$ .

*Proof.* Indeed  $S$  is closed and bounded. Moreover, since for  $\varphi \in S$ ,

$$\|(g')^{-1}\| \leq C_0$$

all the eigenvalues of  $(g')^{-1}$  (which are positive) are uniformly bounded from above, hence those of  $g'$  are uniformly bounded from below, in other words:

$$\exists \varepsilon > 0, \quad \forall \varphi \in S, \quad g' \geq \varepsilon g,$$

or equivalently  $\bar{S}$  lies strictly inside  $A_\lambda$ . Q.E.D.

In the next sections we will show that if  $f$  belongs to some compact (i.e. bounded and closed) subset  $K$  of  $B_\lambda$ , defined by a sequence  $(K_i), i \in \mathbf{N}$ , such that  $\|D^i f\| \leq K_i$ , then for  $\varphi \in A_\lambda$  satisfying  $P_\lambda(\varphi) = f$ , the following a priori estimates hold:

$$\|\varphi\| \leq C_0, \quad \forall i \in \mathbf{N}, \quad \|D^i \nabla \bar{\nabla} \varphi\| \leq C_{i+2}.$$

These estimates imply that  $P_\lambda$  is proper, i.e. that  $S = P_\lambda^{-1}(K)$  is compact, according to the following

PROPOSITION 4.3. *Let  $S$  be a closed subset in  $A_\lambda$ . Suppose that there exists a sequence  $(C_i), i \in \mathbf{N}$ , such that for any  $\varphi$  in  $S$ , the following estimates hold:*

$$\| \varphi \| \leq C_0, \quad \| P_\lambda(\varphi) \| \leq C_0, \quad \forall i \in \mathbf{N}, \quad \| D^i \nabla \bar{\nabla} \varphi \| \leq C_{i+2}.$$

Then  $S$  is compact.

*Proof.* The first two estimates imply a uniform estimate

$$| \text{Log det } (g' g^{-1}) | \leq E.$$

The estimate on  $\| \nabla \bar{\nabla} \varphi \|$  yields another one:

$$\| g' \| \leq F.$$

These two estimates yield

$$\| (g')^{-1} \| \leq G.$$

Now from  $\| D^i \nabla \bar{\nabla} \varphi \| \leq C_{i+2}$  we infer

$$\| D^i \Delta \varphi \| \leq \tilde{C}_{i+2}$$

since  $D$  and  $g^{-1}$  commute ( $\Delta$  denotes the Laplacian in the metric  $g$ ). As  $\Delta$  performs a continuous linear automorphism of the Fréchet space of smooth functions *with zero average* (by Fredholm theory), the Closed Graph Theorem implies the missing estimates. Q.E.D.

*Remark 4.4.* Actually we have been considering two *gradings* of  $C^\infty(X)$  [14]. The usual one, namely the one defined,  $\forall u \in C^\infty(X)$ , by

$$\begin{aligned} \| u \|_0 &= \sup_X | u |, \\ \| u \|_i &= \| u_i \|_{i-1} + \| D^i u \|, \quad i \geq 1, \end{aligned}$$

and another one, well-adapted here since the true unknown is a Kähler metric, defined by

$$\begin{aligned} \| u \|_0^* &= \| u \|_0, \quad \| u \|_1^* = \| u \|_1, \\ \| u \|_i^* &= \| u \|_{i-1}^* + \| D^{i-2}(\nabla \bar{\nabla} u) \|, \quad i \geq 2. \end{aligned}$$

Although it is unnecessary for the purpose of proposition 4.3, it can be shown globally (without Schauder theory) that these two gradings are *tamely* equivalent [14] of degree 2 and base 0 [10] (section 5). Hence, they define the same topology.