Appendix 2. The Hopf fibering and mutually isoclinic planes

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(ii) not commutative, i.e., generally, $XY \neq YX$ (but see (4) (iv) below);

(iii) not associative, i.e., generally, $(XY)W \neq X(YW)$ (but see (7) below).

(4) The *real part* of $X \equiv (q_1, q_2)$ is $\operatorname{Re} X = (\operatorname{Re} q_1, O) \equiv \operatorname{Re} q_1$. X is said to be *real* if $X = \operatorname{Re} X$; i.e., (q_1, q_2) is real iff q_1 is real and $q_2 = 0$.

(i) $\operatorname{Re}(X+Y) = \operatorname{Re}(X) + \operatorname{Re}(Y)$.

(ii) $\operatorname{Re}(XY) = \operatorname{Re}(YX)$.

(iii) $\operatorname{Re}(CX) = 0$ for all X implies that $C = 0$.

(iv) $CX = XC$ for all X iff C is real. In this case, $C = (c_1, 0)$, where c_1 = real, and $CX = (c_1q_1, c_1q_2) = XC$.

(5) The *conjugate* of $X \equiv (q_1, q_2)$ is $X^* = (q_1^*, -q_2)$.

(i) $(X+Y)^* = X^* + Y^*$,

(ii) $(XY)^* = Y^*X^*$.

(iii) $X^* = X$ iff X is real.

(6) The *norm* of X is the non-negative real number $N(X) \equiv XX^*$, which is also equal to X^*X. The *length* of X is the non-negative real number $|X| \equiv N(X)^{1/2} = (XX^*)^{1/2}$.

(i) $N(X) = 0$ iff $X = 0$.

(ii) If $X \neq 0$, then $X^{-1} \equiv X^*/N(X)$ is a right and left inverse of X.

(iii) $N(XY) = N(X)N(Y)$. It follows from this that $XY = 0$ iff $X = 0$ or $Y = 0$.

(7) Though multiplication is generally non-associative,

(i) $(XY)Y^* = X(YY^*)$.

(ii) If $Y \neq 0$, then $(XY)Y^{-1} = X = Y^{-1}(YX)$.

(iii) $\operatorname{Re}((XY)W) = \operatorname{Re}(X(YW))$.

APPENDIX 2. THE HOPF FIBERING AND MUTUALLY ISOCLINIC PLANES

At the beginning of § 4, we described how H. Hopf obtained his fibering of S^{2n-1} by S^{n-1} over S^n, $n = 2, 4$, or 8, by intersecting the unit sphere S^{2n-1} in $R^{2n} = Q_n \times Q_n$ with the Q_n-lines $Y = CX$ and $X = 0$. In Theorem 5.2, we proved that the Hopf fibering and maximal set of mutually isoclinic n-planes in R^{2n} are equivalent concepts. Here we prove, directly, the

THEOREM A2.1. *The set of Q_n-lines $\{Y = CX, X = 0\}$ in $Q_n \times Q_n$, when viewed as n-planes in R^{2n}, are mutually isoclinic n-planes.*

Proof. We shall prove the theorem for the case $n = 8$ only. The proof for the cases $n = 2, 4$ follows the same line and is simpler.

Some preliminaries are necessary. Suppose that under the identification of $Q_8 \times Q_8$ with R^{16} as in Theorem 5.1, the elements (X, Y), (X', Y') of $Q_8 \times Q_8$ become the vectors (X, Y), (X', Y') in R^{16} with respectively the components $(x_1, ..., x_{16})$, $(x'_1, ..., x'_{16})$. Then it can easily be verified that the inner product of the two vectors (X, Y) and (X', Y') is

$$<(X, Y), (X', Y')> \equiv \sum_{i=1}^{16} x_i x'_i = \operatorname{Re}(XX'^* + YY'^*) .$$

It follows from this that the length of the vector (X, Y) is

$$|(X, Y)| = <(X, Y), (X, Y)>^{1/2} = (XX^* + YY^*)^{1/2} ,$$

and that the two vectors (X, Y) and (X', Y') are orthogonal if and only if $\operatorname{Re}(XX'^* + YY'^*) = 0$.

We can now prove our theorem by showing that in R^{16}, the 8-plane $\mathbf{A}: Y = AX$ is isoclinic with the 8-planes $\mathbf{B}: Y = BX$ and $\mathbf{O}^{\perp}: X = 0$.

Let $(T, BT) \in \mathbf{B}$ be the projection of any nonzero vector $(X, AX) \in \mathbf{A}$ on $\mathbf{B}$. Then the vector $(X - T, AX - BT)$ is orthogonal to $\mathbf{B}$, i.e., it is orthogonal to all the vectors $(W, BW) \in \mathbf{B}$, where W is an arbitrary Cayley number. Therefore,

$$\text{(A.1)} \qquad \operatorname{Re}\{(X - T)W^* + (AX - BT)(BW)^*\} = 0 \quad \text{for all} \quad W \in Q_8 .$$

Since, by (4) (ii) and (7) (iii) in Appendix 1, the terms inside the brackets in Re { } are commutative and associative, the left-hand side of (A.1) is equal to

$$\begin{aligned}
&\operatorname{Re}\{(X - T)W^* + [(AX - BT)W^*]B^*\} \\
&= \operatorname{Re}\{(X - T)W^* + [B^*(AX - BT)]W^*\} \\
&= \operatorname{Re}\{(X - T)W^* + [(B^*A)X - (B^*B)T]W^*\} \\
&= \operatorname{Re}\{[X - T + (B^*A)X - (B^*B)T]W^*\} .
\end{aligned}$$

Therefore, by (4) (iii) in Appendix 1, condition (A.1) implies that

$$X - T + (B^*A)X - (B^*B)T = 0 ,$$

and hence

$$\text{(A.2)} \qquad T = (1 + B^*A)X/(1 + B^*B) .$$

Now, the squared length of the vector (X, AX) is

$$|(X, AX)|^2 = XX^* + (AX)(AX)^* = XX^* + AA^*XX^*,$$

i.e.,

(A.3) $$|(X, AX)|^2 = (1+A^*A)XX^*.$$

Similarly,

$$|(T, BT)|^2 = (1+B^*B)TT^*.$$

But by (A.2),

$$TT^* = (1+B^*A)X[(1+B^*A)X]^*/(1+B^*B)^2 = (1+B^*A)(1+A^*B)XX^*/(1+B^*B)^2.$$

Therefore,

(A.4) $$|(T, BT)|^2 = (1+B^*A)(1+A^*B)XX^*/(1+B^*B).$$

Hence, it follows from (A.3) and (A.4) that the angle θ between the vector $(X, AX) \in \mathbf{A}$ and its projection on $\mathbf{B}$ is given by

$$\cos^2\theta = \frac{|(T, BT)|^2}{|(X, AX)|^2} = \frac{(1+A^*B)(1+B^*A)}{(1+A^*A)(1+B^*B)},$$

which shows that the angle between any nonzero vector $(X, AX) \in \mathbf{A}$ and its projection on $\mathbf{B}$ is independent of the choice of X; that is, the 8-plane $\mathbf{A}$ is isoclinic with the 8-plane $\mathbf{B}$.

Finally, to show that the 8-plane $\mathbf{A}$: $Y = AX$ is isoclinic with the 8-plane $\mathbf{O}^\perp$: $X = 0$, we need only observe that the projection of the nonzero vector $(X, AX) \in \mathbf{A}$ on $\mathbf{O}^\perp$ is the vector (O, AX), and

$$\frac{|(O, AX)|^2}{|(X, AX)|^2} = \frac{(AX)(AX)^*}{(1+A^*A)XX^*} = \frac{AA^*}{1 + AA^*}$$

is independent of X.