2. A Topological Lemma

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **34 (1988)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

from $\{u \in C^\infty(X), \int u dX_g = 0\}$ to $\{v \in C^\infty(X), \int v dX_{g'} = 0\}$ ($dX_{g'}$ denotes the volume form in the metric g') when $\lambda = 0$.

For completeness, let us indicate how, for instance theorem 0.2, can be reduced to equation (1) with $\lambda = 0$. It is quite straightforward. First of all we are given a cohomology class $c \in H^2(X, \mathbf{R})$ such that there exists a Kähler form ω in c; let ρ be the Ricci form of ω: $\rho \in C_1(X)$, the first Chern class of X.

Then we are *given* $\gamma \in C_1(X)$ and hence $f \in C^\infty(X)$ a real function (defined up to an additive constant), which measures the deviation for ω from satisfying 0.2:

$$\gamma - \rho = \sqrt{-1}\, \partial\bar{\partial} f .$$

Now we look for another Kähler form $\omega' \in c$, i.e. we look for a smooth real function φ (also defined up to an additive constant), where

$$\omega' - \omega = \sqrt{-1}\, \partial\bar{\partial}\varphi$$

such that the Ricci form ρ' of ω' *coincides with* γ.

In other words, we want φ to satisfy

$$\rho' - \rho \equiv \sqrt{-1}\, \partial\bar{\partial} f ,$$

or equivalently, if g and g' are the Kähler metrics respectively associated with ω and ω',

$$\partial\bar{\partial}\{- \operatorname{Log}\det(g' g^{-1})\} \equiv \partial\bar{\partial} f$$

which immediately yields equation (1) with $\lambda = 0$:

$$- \operatorname{Log}\det(g' g^{-1}) = f ,$$

since anyway f is only defined up to an additive constant.

As ω and ω' are cohomologous and closed, so are the corresponding volume forms, therefore X has same volume measured with the metrics g and g'; this defines completely f, subject to the natural constraint mentioned above.

2. A Topological Lemma

In our setting, the continuity method becomes a "surjectivity method" since it is based on the following

LEMMA 2.1. *Let* A, B *be metric spaces, with* $A \neq \emptyset$ *and* B *connected. Let* $P: A \to B$ *be a continuous map. Assume:*

(i) P *is open,*

(ii) P *is proper, that is, for any compact subset* K *in* $B, P^{-1}(K)$ *is compact. Then* P *is surjective.*

Proof. We only need to prove that $P(A)$ is closed. Let b_0 be a point in $\overline{P(A)}$. Since B is a metric space, there exists a sequence $(b_i)_{i>0}$ in $P(A)$ converging to b_0. The subset $K = \{b_0, b_1, b_2, ...\}$ is compact, hence so is $PP^{-1}(K)$. The latter contains $b_1, ..., b_i, ...,$ hence b_0, and it is obviously contained in $P(A)$. Q.E.D.

In order to make use of this lemma, we shall need some inverse function theorem for (i), and some *a priori* estimates for (ii).

3. Local inversion

THEOREM 3.1. *Let* X *be a smooth compact manifold,* V *and* W *smooth vector bundles on* X, U *an open set in* $C^\infty(X, V)$, *and* $P: U \to C^\infty(X, W)$, *a smooth nonlinear elliptic partial differential operator. Let* A *and* B *be LCFC submanifolds of* U *and of* $C^\infty(X, W)$ *respectively, such that the restriction* P_A *of* P *to* A, *sends* A *into* B. *Then the Jacobian criterion holds for* P_A, *namely, if the derivative of* $P_A: A \to B$ *is invertible at* $\varphi_0 \in A$, *then* P_A *is a local diffeomorphism near* φ_0.

This is a convenient variant of the Nash-Moser theorem (e.g. [14]) regarding suitable restrictions of elliptic operators. It is established in a separate paper [11] (see also [22]). It relies only on the *classical* (Banach) inverse function theorem combined with *elliptic regularity.*

Remark 3.2. The Nash-Moser theorem has been studied by many authors, see the bibliography below and further references in [14] [15] [25].

4. Properness

In view of (2), theorem 3.1 implies that P_λ is open. We want to apply lemma 2.1 in order to prove that P_λ is surjective from A_λ to B_λ. Since $P_\lambda(A_\lambda) \neq \emptyset$ (it contains 0), and since B_λ is connected, this amounts to proving that P_λ is *proper.* Let us explain why *a priori* estimates imply properness.

Concerning subsets in A_λ we have