2. Geodesics in US^2

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2. Geodesics in US^2

If $(p(t), v(t))$ is a geodesic in the unit tangent bundle US^2, then by the discussion in the preceding section, there must be a geodesic $h(t)$ through the identity in $SO(3)$ such that

$$h(t)\,(p(0)) = p(t) \quad \text{and} \quad h(t)\,(v(0)) = v(t)\,.$$

But $h(t)$ must fix a line in three-space, and rotate the orthogonal two-plane at constant speed. Hence $p(t)$, if it moves at all, must travel along a great or small circle, and $v(t)$ must make a constant angle with this circle.

A concrete distance formula between points (p, v) and (q, w) in US^2 is easily obtained. Let δ denote the distance between p and q on S^2, with $0 \leqslant \delta \leqslant \pi$. If this distance is less than π, that is, if p and q are not antipodal, then parallel translate v along the smaller arc of the unique great circle between p and q, and let ε denote the angle at q between this parallel translate of v and the vector w, as shown in Figure 3. If $\delta = \pi$, set $\varepsilon = 0$. Finally, let d denote the distance between (p, v) and (q, w) in US^2. Then a straightforward calculation reveals the formula

$$\cos(d/2) = \cos(\delta/2)\cos(\varepsilon/2)\,,$$

which is just the Pythagorean formula on a round sphere of radius 2, as indicated in Figure 4. Indeed, we have

$$US^2 = SO(3)/SO(1) = SO(3)\,,$$

a round, real projective 3-space.

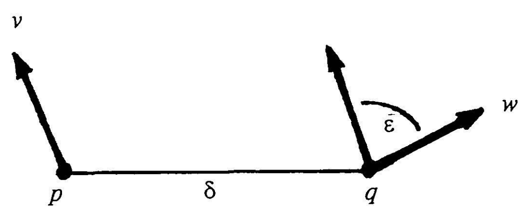

Figure 3

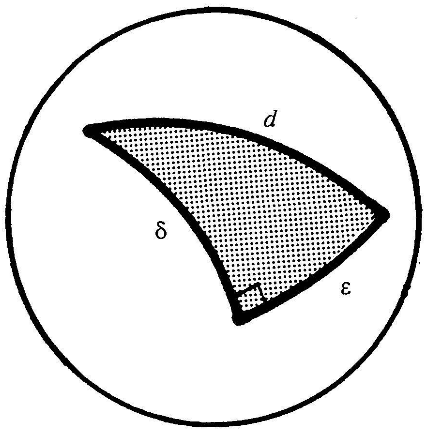

Figure 4