The group of all invertible $n \times n$ upper triangular matrices will be denoted by B_n. Its subgroup consisting of all diagonal matrices is denoted by D_n. We have $B_n = U_n \rtimes D_n$ where U_n is the closed connected subgroup of B_n consisting of all unipotent elements of B_n.

We start with some preliminary facts.

THEOREM 1 (Lie-Kolchin). *Every connected solvable affine algebraic group can be embedded in some* B_n *as a closed subgroup.* □

COROLLARY. *If* G *is a connected solvable affine group then* $G' \subset G_u$. □

THEOREM 2 (Chevalley). *If* N *is a closed normal subgroup of an affine group* G *then there exists a homomorphism* $f: G \to GL_n(k)$ *such that* $\operatorname{Ker} f = N$. □

For the proofs of Theorems 1 and 2 see, for instance, [5, Theorems 6.7 and 5.1.3].

LEMMA 1. *If* $f: G \to H$ *is a surjective homomorphism of affine algebraic groups and* $N := \operatorname{Ker} f$ *then:*

(i) $f(G^0) = H^0$;

(ii) $f(G_u) = H_u$ *and* $f(G_s) = H_s$;

(iii) $\dim G = \dim N + \dim H$;

(iv) *If* N *and* H *are connected then* G *is connected.*

Proof. For the proofs of (i) and (iii) see for instance [4, Section 7.4]. (ii) follows from the fact that f preserves the Jordan decomposition [4, Theorem 2.4.8]. We shall sketch the proof of (iv). Since N is connected, we have $N \subset G^0$. By (i) we have $f(G^0) = H^0 = H$, and consequently $G = NG^0 = G^0$. □

We need a lemma to prove the centralizer theorem. For a more general version of this lemma see [2, Proposition (9.3)].

LEMMA 2. *Let* N *be a closed normal connected abelian unipotent subgroup of an affine group* G *and let* $s \in G_s$. *Then* $M := \{sus^{-1}u^{-1} : u \in N\}$ *is a closed connected subgroup of* N, *the multiplication map* $\mu: M \times Z_N(s) \to N$ *is bijective, and* $Z_N(s)$ *is connected.*

Proof. Since N is abelian, the map $f: N \to N$, defined by $f(u) = sus^{-1}u^{-1}$, is a morphism of algebraic groups whose kernel is $Z_N(s)$ and image M, so M is a closed connected subgroup of N. If $x \in M \cap Z_N(s)$ then $x = sus^{-1}u^{-1}$ for some $u \in N$. Since $usu^{-1} = x^{-1}s = sx^{-1}$ is semisimple and x is unipotent, the uniqueness of the Jordan decomposition implies that $x = 1$. Hence $M \cap Z_N(s) = 1$ and so μ is injective. By Lemma 1 (iii) we have $\dim N = \dim M + \dim Z_N(s)$, which implies that the homomorphism μ is also surjective, i.e., $MZ_N(s) = N$. The same argument shows that $MZ_N(s)^0 = N$, and so $Z_N(s)$ must be connected. □

THEOREM 3. *If G is a connected solvable affine group and $s \in G_s$ then $Z_G(s)$ is connected and $G = G_uZ_G(s)$.*

Proof. We use induction on $\dim G$. If G is abelian the assertions are trivial. Otherwise let N be the last non-trivial term of the derived series of G. By the Corollary of Theorem 1, N is unipotent. We now apply Theorem 2 to this G and N. Let f be as in that theorem. We shall write $\bar{x}$ for $f(x)$ and $\bar{G}$ for $f(G)$.

Let $z \in G$ be such that $\bar{z} \in Z_{\bar{G}}(\bar{s})$. Then $szs^{-1}z^{-1} \in N$. By Lemma 2 there exists $u \in N$ and $v \in Z_N(s)$ such that $szs^{-1}z^{-1} = sus^{-1}u^{-1} \cdot v$. Since v commutes with u and s, and $zsz^{-1} = v^{-1} \cdot usu^{-1}$, it follows that $v = 1$. Thus $u^{-1}z \in Z_G(s)$ and consequently we have a short exact sequence

$$1 \to Z_N(s) \hookrightarrow Z_G(s) \to Z_{\bar{G}}(\bar{s}) \to 1 .$$

By Lemma 2, $Z_N(s)$ is connected. By Lemma 1 (iii) we have $\dim \bar{G} < \dim G$. By induction hypothesis, we conclude that $Z_{\bar{G}}(\bar{s})$ is connected and that $\bar{G} = (\bar{G})_u \cdot Z_{\bar{G}}(\bar{s})$. Now Lemma 1 (iv) implies that $Z_G(s)$ is connected. By part (ii) of the same lemma we have $f(G_u) = (\bar{G})_u$ and so $f(G_uZ_G(s)) = (\bar{G})_uZ_{\bar{G}}(\bar{s}) = \bar{G}$. Since $N \subset G_u$, it follows that $G = G_uZ_G(s)$. □

We now proceed to prove the main results about the structure of connected solvable affine groups. But first we need two lemmas.

LEMMA 3. *Let $S \subset B_n$ be a commuting set of semisimple elements. Then there exists $b \in B_n$ such that $b^{-1}Sb \subset D_n$.*

Proof. It is an elementary fact of linear algebra that there exists $a \in GL_n(k)$ such that $a^{-1}Sa \subset D_n$. Hence if $M_n(k)$ is the algebra of n by n matrices over k and A its subalgebra generated by S, we know that A is semisimple (and commutative). Let $V := k^n$ be the space of column vectors and let $e_1, \ldots, e_n$ be its standard basis. We shall view V as a left $M_n(k)$-module via matrix multiplication. The subspace V_i spanned by the vectors $e_1, \ldots, e_i$ is an A-submodule of V for each i. Since A is semisimple, there exist $v_i \in V_i \backslash V_{i-1}$, $1 \leqslant i \leqslant n$, such that $Av_i = kv_i$. Thus if b is the matrix whose i-th column is v_i, $1 \leqslant i \leqslant n$, then $b \in B_n$ and $b^{-1}Sb \subset D_n$. □

LEMMA 4. *If G is a connected solvable affine group, $T \subset G_s$ a closed subgroup, and $G = G_uT$ then T is a torus and $G = G_u \rtimes T$.*

Proof. By the Lie-Kolchin theorem we may assume that G is a closed subgroup of some B_n. By using the projection map $B_n \to D_n$ we obtain a short exact sequence $1 \to G_u \hookrightarrow G \xrightarrow{p} D \to 1$, where $D \subset D_n$ is a torus. Since $D = p(G) = p(G_uT) = p(T)$, Lemma 1 (i) implies that $p(T^0) = D$. Thus $G = G_uT^0$ and using $T \cap G_u = 1$ we conclude that $T = T^0$. In particular T is abelian and by Lemma 3 we may assume that $T \subset D_n$, i.e., $T = D$. Since $B_n = U_n \rtimes D_n$, $G_u \subset U_n$, $T = D \subset D_n$, and $G = G_uT$, it follows that $G = G_u \rtimes T$. □

THEOREM 4. *Let G be a connected solvable affine group. Then $G = G_u \rtimes T$ where T is a maximal torus. In particular, G_u is connected.*

Proof. We use induction on dim G. Assume first that $G_s \subset Z(G)$. Then $G_s = Z(G)_s$ is a closed subgroup of G and $G = G_uG_s$. The assertion then follows from Lemma 4. Now assume that there exists $s \in G_s \backslash Z(G)$. Then $Z_G(s)$ is a proper closed subgroup of G, see e.g. [4, Section 8.2]. By Theorem 3 it is connected and $G = G_uZ_G(s)$. By induction hypothesis there exists a torus T such that $Z_G(s) = Z_G(s)_uT$. Then $G = G_uZ_G(s) = G_uT$ and $G = G_u \rtimes T$ by Lemma 4. □

THEOREM 5. *Let $G = G_u \rtimes T$ be a connected solvable affine group. Then every $s \in G_s$ is conjugate to an element of T.*

Proof. We use induction on dim G. We have $s = ut$ where $u \in G_u$ and $t \in T$. If G is abelian then $u = 1$ and $s = t$. Otherwise let N be the last non-trivial term of the derived series of G. By the corollary of Theorem 1 we have $N \subset G_u$. Hence N is a closed connected normal abelian unipotent subgroup of G. By Theorem 2 and the induction hypothesis there exists $x \in G$ such that $xsx^{-1} = tv$ where $v \in N$. By Lemma 2, $v = t^{-1}utu^{-1}z$ where $u \in N$ and $z \in Z_N(t)$. Hence $xsx^{-1} = utu^{-1}z$. Since $xsx^{-1}, utu^{-1} \in G_s$, $z \in G_u$, and z commutes with u and t, it follows that $z = 1$ and consequently $xsx^{-1} = utu^{-1}$. □