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Symmetry Operations in Euclidean Spaces

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Summary: It is demonstrated that a small number of symmetry operations are both necessary and sufficient for the description of all groups of euclidean geometric symmetry.

Introduction

Many teachers of crystallography would be embarrassed if a student were to ask for proof that the small number of symmetry operations taught him are both necessary and sufficient to describe all the symmetry groups of crystallography. The fact that this is most unlikely to occur, does not deprive the question of legitimacy. The general difficulty with which students absorb the subject of crystallographical symmetry should be interpreted as mainly due to a poor presentation of a relatively simple subject matter.

In the following an elementary, geometrically visual description of symmetry operations is offered. A rigorously explicit development would be unnecessarily lengthy; it is, however, helpful to quote two theorems and two rules.

1. Equivalent asymmetric parts of a configuration (any collection of points) are either congruent or enantiomorphous (one can be moved so that it becomes the mirror-image of the other). See proof in appendix.

2. The commutative law is valid for coupled symmetry operations, i. e. the result is independent of the sequence of performance of the two component operations.

3. In order to avoid inconsistencies, symmetry operations must be performed without transcending the dimension of the configuration (Bader, 1948, *Am. Min.* 33, 642—644).

4. A symmetry operation (coverage of one part of a configuration by another) shall be performed in the simplest manner possible.

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I. Three dimensional configurations

Let the configuration consist of two asymmetric equivalent tetrahedra. This is representative of the non-specialized general case, since the points of an asymmetric configuration can be conceived as being corners of tetrahedra.

1. The tetrahedra are congruent

It is always possible to cover one tetrahedron with the other by means of two consecutive rotations around two lines. There is then the choice of an infinite number of pairs of lines and corresponding pairs of angles of rotation. One unique case exists, however, namely when the lines are normal to each other and one of them lies at infinity. Rotation around the latter produces a parallel shift, a *translation*, and the former is parallel to the line of translation. The operation becomes a *translatory-rotation*, symbolized by a *screw axis.*

Special cases

(a) The translatory component of the screw axis is zero. Then the operation is a simple *rotation* around a line, indicated by a *rotary axis.*
(b) The rotary component of the screw axis is zero. Then the operation is a simple *translation* parallel to a line.

2. The tetrahedra are enantiomorphous

Coverage can always be achieved by a reflection in a plane, which produces a virtual tetrahedron congruent with the one to be covered, followed by rotation around a line, or vice versa. But the operation is not commutative unless the line is normal to the plane, in which case the operation is a *rotary-reflection.* (It is a matter of taste if one wishes to substitute the equivalent *rotary-inversion.*) The operator (element of symmetry) is the *axis and plane of rotary-reflection* (or the *axis and point of rotary-inversion).*

Special cases

(a) The rotary component of rotary-reflection is zero. The operation is a simple *reflection* in a plane (operator is a *mirror plane* or *plane of symmetry.*) This is equivalent to a 180°-rotary-inversion, but then the position of the axis of rotary-inversion is indeterminate, except that it is normal to the plane of reflection it replaces.

(b) The rotary component of rotary-reflection is 180°. The position of the operator (axis and plane of rotary-reflection) is then indeterminate, except that it must include a given point. It is therefore simpler to define the operator by this point which is a *point of inversion (center of symmetry)* corresponding to a rotary-inversion, with zero component.

(c) The rotary component of rotary-reflection refers to rotation around a line at infinity. The operation becomes a *translatory-reflection* the operator a *glide plane*. It is noted that this special case is not imaginatively derivable from rotary-inversion.

II. Two dimensional configurations

Let the configuration consist of two asymmetric equivalent triangles.

1. The triangles are congruent

It is always possible to cover one with the other by means of *rotation* around a point. The element of symmetry is a *point of rotation* (not an axis).

Special case

Corresponding sides of the triangle are parallel. The point of rotation is at infinity. The operation becomes a *translation.*

2. The triangles are enantiomorphous

The covering operation consists of two coupled operations. First, congruence is established by producing a virtual triangle by reflection in a line, and coverage is then achieved by rotation around a point, or vice versa. However, the commutative law is not valid unless the point of rotation is at infinity, i. e. unless the first order component operation is a translation. It follows that the second order component (reflection) must be in a line parallel to the directions of translation. In consequence the covering operation is a *translatory-reflection* in a line. The indicating element of symmetry is a *glide line* (not a glide plane).

Special cases

The translatory component is zero. The operation is a simple *reflection* in a line, the operator a *mirror line.*

III. One dimensional configurations

All the points of the configuration lie on a line, and only two operations are possible: *translation* and *reflection* in a point. Here the concept of translation as a special case of rotation breaks down as it would transcend the dimensions of the configuration.

Conclusion

Keeping in mind the mandate that a given coverage shall be achieved by the simplest possible geometrical operation, we can now enumerate, the necessary and sufficient operations of symmetry.

Operation	Element of symmetry
translatory-rotation	screw axis
translation	directions of translation
rotation	axis of rotation point of rotation
(rotary-reflection or rotary-inversion)	axis-plane or rotary-reflection, axis-point of rotary-inversion
reflection	mirror plane mirror line mirror point
inversion	center of symmetry
translatory-reflection	glide plane glide line

This adds up to seven operations and eleven elements of symmetry.

Appendix

Equivalent parts of configuration are either congruent or enantimorphous.

The asymmetric tetrahedron (corner points $P_1P_2P_3P_4$) to be equivalent (coverable by symmetry operation) with the tetrahedron $P'_1P'_2P'_3P'_4$. Since one of the conditions of symmetry is that corresponding edges of the tetrahedra shall be of equal length, it is possible to move one tetrahedron so that the triangle $P_1P_2P_3$ covers the corresponding triangle $P'_1P'_2P'_3$. The points P_4 and P'_4 will then either coincide (tetrahedra are *congruent*) or lie on opposite sides of the plane of the triangles $P_1P_2P_3$ and $P'_1P'_2P'_3$. In the latter case they both lie on a single normal to the plane of the triangles and at equal distances from the plane. Each of the tetrahedra is then a mirror image of the other; they are said to be *enantiomorphous*.

Chart of symmetry operations

Dimensions of configuration	Simple		Coupled	
	1st Order	2nd Order	1st Order	2nd Order
1	Translation	Reflection in a point	—	—
2	Translation. Rotation around a point	Reflection in a line	—	Translatory-reflection in a line
3	Translation. Rotation around a line	Reflection in a plane. Inversion	Translatory-rotation in and around a line	Translatory-reflection in a plane. Rotary-reflection normal to and in a plane. Rotary-inversion in a line and point

Note: Operations are of the first order if they can be performed by moving a model without deforming it. Second Order operations, involve either reflection or inversion.

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