# An elementary unified approach to some loss variance bounds

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# An elementary unified approach to some loss variance bounds

## 1 Introduction: a financial loss structure

Let $(\Omega, P, A)$ be a probability space such that $\Omega$ is the sample space, $P$ the probability measure, and $A$ the $\sigma$-field of events of $\Omega$. Let $E \in A$ be an event such that $0 < P(E) < 1$. Then the complement event $\overline{E} \in A$ and $0 < P(\overline{E}) = 1 - P(E) < 1$. Consider $X : \Omega \to R$ a real random variable on this probability space with finite mean $\mu$ and variance $\sigma^2$, which one endows with a structure of *"financial loss"* as follows.

One assumes that $0 < Pr(X > 0)$, $Pr(X \leq 0) < 1$. Since the random variable $X$ takes both non-positive and positive values (and by assumption on the probability of the event $E$), it can be written (in a way depending on $E$) as difference of a positive random variable, representing financial liabilities, and a non-negative random variable, representing financial assets. Any such difference can be interpreted as a financial loss. This intuitive notion can be modelled by a precise mathematical object. Let $I : A \to \{0, 1\}$ be the indicator function and define

| | |
|---|---|
| $X(E) := X \cdot I(E)$ | : the amount to be paid if the event $E$ occurs, called $E$-*loss*, or simply *loss* if $E$ is clear from the context |
| $\overline{X}(E) := -X(\overline{E}) = (-X) \cdot I(\overline{E})$ | : the amount gained if the event $\overline{E}$ occurs, called $E$-*gain*, or simply *gain* |

Since $I(E) + I(\overline{E}) = 1$ one has $X = X \cdot I(E) + X \cdot I(\overline{E}) = X(E) - \overline{X}(E)$, which as a difference between loss and gain justifies the interpretation of $X$ as a financial loss. The negative value $G := -X = \overline{X}(E) - X(E)$ is called *financial gain.*

**Example 1.1.** Let $E = \{X > 0\}$, $\overline{E} = \{X \leq 0\}$, a situation often encountered. Let $F(x)$ be the distribution function of $X$. Then $P(\overline{E}) = Pr(X \leq 0) = F(0)$ is the no loss probability and $P(E) = \overline{F}(0) = 1 - F(0)$ is the loss probability. By assumption $X$ takes both negative and positive values. The random variable $X(E) = X_+$ is the loss, $\overline{X}(E) = G \cdot I(\overline{E}) = G_+$

is the gain. Clearly one has $X_- = G_+$, $G_- = X_+$, that is the negative part of the loss equals the gain, and the negative part of the gain equals the loss.

In general one has the following loss and gain identities of any order $n = 1, 2, \ldots$:

$$X(E)^n + (-1)^n \cdot \overline{X}(E)^n = X^n\,, \tag{1.1}$$
$$\overline{X}(E)^n + (-1)^n \cdot X(E)^n = G^n\,. \tag{1.2}$$

The second set of relations is obtained from the first just by changing signs.

**Example 1.1** (continued). In Section 5 (a brief outlook on applications) it is argued that the first-order gain identity $G + X_+ = G_+$, that is the relation

financial gain + loss = gain,

is of fundamental importance for both Actuarial Science and Finance. Its general study has been named "AFIR problem" by Bühlmann(1995).

## 2 Loss and gain variance bounds

At an elementary level one is only interested in first and second order moments. The following notations are used:

$$\begin{array}{lll}
M(E) & = E[X(E)] & : \text{mean loss} \\
M_2(E) & = E[X(E)^2] & : \text{mean squared loss} \\
V(E) & = M_2(E) - M(E)^2 = \mathrm{Var}[X(E)] & : \text{loss variance} \\
\overline{M}(E) & = E[\overline{X}(E)] = E[G(\overline{E})] & : \text{mean gain} \\
\overline{M}_2(E) & = E[\overline{X}(E)^2] = E[G(\overline{E})^2] & : \text{mean squared gain} \\
\overline{V}(E) & = \mathrm{Var}[\overline{X}(E)] = \mathrm{Var}[G(\overline{E})] & : \text{gain variance}
\end{array}$$

Besides the loss and gain random variables, it appears very useful to consider the same conditional quantities given that the events $E$ and $\overline{E}$ have occurred:

$X_c(E) = (X|E)$ : amount paid given that the event $E$ has occurred, called *conditional loss*

$\overline{X}_c(E) = (-X|\overline{E}) = (G|\overline{E})$ : amount gained given that the event $\overline{E}$ has occurred, called *conditional gain*

The corresponding first and second order conditional moments are:

$$\begin{aligned}
m(E) &= E[X|E] && \text{: mean conditional loss} \\
m_2(E) &= E[X^2|E] && \text{: mean squared conditional loss} \\
v(E) &= m_2(E) - m(E)^2 && \\
&= \mathrm{Var}[X|E] && \text{: conditional loss variance} \\
\overline{m}(E) &= E[G|E] && \text{: mean conditional gain} \\
\overline{m}_2(E) &= E[G^2|\overline{E}] && \text{: mean squared conditional gain} \\
\overline{v}(E) &= \mathrm{Var}[G|\overline{E}] && \text{: conditional gain variance}
\end{aligned}$$

The conditional and unconditional values are related as follows:

$$M(E) = P(E) \cdot m(E) \tag{2.1}$$

$$\begin{aligned}
M_2(E) &= P(E) \cdot m_2(E)\,, \\
V(E) &= P(E) \cdot (v(E) + P(\overline{E}) \cdot m(E)^2)
\end{aligned} \tag{2.2}$$

$$\overline{M}(E) = P(\overline{E}) \cdot \overline{m}(E) \tag{2.3}$$

$$\begin{aligned}
\overline{M}_2(E) &= P(\overline{E}) \cdot \overline{m}_2(E)\,, \\
\overline{V}(E) &= P(\overline{E}) \cdot (\overline{v}(E) + P(E) \cdot \overline{m}(E)^2)
\end{aligned} \tag{2.4}$$

The fundamental identities (1.1), ( 1.2) imply "loss and gain parity relations" of any order $n = 1, 2, \ldots$ For the first two orders one has:

$$M(E) - \overline{M}(E) = \mu \tag{2.5}$$

$$M_2(E) + \overline{M}_2(E) = \mu^2 + \sigma^2 \tag{2.6}$$

$$V(E) + \overline{V}(E) = \sigma^2 - 2M(E) \cdot \overline{M}(E) \tag{2.7}$$

The latter relation expresses the fact that the second order *total loss and gain variance* depends on the variance $\sigma^2$, which is often assumed to be known, and on $M(E)$, $\overline{M}(E)$, which are first order quantities.

**Example 1.1** (continued). In this main special situation the unconditional quantities are the mean loss $M^+ = E[X_+]$, the mean squared loss $M_2^+ = E[X_+^2]$, the loss variance $V^+ = \mathrm{Var}[X_+]$, the mean gain $M^- = E[X_-] = E[G_+]$, the mean squared gain $M_2^- = E[X_-^2] = E[G_+^2]$, and the gain variance $V^- = \mathrm{Var}[X_-] = \mathrm{Var}[G_+]$. Without the obvious names the conditional values are $m^+ = E[X|X > 0]$, $m_2^+ = E[X^2|X > 0]$, $v^+ = \mathrm{Var}[X|X > 0]$, $m^- = E[G|X \le 0]$, $m_2^- = E[G^2|X \le 0]$, $v^- = \mathrm{Var}[G|X \le 0]$. The relations (2.1) to (2.7) specialize to

$$M^+ = \overline{F}(0) \cdot m^+ \tag{2.1'}$$

$$\begin{aligned} M_2^+ &= \overline{F}(0) \cdot m_2^+ , \\ V^+ &= \overline{F}(0) \cdot (v^+ + F(0) \cdot (m^+)^2) \end{aligned} \tag{2.2'}$$

$$M^- = F(0) \cdot m^- \tag{2.3'}$$

$$\begin{aligned} M_2^- &= F(0) \cdot m_2^- , \\ V^- &= F(0) \cdot (v^- + \overline{F}(0)(m^-)^2) \end{aligned} \tag{2.4'}$$

$$M^+ - M^- = \mu \tag{2.5'}$$

$$M_2^+ + M_2^- = \mu^2 + \sigma^2 \tag{2.6'}$$

$$V^+ + V^- = \sigma^2 - 2M^+M^- \tag{2.7'}$$

The considered elementary notions imply immediately the following loss and gain variance bounds.

**Theorem 2.1.** If the loss probability $P(E)$ is unknown, one has the upper bounds

$$V(E) \le \sigma^2 - 2M(E) \cdot \overline{M}(E) \tag{2.8}$$

$$\overline{V}(E) \le \sigma^2 - 2M(E) \cdot \overline{M}(E) \tag{2.9}$$

**Proof.** Since variances are non-negative this is an obvious consequence of (2.7).

**Theorem 2.2.** If the loss probability $P(E)$ is known, one has the lower and upper bounds

$$\frac{P(\overline{E})}{P(E)} \cdot M(E)^2 \le V(E) \le \sigma^2 - 2M(E) \cdot \overline{M}(E) - \frac{P(E)}{P(\overline{E})} \cdot \overline{M}(E)^2 \tag{2.10}$$

$$\frac{P(E)}{P(\overline{E})} \cdot \overline{M}(E)^2 \le \overline{V}(E) \le \sigma^2 - 2M(E) \cdot \overline{M}(E) - \frac{P(\overline{E})}{P(E)} \cdot \overline{M}(E)^2 \tag{2.11}$$

**Proof.** From the relationships (2.1), (2.2) one gets

$$v(E) = m_2(E) - m(E)^2 = \frac{1}{P(E)} \cdot \left( M_2(E) - \frac{M(E)^2}{P(E)} \right).$$

Since $v(E) \ge 0$ one obtains

$$V(E) = M_2(E) - M(E)^2 \ge \frac{P(\overline{E})}{P(E)} \cdot M(E)^2,$$

which is the lower bound in (2.10). The lower bound in (2.11) is shown similarly. The upper bounds follow from the identity (2.7) using the lower bounds.

Because of their practical importance, the special cases obtained when $E = \{X > 0\}$ are stated separately.

**Corollary 2.1.** If the loss probability $\overline{F}(0)$ is unknown, one has the upper bounds

$$V^+ \le \sigma^2 - 2M^+M^- \tag{2.8'}$$

$$V^- \le \sigma^2 - 2M^+M^- \tag{2.9'}$$

**Corollary 2.2.** If the loss probability $\overline{F}(0)$ is known, one has the lower and upper bounds

$$\frac{F(0)}{\overline{F}(0)} \cdot (M^+)^2 \leq V^+ \leq \sigma^2 - 2M^+M^- - \frac{\overline{F}(0)}{F(0)} \cdot (M^-)^2 \tag{2.10'}$$

$$\frac{\overline{F}(0)}{F(0)} \cdot (M^-)^2 \leq V^- \leq \sigma^2 - 2M^+M^- - \frac{F(0)}{\overline{F}(0)} \cdot (M^+)^2 \tag{2.11'}$$

**Remarks 2.1.**

(a) Up to an obvious location or shift transformation, the inequality (2.8′) has been mentioned in Hürlimann(1993a) (see also Birkel(1994), Hesselager(1993), Sundt(1993), Exercise 10.1).
(b) Birkel(1994) considers a loss random variable $X = Z - \varphi(Z)$, $Z$ a non-negative claims amount, $\varphi$ some non-negative transformation, and interprets the loss $X_+$ as claims amount of a "general" reinsurance contract. If $\varphi(z) = d$ is a constant deductible, one recovers the stop-loss contract. If $\varphi(z)$ is not constant one has

$$\begin{aligned}\sigma^2 &= \mathrm{Var}[Z] - 2\,\mathrm{Cov}[Z, \varphi(Z)] + \mathrm{Var}[\varphi(Z)] \\ &= \mathrm{Var}[Z] - 2\,\mathrm{Cov}[Z - \varphi(Z), \varphi(Z)] - \mathrm{Var}[\varphi(Z)]\end{aligned}$$

In case $\varphi(z)$ and the function $f(z) = z - \varphi(z)$ are non-decreasing, one has $\mathrm{Cov}[f(Z), \varphi(Z)] \geq 0$ because the pair $(Z, Z)$ is positively quadrant dependent. Recall that a pair of random variables $(X, Y)$ is called *positively quadrant dependent*, a relation written $PQD(X, Y)$, if $Pr(X > x, Y > y) \geq Pr(X > x)Pr(Y > y)$ for all $x$, $y$. It is well-known that the relation $PQD(X, Y)$ is equivalent with the property $\mathrm{Cov}[f(X), g(Y)] \geq 0$ for all nondecreasing real functions $f$ and $g$ for which the covariance exists. For appropriate references consult Jogdeo(1982). The second equality implies the upper bound $\sigma^2 \leq \mathrm{Var}[Z]$. In case only $\varphi(z)$ is non-decreasing, one has $\mathrm{Cov}[Z, \varphi(Z)] \geq 0$ (for the same reason) and the first equality implies the upper bound $\sigma^2 \leq \mathrm{Var}[Z] + \mathrm{Var}[\varphi(Z)]$. Inserting these bounds into (2.8′), (2.10′), one recovers Birkel's main result. These simple examples illustrate the fact that in more general applications of the loss and gain variance bounds, the difficulty will be to calculate or at least to estimate appropriately the financial loss variance function $\sigma^2$.

(c) Equivalent statements can be made for the second-order moments:

$$\frac{M(E)^2}{P(E)} \leq M_2(E) \leq \mu^2 + \sigma^2 - \frac{\overline{M}(E)^2}{P(\overline{E})} \tag{2.12}$$

$$\frac{\overline{M}(E)^2}{P(\overline{E})} \leq \overline{M}_2(E) \leq \mu^2 + \sigma^2 - \frac{M(E)^2}{P(E)} \tag{2.13}$$

$$\frac{(M^+)^2}{\overline{F}(0)} \leq M_2^+ \leq \mu^2 + \sigma^2 - \frac{(M^-)^2}{F(0)} \tag{2.12'}$$

$$\frac{(M^-)^2}{F(0)} \leq M_2^- \leq \mu^2 + \sigma^2 - \frac{(M^+)^2}{\overline{F}(0)} \tag{2.13'}$$

To derive (2.12) insert $V(E) = M_2(E) - M(E)^2$ into (2.10). The left hand side is then obtained by noting that $P(E) + P(\overline{E}) = 1$. For the right hand side add and subtract $\overline{M}(E)^2$, and then use that $M(E) - \overline{M}(E) = \mu$ by (2.5) and also $P(E) + P(\overline{E}) = 1$. The bounds (2.13) are derived similarly from (2.11) while (2.12′) and (2.13′) are restatements for the special case $E = \{X > 0\}$.

(d) There is a "dual" or "conjugate" property relating all these inequalities, which render them easy to remind of. To pass from one inequality to the other, it suffices to regard loss quantities without a "bar" (resp. with a plus sign) as conjugates of gain quantities with a bar (resp. with a minus sign). The "algebraic" bar reflects the property that $E$ and $\overline{E}$ are complementary events. In case $E = \{X > 0\}$ one adopts a formal "bar" mathematical operation such that $M^- = \overline{M}^+$, $M_2^- = \overline{M}_2^+$, $V^- = \overline{V}^+$, $\overline{F}(x) = 1 - F(x)$. One makes the further conventions $\overline{\overline{M}}^+ = M^+$, $\overline{\overline{M}}_2^+ = M_2^+$, $\overline{\overline{V}}^+ = V^+$, $\overline{\overline{F}}(x) = F(x)$, $\overline{\sigma} = \sigma$, $\overline{\mu} = -\mu$. Then the pairs (2.10), (2.11), resp. (2.10′), (2.11′), and (2.12), (2.13), resp. (2.12′), (2.13′), are conjugate pairs, while the relations (2.5) to (2.9), resp. (2.5′) to (2.9′) are self-conjugate. In case $E = \{X > 0\}$ the dual counterparts have been derived and applied in Hürlimann(1994a).

## 3 Sharpness and extremal variance bounds

It is natural to ask when the obtained variance bounds are sharp, that is attained for some financial loss random variable. In Lemma 3.1 below,

one shows this is the case for diatomic financial losses. In practice one often knows the mean loss $M(E)$ or equivalently by (2.5) the mean gain $\overline{M}(E)$. In this situation one asks for extremal bounds over the space $D := D(\mu, \sigma, M(E))$ of all financial loss random variables with fixed mean, variance and mean loss. In the special case $X = Z - d$, $Z$ a random variable, $d$ a constant, with $E = \{X > 0\}$, $X(E) = X_+ = (Z - d)_+$ the stop-loss transform, the present problem has been posed and tackled by Schmitter(1993/95). Since his proof appears to be somewhat complicated, our aim is to present a simpler and also more general probabilistic proof in the spirit of Hürlimann(1994b) but much simplified, more rigorous and shedding some additional insight.
From (2.7) and Theorem 2.2 one observes that the loss variance $V(E) = \sigma^2 - 2M(E)\overline{M}(E) - \overline{V}(E)$ is *maximum* over the space $D$ if the following condition can be fulfilled:

(C) The lower bound of the inequality $\overline{V}(E) \geq \frac{P(E)}{P(\overline{E})} \cdot \overline{M}(E)^2$ is attained and this quantity is a *minimum*.

The set of all diatomic financial losses with fixed mean and variance is denoted by $D_2 := D_2(\mu, \sigma)$. It is well-known that a diatomic random variable with fixed mean and variance is uniquely determined by the one-parametric family of supports $\{x, x^*\}$ such that

$$x = \mu - \sigma\sqrt{\frac{1-p}{p}}, \quad x^* = \mu + \sigma\sqrt{\frac{p}{1-p}}, \tag{3.1}$$

where $p$, later set equal to $P(\overline{E})$, is the probability at the mass point $x$, and $x^* = \mu + \frac{\sigma^2}{\mu - x}$ is an involution mapping reflecting the equation of variance $(\mu - x)(x^* - \mu) = \sigma^2$.

**Lemma 3.1.** Let $X = \{x, x^*\} \in D_2$, be a diatomic financial loss of the form (3.1). Then the following equalities hold simultaneously:

$$V(E) = \left(\frac{p}{1-p}\right) \cdot M(E)^2, \quad \overline{V}(E) = \left(\frac{1-p}{p}\right) \cdot \overline{M}(E)^2. \tag{3.2}$$

**Proof.** Since $X$ is endowed with the financial loss structure, one has by assumption $x \le 0 < x^*$. For arbitrary $p$ one has $M(E) = (1-p)x^*$, $M_2(E) = (1-p)(x^*)^2$, $V(E) = p(1-p)(x^*)^2$, hence the equality $V(E) = \left(\frac{p}{1-p}\right) \cdot M(E)^2$ holds. The other equality is similarly obvious.
Suppose now that $M(E)$ is known, and consider a diatomic financial loss for which (3.2) holds with $p = P(\overline{E})$. As a function of $\left(\frac{p}{1-p}\right)$ only, the first quantity is monotone increasing, and the second one monotone decreasing. The maximum of $V(E)$ is thus obtained at the greatest value of $\left(\frac{p}{1-p}\right)$, which solves (2.7):

$$\left(\frac{p}{1-p}\right) \cdot M(E)^2 + \left(\frac{1-p}{p}\right) \cdot \overline{M}(E)^2 = \sigma^2 - 2M(E)\overline{M}(E)\,. \tag{3.3}$$

If $M(E) = 0$ then $\overline{V}(E) = \sigma^2$ is maximum, hence condition (C) cannot hold. Therefore one can assume $M(E) > 0$. Multiplying (3.3) with $\left(\frac{p}{1-p}\right)$, one obtains a quadratic equation, whose greatest solution is

$$\left(\frac{p}{1-p}\right) = \frac{1}{2} \cdot \frac{\sigma^2 - 2M(E)\overline{M}(E) + \sigma\sqrt{\sigma^2 - 4M(E)\overline{M}(E)}}{M(E)^2}\,. \tag{3.4}$$

The "inequality of Bowers" for $E$-losses (see Theorem 4.2 in Section 4)

$$M(E) \le \frac{1}{2}\left(\sqrt{\mu^2 + \sigma^2} + \mu\right), \tag{3.5}$$

which implies

$$4M(E)\overline{M}(E) \le \sigma^2\,, \tag{3.6}$$

guarantees a real solution. It remains to show that the obtained diatomic financial loss belongs to $D(\mu, \sigma, M(E))$, that is $E[X(E)]$ equals the given mean loss $M(E)$. Since $x \le 0 < x^*$ and by (3.1), one must satisfy

$$E[X(E)] = (1-p)x^* = \mu(1-p) + \sigma\sqrt{p(1-p)} = M(E)\,, \tag{3.7}$$

an equation, which in virtue of (2.5), is equivalent to

$$pM(E) + (1-p)\overline{M}(E) = \sigma\sqrt{p(1-p)}\,. \tag{3.8}$$

Taking squares one sees that $p$ is indeed solution of (3.3). The above considerations show that Kremer's generalized upper bound for $V(E)$ in (2.10) is sharp and yields the maximum value of $V(E)$ over the space of financial losses $D(\mu, \sigma, M(E))$. This is in essence the result of Schmitter(1993/95). The other extremal values are obtained similarly. The applied method and the symmetric character of the bounds reveals that $V(E) =$ max. exactly when $\overline{V}(E) =$ min., and $\overline{V}(E) =$ max. when $V(E) =$ min., extremal values taken with respect to the space $D$. These facts are summarized as follows.

**Theorem 3.1.** Suppose that $0 < p = P(\overline{E}) < 1$. Then the loss and gain variance extremal bounds over the space of financial losses $D(\mu, \sigma, M(E))$ are attained for diatomic financial losses as follows:

*Case 1:* maximal loss and minimal gain variance

If $M(E) > 0$ the extremal values

$$\max_D\{V(E)\} = \frac{1}{2}\left\{\sigma^2 - 2M(E)\overline{M}(E) + \sigma\sqrt{\sigma^2 - 4M(E)\overline{M}(E)}\right\} \quad (3.9)$$

$$\min_D\{\overline{V}(E)\} = \frac{1}{2}\left\{\sigma^2 - 2M(E)\overline{M}(E) - \sigma\sqrt{\sigma^2 - 4M(E)\overline{M}(E)}\right\} \quad (3.10)$$

are attained at $X = \left\{\mu - \sigma\sqrt{\frac{1-p}{p}}, \mu + \sigma\sqrt{\frac{p}{1-p}}\right\}$ with

$$\left(\frac{p}{1-p}\right) = \frac{1}{2} \cdot \frac{\sigma^2 - 2M(E)\overline{M}(E) + \sigma\sqrt{\sigma^2 - 4M(E)\overline{M}(E)}}{M(E)^2}. \quad (3.11)$$

*Case 2:* maximal gain and minimal loss variance

If $\overline{M}(E) > 0$ the extremal values

$$\max_D\{\overline{V}(E)\} = \frac{1}{2}\left\{\sigma^2 - 2M(E)\overline{M}(E) + \sigma\sqrt{\sigma^2 - 4M(E)\overline{M}(E)}\right\} \quad (3.12)$$

$$\min_D\{V(E)\} = \frac{1}{2}\left\{\sigma^2 - 2M(E)\overline{M}(E) - \sigma\sqrt{\sigma^2 - 4M(E)\overline{M}(E)}\right\} \quad (3.13)$$

are attained at $X = \left\{\mu - \sigma\sqrt{\frac{1-p}{p}}, \mu + \sigma\sqrt{\frac{p}{1-p}}\right\}$ with

$$\left(\frac{1-p}{p}\right) = \frac{1}{2} \cdot \frac{\sigma^2 - 2M(E)\overline{M}(E) + \sigma\sqrt{\sigma^2 - 4M(E)\overline{M}(E)}}{\overline{M}(E)^2}. \quad (3.14)$$

**Remarks 3.1.**

In the limiting case as $M(E) \to 0$, but $M(E)$ does not attain zero, Case 1 still applies. One finds a diatomic maximizing financial loss $X = \{x, x^*\}$ such that $x \to \mu(\le 0)$, $x^* \to \infty$, $p \to 1$, $\max\{V(E)\} \to \sigma^2$, $\min\{\overline{V}(E)\} \to 0$. On the other side $V(E) = \sigma^2$, $\overline{V}(E) = 0$ is exactly attained if in Case 1 one chooses $\overline{M}(E) = 0$, $M(E) = \mu > 0$. One finds the diatomic maximizing financial loss $X = \{0, \mu + \frac{\sigma^2}{\mu}\}$ with $p = \frac{\sigma^2}{\mu^2+\sigma^2}$, which is somewhat pathological because $x = 0$ is not strictly negative, but allowed by our modelling assumptions. As shown below this is the unique diatomic financial loss for which equality holds in the weaker inequality (2.8). The conjugate observations can be made in Case 2.

It may be useful to know when the bounds of Theorem 2.1 are sharp, a result applied in Hürlimann(1994c).

**Theorem 3.2.** The upper bounds in (2.8), (2.9) are attained as follows:

*Case 1:* maximal loss variance

The upper bound (2.8) is sharp provided $X = \{0, \mu + \frac{\sigma^2}{\mu}\}$, $p = \frac{\sigma^2}{\mu^2+\sigma^2}$, $\overline{M}(E) = 0$, $M(E) = \mu > 0$, $\overline{V}(E) = 0$, $V(E) = \sigma^2$,

*Case 2:* maximal gain variance

The upper bound (2.9) is sharp provided $G = \{0, -(\mu + \frac{\sigma^2}{\mu})\}$, $p = \frac{\sigma^2}{\mu^2+\sigma^2}$. $M(E) = 0$, $\overline{M}(E) = -\mu > 0$, $V(E) = 0$, $\overline{V}(E) = \sigma^2$.

**Proof.** Looking at (2.10), one observes that the upper bound in the weaker inequality (2.8) is attained only if $\overline{M}(E) = 0$. The maximizing diatomic financial loss is found from the Case 1 in Theorem 3.1. Case 2 follows similarly.

## 4 A variant of the inequality of Bowers

In Actuarial Science the following result is attributed to Bowers(1969).

**Theorem 4.1.** Let $Z$ be a random variable with finite mean $\mu$ and variance $\sigma^2$. Then the net stop-loss premium at the deductible $d$ satisfies the inequality

$$E[(Z-d)_+] \le \frac{1}{2}\left\{\sqrt{(d-\mu)^2+\sigma^2}-(d-\mu)\right\}, \tag{4.1}$$

whose upper bound is attained for a diatomic random variable with support $\{z, z^*\}$ such that $z = d - \sqrt{(d-\mu)^2+\sigma^2}$, $z^* = d + \sqrt{(d-\mu)^2+\sigma^2}$.

In the context of a general financial loss theory, there is a need for further variants of this result, as for example (3.5), which has been used for a proof of Theorem 3.1.

**Theorem 4.2.** Let $X$ be a financial loss random variable with finite mean $\mu$ and variance $\sigma^2$, such that $X = X(E) - \overline{X}(E)$, $E$ an event with probability $0 < P(E) < 1$. Then the $E$-loss satisfies the inequality

$$M(E) \le \frac{1}{2}\left(\sqrt{\mu^2+\sigma^2}+\mu\right), \tag{4.2}$$

whose upper bound is attained for a diatomic financial loss with support $\{x, x^*\}$ such that $x = -\sqrt{\mu^2+\sigma^2}$, $x^* = \sqrt{\mu^2+\sigma^2}$.

**Proof.** The verification is done in two steps.

*Step 1:* The inequality (4.2) holds.

Besides the financial loss $X = X(E) - \overline{X}(E)$ consider its "absolute" value $|X| := X(E) + \overline{X}(E)$. Since $E$, $\overline{E}$ are complementary events, one has $X(E)\cdot\overline{X}(E) = 0$, hence $|X|^2 = X^2$. This property justifies in particular the interpretation of $|X|$ as absolute value of the financial loss. It follows that

$$\mathrm{Var}[|X|] = E[X^2] - E[|X|]^2 = \mu^2 + \sigma^2 - (M(E) + \overline{M}(E))^2 \ge 0,$$

and thus

$$M(E) + \overline{M}(E) \le \sqrt{\mu^2+\sigma^2}.$$

From (2.5) one has $\overline{M}(E) = M(E) - \mu$, which by insertion yields the desired inequality.

*Step 2:* The upper bound is attained for a diatomic financial loss.

Consider a diatomic financial loss $\{x, x^*\}$ of the type (3.1). The financial loss structure implies that one has to maximize (with respect to $p$) the function

$$f(p) = M(E) = (1-p)x^* = \mu(1-p) + \sigma\sqrt{p(1-p)}\,.$$

A calculation shows that $f(p_{\max}) = \frac{1}{2}\{\mu + \sqrt{\mu^2+\sigma^2}\}$ with $p_{\max} = \frac{1}{2}\{1 - \frac{\mu}{\sqrt{\mu^2+\sigma^2}}\}$. This completes the proof.

**Remark 4.1**

In the special case $X = Z - d$, $Z$ with mean $\mu$ and variance $\sigma^2$, $E = \{X > 0\}$, the $E$-loss $X(E) = (Z-d)_+$ is the stop-loss transform. One recovers the original inequality of Bowers.

## 5 Outlook on some applications

Our aim is to suggest the potential usefulness of the present elementary approach to various problems dealing with financial losses and gains. As a general preliminary remark, let us state the first-order gain identity $G + X_+ = G_+$ for use in ALM=Asset and Liability Management. Let $A = \{A(t)\}$, $t \geq 0$, $L = \{L(t)\}$, $t \geq 0$, be stochastic processes representing accumulated values of assets and liabilities at a future time $t \geq 0$. Then the stochastic process $G = \{G(t)\}$, $t \geq 0$, defined by $G(t) = A(t) - L(t)$, represent the financial gain, while $X = \{X(t)\}$, $t \geq 0$, defined by $X(t) = L(t) - A(t)$, is the financial loss. As a convention indices will be omitted in case an affirmation can be made whatever the time parameter is. The gain identity for ALM can be rewritten as

$$A + (L-A)_+ = L + (A-L)_+\,, \tag{5.1}$$

and states that the asset value $A$ plus the option to exchange $A$ for $L$ meets the liability $L$ plus the option to exchange $L$ for $A$, which is the basic idea underlying Portfolio Insurance, created by Leland in the night of September 11, 1976 (cf. Luskin(1988)). In a practical meanvariance framework, the

extremal financial risks of the ALM equilibrium (5.1) depend, by knowledge of the derivative (future) prices $M^+ = E[(L-A)_+]$, $M^- = E[(A-L)_+]$ upon the extremal bounds for $V^+$, $V^-$ described by our main results in Section 3. Note that the exchange option has been first priced by Margrabe(1978). In the special cases when $A$ or $L$ are deterministic, one recovers call- and put-options first priced by Black and Scholes(1973).
Some actuarial hedging models may also be viewed as *transformed models* of the ALM equilibrium relationship (5.1). Suppose a risk manager retains an amount $R$ on individual contracts with positive gain $G_+$, called *loss reserve,* to cover potential losses on individual contracts with positive losses $X_+$. Then the financial gain $G$ and the positive gain $G_+$ are reduced by the amount $R$. The obtained *shifted* gain identity may be rewritten as

$$A + \{(L-A)_+ - R\} = L + \{(A-L)_+ - R\}. \tag{5.2}$$

The component $A^h = R - (L-A)_+$ is interpreted as actuarial hedging model, and $D = (A-L)_+ - R \geq 0$ plays the role of a dividend to be paid out by good experience (cf. Hürlimann(1991a/9lb)). One observes that loss reserves and dividends are *invariant* with respect to ALM operations made on the equilibrium relationship (5.1). For the "optimal" choice of a *stable loss reserve* $R = \min\{B, G_+\}$ such that $\mathrm{Var}[R] = \min.$ under the restriction $E[R] = M^+$, $B$ some deterministic process, one has $R - X_+ = B - (B-G)_+$. Therefore this *stable hedging model* may also be obtained from the gain identity by shifting the financial gain from $G$ to $G-B$. In this situation the risk-adjusted gain identity $G - B + (B-G)_+ = (G-B)_+$ may be rewritten as

$$A + \{(B-G)_+ - B\} = L + (G-B)_+, \tag{5.3}$$

where $A^h = B - (B-G)_+$ is the hedging component and $D = (G-B)_+$ the dividend. These actuarial hedging models have been studied further by the author(l993b/95a/95b).

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## Summary

A structure of financial loss on a probability space is considered. The financial loss is represented as difference between loss and gain. In this general mathematical framework, the inequality of Bowers(1969) on the mean loss, as well as inequalities by Kremer(1990), Hürlimann(1993a/94a) and Birkel(1994) on the loss variance, are all simple consequences of the non-negative property of certain variance functions. Similar inequalities for the gain are obtained via a conjugate operation. Sharpness and extremal properties of the variance bounds are discussed. By given mean, variance of the financial loss, and fixed mean loss, it is shown that the loss and gain variance extremal bounds are attained for diatomic financial losses. In the special case of the stop-loss variance upper bound, this result is originally due to Schmitter(1993/95). The potential usefulness of our approach for Asset and Liability Management, including in particular some actuarial hedging models, is briefly mentioned.

## Zusammenfassung

Der finanzielle Verlust wird als Struktur auf einen Wahrscheinlichkeitsraum modelliert. Er wird als Differenz zwischen Verlust und Gewinn dargestellt. In dieser allgemeinen mathematischen Umgebung sind die Ungleichung von Bowers(1969) über den erwarteten Verlust, sowie Ungleichungen von Kremer(1990), Hürlimann(1993a/94a) und Birkel(1994) über die Varianz des Verlusts, einfache Folgerungen der nicht-negativen Eigenschaft von gewissen Varianzfunktionen. Ähnliche Ungleichungen für den Gewinn werden mit Hilfe einer Konjugation erhalten. Die Scharfheit und extremale Eigenschaften der Schranken für die Varianz werden diskutiert. Bei gegebenem Erwartungswert, Varianz des finanziellen Verlusts, und festem erwarteten Verlust, werden die extremalen Schranken für den Verlust und Gewinn durch einen zweipünktigen finanziellen Verlust erreicht. Im Spezialfall der oberen Schranke für die Stop-Loss Varianz wurde das Resultat von Schmitter(1993/95) hergeleitet. Die potentielle Nützlichkeit unserer Methode für die Verwaltung von Aktiven und Passiven, insbesondere für einige aktuarielle Absicherungsmodelle, wird kurz erwähnt.

## Résumé

On considère une structure de perte financière sur un espace de probabilité. La perte financière est représentée comme différence entre perte et gain. Dans cet environnement mathématique général, l'inégalité de Bowers(1969) sur la perte moyenne, ainsi que des inégalités de Kremer(1990), Hürlimann(1993a/94a) et Birkel(1994) sur la variance de la perte, sont toutes des conséquences simples de la proprieté de non-négativité de certaines fonctions variance. Des inégalités semblables pour le gain sont obtenues par conjugaison. Les cas d'égalités et les propriétés extrémales de ces bornes pour la variance sont discutées. Etant donné la moyenne et variance de la perte financière, ainsi que la perte moyenne, on montre que les bornes extrémales pour la variance de la perte et du gain sont atteintes par des pertes financières biatomiques. Dans le cas particulier de la borne supérieure pour la variance stop-loss, ce résultat est dû à Schmitter(1993/95). L'utilité potentielle de notre approche pour la gestion des actifs et passifs, en particulier pour quelques modèles actuariels de couverture du risque, est brièvement mentionnée.