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Autor(en): **Schmidli, Hanspeter**

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Hanspeter Schmidli,* Århus

Risk Theory in an Economic Environment and Markov Processes

1 Introduction

Already at the beginning of the century Filip Lundberg considered in his Ph.D. thesis the model which forms the basis to the models considered in non-life insurance risk theory. He assumed that the income process is linear with a premium rate c and the outflow forms a compound Poisson Process; i.e. the claim arrival times form a Poisson process and the claim sizes are iid. and independent of the arrival process. We will call this model Cramér-Lundberg model. During the years a lot of generalizations have been considered. For instance Sparre Andersen [1] considered the case where the claim arrival process is a renewal process, Björk and Grandell [4] the case of a doubly stochastic claim arrival process and Asmussen [2] let the claim arrival process and the claim sizes be dependent via an external continuous time Markov chain.
In the present work we are interested in so-called *risk processes in an economic environment* where interest and inflation are present. Such a model was considered by Delbaen and Haezendonck [8]. As in the Cramér-Lundberg model (technical) ruin occurs when the process first enters the negative half-plain. It is clear that the level 0 plays here an unrealistic role. Gerber [13] considered a risk model, where the company is allowed to borrow money and has to pay interest for it with the same rate of interest as the company receives for positive surplus. He introduced the notion of *absolute ruin* which occurs at the first epoch where the premium income becomes smaller than the outflow for paying interest. Dassios and Embrechts [5] modified the classical Cramér-Lundberg model by allowing borrowing but not investment. They computed the (absolute) ruin probability for the case of exponentially distributed claims. Recently Embrechts and Schmidli [11] extended the model with the possibility of investment above some 'liquid reserve' level Δ. Figure 1 shows a path of this process. The Laplace transform of the ruin probabilities was computed, but explicit results were only obtainable for special cases. The results can also be found in [19]. In the present

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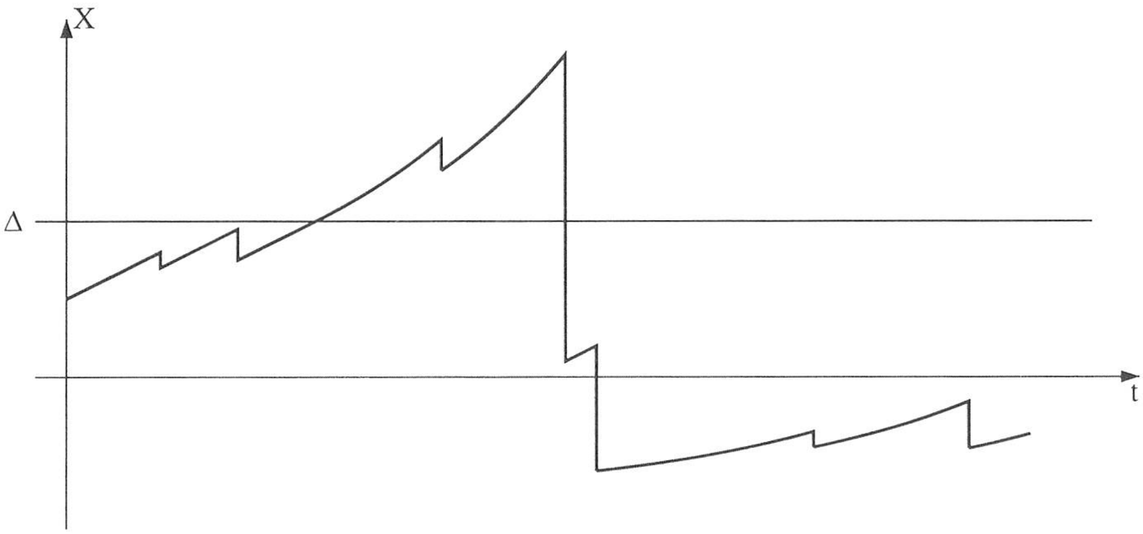

Figure 1: Sample path of a general risk process

paper we will recall the method and the results of [11] and we will give an explicit inversion formula for the special case of equal interest rates for invested and borrowed money. This will lead to a numeric procedure for computing the ruin probabilities.
In practice there are solvency restrictions by law. It seems therefore not realistic to allow an insurance company to borrow money. The process described above admits another interpretation. Note that we only model one kind of insurance contracts within a company. In the middle part between 0 and Δ only investment possibilities exist to cancel the effect of the inflation. In the part above Δ investments with a higher net yield are possible. In the negative part one needs capital which cannot furthermore be invested by the company and therefore yields a loss for the company.

2 Piecewise deterministic Markov processes and martingales

The process introduced above has some similarities with the Cramér-Lundberg process. Between the jump times the paths of the process are deterministic (but no longer linear) and it is also a Markov process. This sort of processes, the so-called piecewise deterministic Markov processes (PDMP) were introduced by Davis [6]. It turned out, that the theory of PDMP's yields an ideal tool in risk theory, see for instance [5], [10] or [19]. We will give here an idea, how PDMP's are constructed. For an exact definition of the process see [6], [7] or [19].

Let E be an open manifold, typically

$$E = \bigcup_{\iota \in I} \{(\iota, x) : x \in M_\iota\}$$

where each M_ι is an open subset of $\mathbb{R}^{d_\iota}$ ($d_\iota \in \mathbb{N}$) and I is a countable set endowed with the discrete topology. We denote by $\mathcal{E}$ the Borel sets of E. The deterministic paths of the process (X_t) between the jumps are determined by the integral curves of a vectorfield χ on E. We assume that through every point of the manifold there exists exactly one integral curve, and that the integral curve can only leave the manifold through the boundary. Note that, as in differential geometry, χ is a first order differential operator, such that $\frac{\partial}{\partial t} f(X_t) = (\chi f)(X_t)$ along an integral curve for all f's absolutely continuous along integral curves. On the path there is given a jump intensity $\lambda : E \to \mathbb{R}^+$, which determines the jump times. If the process reaches the boundary ∂E of E the process jumps back into the interior of the manifold. Therefore, denoting the jump times by (T_k) and setting $T_0 = 0$

$$P[T_{k+1} \leq T_k + h | \mathfrak{F}_{T_k}] \\ = \begin{cases} 1 - \exp\left\{ - \int\limits_0^h \lambda(X_{T_k + s})\, ds \right\} & \text{if } X_{T_k+s} \in E \, \forall s \in [0, h] \\ 1 & \text{otherwise} \end{cases}$$

where $(\mathfrak{F}_t)$ denotes the natural filtration of the process (X_t). The position after the jump will be given by a jump measure $Q : \mathcal{E} \times (E \cup \partial E) \to [0, 1]$, i.e. $Q(A, x)$ denotes the probability that a jump from x leads to the Borel set A. We denote by

$$N_t := \inf\{k \in \mathbb{N} \cup \{0\} : T_k \leq t\}$$

the number of jumps in the interval $[0, t]$. Furthermore we assume that $E[N_t] < \infty$ for all $t \geq 0$.

Example 1. In the classical Cramér-Lundberg model (C_t) the state space of the process is $E = \mathbb{R}$. The deterministic paths are linear with slope c. $\frac{\partial}{\partial t} f(x + ct) = c\frac{\partial}{\partial x} f(x + ct)$, i.e. the vectorfield has the form $\chi = c\frac{\partial}{\partial x}$. Because the claim times form a Poisson process, the jump intensity $\lambda(x) = \lambda$ is constant. Denoting the claim size distribution by G the jump measure becomes $Q(dy, x) = dG(x - y)$. In order to get time dependent martingales one can also consider the PDMP (C_t, t), where χ becomes $\chi = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}$. ■

The main reason why the theory of PDMP turns out to be an ideal tool in risk theory is the easy way one can construct martingales. To do that we first need the following definition.

Definition 1. *If for some measurable real functions* $f, \widetilde{f} : E \to \mathbb{R}$

$$f(X_t) - f(X_0) - \int_0^t \widetilde{f}(X_s)\, ds \tag{1}$$

is an $\mathfrak{F}_t$*-martingale then wet set* $\mathfrak{A}f := \widetilde{f}$. *The operator* $\mathfrak{A}$ *is called the (full) generator of* (X_t) *and* f *is said to be in the domain of the generator* $\mathcal{D}(\mathfrak{A})$.

Remarks.

i) Note that $\mathfrak{A}$ is in fact a multi-valued operator. We identify therefore all versions of $\widetilde{f}$ in the image space of $\mathfrak{A}$.

ii) For a deeper understanding of the above definition let us consider the classical definition of the infinitesimal generator. The idea is to subtract from a process its drift in order to get a martingale. If for some measurable bounded functions $f, \widetilde{f} : E \to \mathbb{R}$

$$\lim_{t \downarrow 0} \frac{1}{t} E[f(X_t) - f(x) \mid X_0 = x] = \widetilde{f}(x) \tag{2}$$

in the sense of uniform convergence then the so-called Dynkin formula (see [22, p. 129]) assures that (1) is a martingale. Therefore the domain of the infinitesimal generator is contained in the domain of the full generator and, the full generator restricted to the domain of the infinitesimal generator and the infinitesimal generator coincide.
It turns out that for many applications the restriction to bounded functions is not satisfactory. There exist unbounded functions f, such that a function $\widetilde{f}$ satisfying (2) leads to a martingale of type (1). Some examples can be found in [19] or [10]. Using the given definition of the full generator also unbounded functions can be in the domain of the generator. It remains to find conditions for a function f to be in the domain of the full generator and formulae for computing $\mathfrak{A}f$.

iii) A third possible definition of a generator is the extended generator, where '$\mathfrak{F}_t$-martingale' in the above definition is substituted by 'local $\mathfrak{F}_t$-martingale'. The advantage of dealing with the extended generator is that necessary and sufficient conditions can be given for a function to be in the domain of the extended generator. Davis [6] and [7] discusses the extended generator for piecewise deterministic Markov processes. The approach we use will straightforwardly yield martingales. Therefore the definition of the full generator suffices for our purposes. The conditions in the following proposition turn out to be easier to verify than those in [6]. ■

For our purpose it is enough to know a large subset of functions in the domain of the generator. This class is given by the following proposition. Its proof can be found in [19]. Denote by X_{t-} the left-hand limit of the process (X_t) at time t.

Proposition 1. *Let (X_t) be a* PDMP *and $f : (E \cup \partial E) \to \mathbb{R}$ be a real measurable function satisfying*

i) *f is absolutely continuous along integral curves,*

ii) $f(x) = \int_E f(y)Q(dy,x) \quad \forall x \in \partial E,$ *(boundary condition)*

iii) $E[\sum_{T_i \le t} |f(X_{T_i}) - f(X_{T_i -})|] < \infty.$

Then $f \in \mathcal{D}(\mathfrak{A})$ and the generator is given by

$$\mathfrak{A}f(x) = \chi(x) + \lambda(x)\left[\int_E (f(y) - f(x))Q(dy,x)\right]. \tag{3}$$

□

The idea is now to solve the equation $\mathfrak{A}f(x) = 0$ with $\mathfrak{A}f$ given by (3) such that conditions i) – iii) are fulfilled. Then the process $(f(X_t) : t \ge 0)$ will be a martingale.

Remark. The equation $\mathfrak{A}f(x) = 0$ for the Cramér-Lundberg model can also be found in [15, p. 317]. Application of our techniques often leads to increasing functions f. In [15] f is interpreted as an utility function in the sense that $f(X_t)$ becomes a 'fair game', i.e. a martingale. ■

Example 1 (continued)**.** For the Cramér-Lundberg model we have to solve

$$cf'(x) + \lambda\left[\int_0^\infty (f(x-y) - f(x))\, dG(y)\right] = 0\,,$$

compare also [15, p. 317]. We try a function of the form $f(x) = e^{-Rx}$ with $R > 0$ which leads to the condition

$$-cR + \lambda(\widehat{G}(-R) - 1) = 0$$

where $\widehat{G}(s) := \int_0^\infty e^{-sx}\, dG(x)$ denotes the Laplace-Stieltjes transform (LS-transform) of G. The constant R above has to be therefore the so-called Lundberg exponent (if a solution exists). Condition i) is easily seen to be fulfilled, condition ii) becomes trivial since $\partial E = \emptyset$ and a straightforward calculation shows that also iii) is fulfilled. The martingale $(\exp\{-RX_t\})$ was first computed by Gerber [14]. ■

3 The risk process with the possibility of investment and borrowing

We now turn back to the process introduced in Section 1. We assume as before that the claim arrival process is a Poisson process with rate λ and that the claim sizes are iid. with distribution function G and independent of the claim arrival process. Therefore the parameters for the corresponding PDMP become as in the Cramér-Lundberg model $\lambda(x) = \lambda$ and $Q(dy, x) = dG(x - y)$. We denote by $\widehat{G}(s) := \int_0^\infty e^{-sx}\, dG(x)$ the LS-transform of G. It remains to determine the corresponding vectorfield. Consider a starting point $x_0 \in (-\infty, 0)$. Denote the force of interest for borrowed money by β_2, that means that after some time t, x_0 has changed to $x_0 e^{\beta_2 t}$. The premium income and its interest become $\int_0^t c e^{\beta_2(t-s)}\, ds = c/\beta_2(e^{\beta_2 t} - 1)$. If no jump occurs in $[0, t]$ then the process starting in x_0 can be written as

$$X_s = \left(x_0 + \frac{c}{\beta_2}\right) e^{\beta_2 s} - \frac{c}{\beta_2}$$

which leads to

$$(\chi f)(X_s) = (c + \beta_2 x_0) e^{\beta_2 s} f'(X_s) = (c + \beta_2 X_s) f'(X_s)$$

or $\chi = (c + \beta_2 x)\frac{\partial}{\partial x}$ if $x < 0$. Denoting by β_1 the force of interest for invested money and by Δ the liquid reserve level we get for the vectorfield of the considered process

$$\chi = \begin{cases} (\beta_1(x - \Delta) + c)\frac{\partial}{\partial x} & \Delta \le x\,, \\ c\frac{\partial}{\partial x} & 0 \le x < \Delta\,, \\ (\beta_2 x + c)\frac{\partial}{\partial x} & x < 0\,. \end{cases} \tag{4}$$

It follows immediately that if $X_t < -c/\beta_2$ then the deterministic paths are strictly decreasing and because only downward jumps are possible, the process

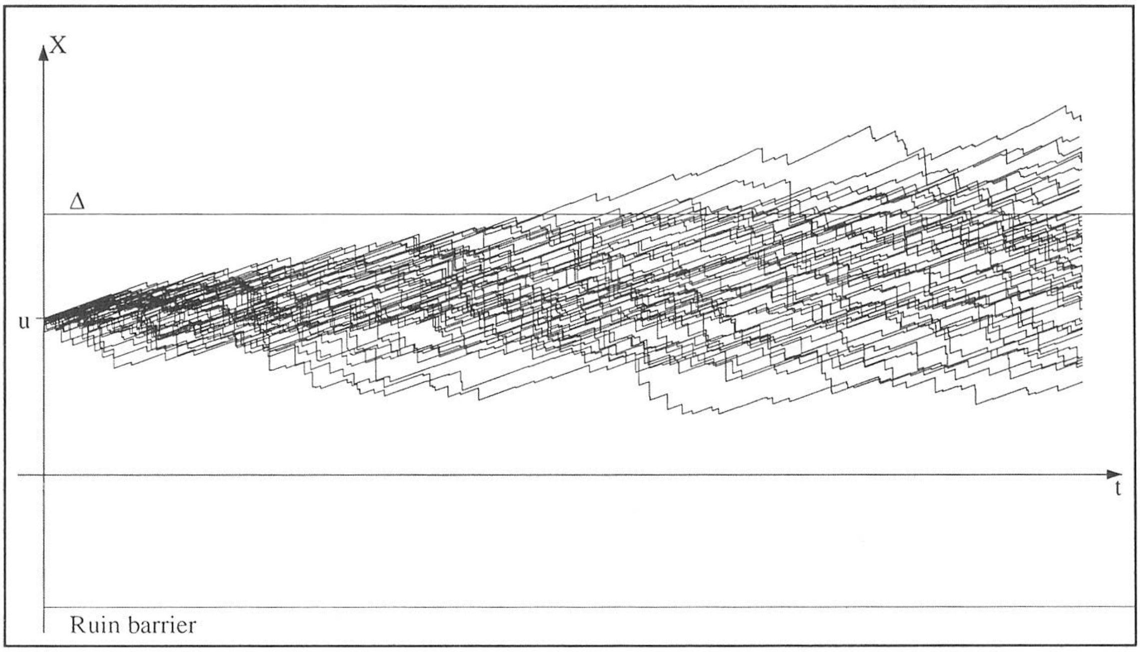

Figure 2: Simulated example of a risk process taking borrowing ($\beta_2 = 0.095$) and investment ($\beta_1 = 0.058$, $\Delta = 250$) into account. Exponential claims.

(X_t) converges to $-\infty$ a.s.. Hence we call the time point

$$\tau := \inf\{t \geq 0 : X_t < -c/\beta_2\}$$

ruin time. Note that τ is the first time where the premium income is strictly smaller than the outflow for paying interest. So the considered ruin time is identical with the absolute ruin time in [13] and [5].

As an illustration we have simulated 50 paths of the above defined process over 30 units of time. The parameters were $u = X_0 = 150$, $\Delta = 250$, $\lambda = 2$ and $c = 12$. As force of interest we have used $\beta_1 = 0.058$ and $\beta_2 = 0.095$ which corresponds to a rate of interest of 6.0 % (10.0 % respectively). Figure 2 shows paths with exponential claims with mean 5. Here the classical net profit condition $c > \lambda\mu$ is fulfilled and most paths are lying between 0 and Δ. A more illustrative picture is derived by using heavy-tailed claim sizes. Figure 3 shows the case of lognormally distributed claims with mean 7.39 and variance 2981, that means the classical net profit condition is not fulfilled. Above the liquid reserve level Δ the paths are exponentially increasing. Between 0 and Δ their behaviour is as in the classical case and below 0 the slopes of the paths are smaller. Under a certain level there is practically no chance to survive.

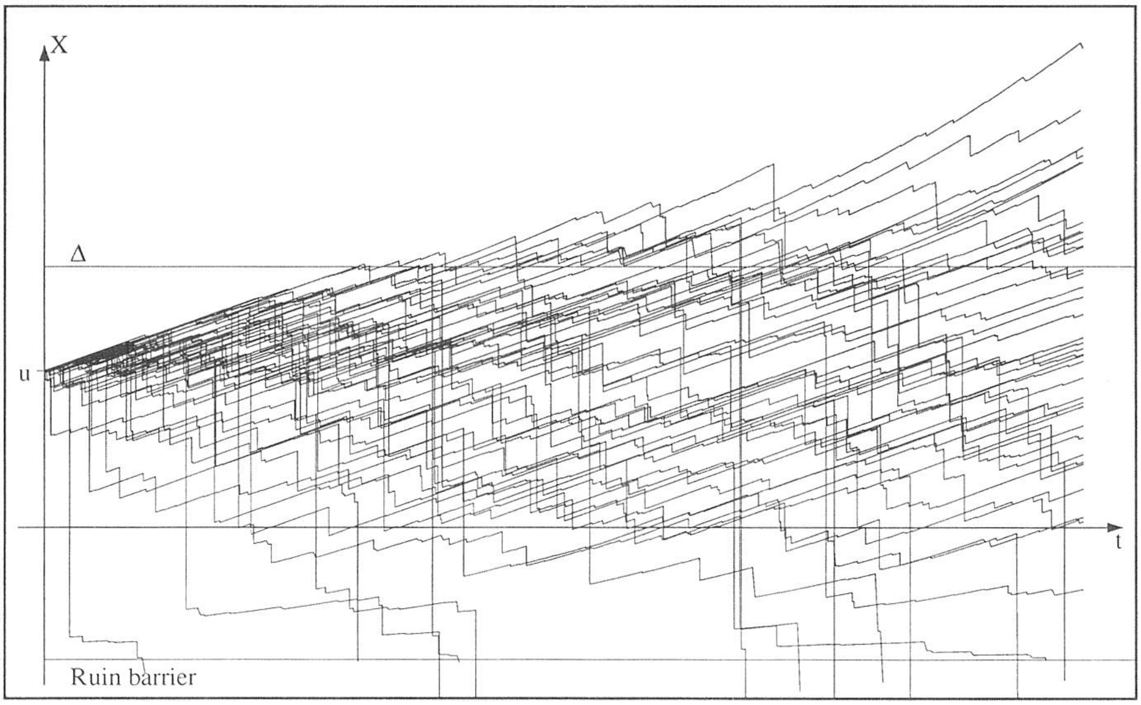

Figure 3: Idem as in Fig. 2. Lognormal claims.

We are interested in the probability of ultimate ruin $P[\tau < \infty]$. So we do not have to care about what is happening with the process in $(-\infty, -c/\beta_2)$ and hence we are only looking for functions f with $f(x) = 0$ for $x < -c/\beta_2$. Because the behaviour of the process is different in the three regions $[-c/\beta_2, 0)$, $[0, \Delta)$ and $[\Delta, \infty)$ we split the function f into three parts

$$f(x) = f_1(x)\mathbb{1}_{\{x \geq \Delta\}} + f_2(x)\mathbb{1}_{\{0 \leq x < \Delta\}} + f_3(x)\mathbb{1}_{\{-c/\beta_2 \leq x < 0\}} \cdot$$

In order to get a solution of $\mathfrak{A}f = 0$ we have by (3) to solve the system

$$\begin{aligned}(c + \beta_1(x - \Delta))f_1'(x) + \lambda\Big[&\int_0^{x-\Delta} f_1(x-y)\,dG(y) \\ + \int_{x-\Delta}^{x} f_2(x-y)\,dG(y) + &\int_x^{x+c/\beta_2} f_3(x-y)\,dG(y) - f_1(x)\Big] = 0 \qquad (5)\end{aligned}$$

$$cf_2'(x) + \lambda\Bigg[\int_0^x f_2(x-y)\,dG(y) + \int_x^{x+c/\beta_2} f_3(x-y)\,dG(y) - f_2(x)\Bigg] = 0 \qquad (6)$$

$$(c+\beta_2 x)f_3'(x) + \lambda\Bigg[\int_0^{x+c/\beta_2} f_3(x-y)\,dG(y) - f_3(x)\Bigg] = 0 \qquad (7)$$

Considering condition i) of Proposition 1 we assume that $f_1(\Delta) = f_2(\Delta)$ and $f_2(0) = f_3(0)$. Regarding (7), it follows that the integro-differential equation restricted to $\{x < 0\}$ can be solved independently from its solution on $\{x \geq 0\}$. This restricted solution is independent of the parameters β_1 and Δ. The same observation applies to the restriction on $\{x < \Delta\}$, hence we proceed in three steps:

i) The special case of no liquid reserve ($\Delta = 0$) and equal interest rates ($\beta_1 = \beta_2$), which was first considered by Gerber [13].

ii) The Dassios-Embrechts (absolute ruin) model, where no investment is allowed ($\Delta = \infty$).

iii) The general model, as defined above.

We will only show how to find the function f_3. f_2 and f_1 can be found in a similar way. For the details see [11] or [19]. Let $\widehat{f_3}(s) := \int_{-c/\beta_2}^{\infty} f_3(x)e^{-sx}\,dx$ denote the Laplace transform of f_3 and assume that it exists. Multiplying equation (7) with e^{-sx} and integrating it from $-c/\beta_2$ to ∞ yields

$$-\beta_2\Big(\widehat{f_3}(s) + s\widehat{f_3}'(s)\Big) + cs\widehat{f_3}(s) - \lambda\widehat{f_3}(s)(1-\widehat{G}(s)) = 0\,.$$

This linear differential equation has the solution

$$\widehat{f_3}(s) = K\frac{1}{s}\exp\left\{\frac{c}{\beta_2}s\right\}\exp\left\{-\frac{\lambda}{\beta_2}\int_0^s \frac{1-\widehat{G}(\xi)}{\xi}\,d\xi\right\} \qquad (8)$$

where K is a (strictly positive) constant. The factor $\exp\{c/\beta_2 s\}$ is due to integrating from $-c/\beta_2$ to ∞ instead from 0 to ∞. The factor s^{-1} shows that the solution is the integral of the inverse transform of the last factor. Using the theory of completely monotone functions one can show that in fact $\widehat{f_3}$ is the Laplace transform of a positive increasing and bounded function which satisfies the conditions of Proposition 1. Furthermore $f_3(-c/\beta_2) = 0$. The functions $f_i (i \in \{1, 2, 3\})$ have the following Laplace transforms.

Lemma 1. *Denote by* $s_0 := \sup\{s \geq 0 : cs - \lambda(1 - \widehat{G}(s)) \leq 0\}$ *and let* $\widehat{f_1}(s) = \int_\Delta^\infty f_1(x)e^{-sx}\,dx$, $\widehat{f_2}(s) = \int_0^\infty f_2(x)e^{-sx}\,dx$, $\widehat{f_3}(s) = \int_{-c/\beta_2}^\infty f_3(x)e^{-sx}\,dx$ *denote the Laplace transforms of the function* f_1, f_2, f_3. *Let furthermore* $\widehat{g}(s) := \exp\{\beta_1^{-1}(cs - \lambda\int_0^s \xi^{-1}(1 - \widehat{G}(\xi))\,d\xi)\}$. *Then*

i) $$\widehat{f_3}(s) = K\frac{1}{s}\exp\left\{\frac{c}{\beta_2}s\right\}\exp\left\{-\frac{\lambda}{\beta_2}\int_0^s \frac{1 - \widehat{G}(\xi)}{\xi}\,d\xi\right\}.$$

ii) $$\widehat{f_2}(s) = \frac{cf_3(0) - h_2(s)}{cs - \lambda(1 - \widehat{G}(s))} \quad \text{for } s > s_0,$$

where $h_2(s) := \lambda\int_0^\infty \int_x^{x+c/\beta_2} f_3(x - y)\,dG(y)e^{-sx}\,dx$.

iii) $$\widehat{f_1}(s) = \frac{\widehat{g}(s)e^{-\Delta s}}{\beta_1 s}\int_s^\infty \frac{cf_2(\Delta) - h_1(\eta)e^{\Delta\eta}}{\widehat{g}(\eta)}\,d\eta,$$

where

$$h_1(s) := \lambda\int_\Delta^\infty\left[\int_{x-\Delta}^x f_2(x - y)\,dG(y) + \int_x^{x+c/\beta_2} f_3(x - y)\,dG(y)\right]e^{-sx}\,dx.$$

□

Remark. In the Dassios-Embrechts model the net profit condition $c > \lambda\mu$ of the classical Cramér-Lundberg model is equivalent to $P[\tau < \infty] < 1$. It will turn out that this is equivalent to $f_2(\infty) < \infty$. Note that $s_0 > 0$ corresponds to $c < \lambda\mu$ which implies (see the classical model) $P[\tau < \infty] = 1$. In this case

$$f_2(x) \approx \frac{cf_3(0) - h_2(s_0)}{c + \lambda\widehat{G}'(s_0)}e^{s_0 x}.$$

The functions f_3 and f_1 will be bounded in any case and therefore $P[\tau < \infty] < 1$ for any $c > 0$ if $\Delta < \infty$ provided $X_0 > -c/\beta_2$. ■

The computed martingale is positive. We know from martingale convergence theorem [9, Thm. 6] that the martingale must converge for $t \to \infty$. But there are only the two possible convergence points $\{0, f(\infty)\}$. Hence $\lim_{t\to\infty} X_t \in \{-\infty, \infty\}$. Furthermore if in the Dassios-Embrechts model $c \leq \lambda\mu$ (which is equivalent to $f(\infty) = \infty$) then $\tau < \infty$ a.s.. If $f(\infty) < \infty$ then

$$\begin{aligned} f(\infty)P[\tau = \infty] &= E[f(X_\infty)] = \lim_{t\to\infty} E[f(X_t)] \\ &= f(X_0) \end{aligned}$$

and we get the following Proposition.

Proposition 2. *For any $\Delta \in [0, \infty]$,*

$$P[\tau < \infty \mid X_0 = u] = 1 - \frac{f(u)}{f(\infty)} \tag{9}$$

where f is determined by Lemma 1. □

Remarks.

i) In the notation of [15] $v(u) = 1 - f(u)/f(\infty)$ and because $v(X_\tau) = 1$ on $\{\tau < \infty\}$ Proposition 2 is identical with Theorem 1 of [15].

ii) In the above model with $\Delta < \infty$ one does not need a net profit condition as in the classical case to assure $P[\tau < \infty] \neq 1$. The reason for this is that, whenever the surplus is large enough, then the income rate becomes $c + \beta_1 X_t > \lambda\mu$ and there is a positive probability for reaching this level.

iii) If one sets $h_2 := 0$ (or equivalently, $f_3(x) = 0$ for $x < 0$) in Lemma 1 one recovers the martingale for the classical risk model which also can be found in [5].

iv) The only case, where the function f and (9) can be calculated explicitly, seems to be when the claim-sizes are distributed corresponding to an Erlang-distribution, e.g. a $\Gamma(\alpha, n)$-distribution for $\alpha > 0$ and $n \in \mathbb{N}$. For more general distribution functions one has to invert the LS-transforms numerically. Alternative numerical techniques based upon the theory of matrix-exponential distributions, are to be found in [3]. ■

Example 2. Consider the case of exponentially distributed claims $\widehat{G}(s) = (1+\mu s)^{-1}$. Then the Laplace transforms of Lemma 1 can be inverted explicitly

$$\begin{aligned}
f_3(x) &= \widetilde{K} \int\limits_0^{x+c/\beta_2} s^{(\lambda/\beta_2)-1} e^{-s/\mu}\, ds\,, \\
f_2(x) &= f_3(0) + \frac{f_3'(0)}{1/\mu - \lambda/c}(1 - e^{-(1/\mu-\lambda/c)x})\,,
\end{aligned} \tag{10}$$

and

$$f_1(x) = f_2(\Delta) + \left(\frac{\beta_1}{c}\right)^{(\lambda/\beta_1)-1} e^{c/(\beta_1\mu)} f_2'(\Delta) \int\limits_{c/\beta_1}^{x+c/\beta_1-\Delta} s^{(\lambda/\beta_1)-1} e^{-s/\mu}\, ds\,.$$

The functions f_1 and f_3 are of Gamma-type and the function f_2 is similar to the solution in the Cramér-Lundberg model (compare with Example 1 and note that $R = 1/\mu - \lambda/c$).

Figure 4 shows as an example the ruin probabilities, calculated via above martingale, depending on the liquid reserve for exponential claims with mean $\mu = 1$, initial capital $u = 50$, hazard rate $\lambda = 8$, premium rate $c = 8.2$, force of interest $\beta_1 = 0.058$ (6.0 %) for invested money and $\beta_2 = 0.095$ (10.0 %) for borrowed money. The ruin-probability tends asymptotically to the (absolute) ruin probability in the Dassios-Embrechts model. The straight line is the ruin probability in the classical Cramér-Lundberg model. ■

Remarks.

i) The solution for f_3 in (10) was first computed by Gerber [13, p. 65].

ii) Considering the surplus process $\widetilde{X}_t = X_t + c/\beta_2$ the model can be interpreted as a Cramér-Lundberg model with a state dependent premium rate

$$c(\widetilde{x}) = \begin{cases} \beta_2\widetilde{x} & 0 \le \widetilde{x} < c/\beta_2\,, \\ c & c/\beta_2 \le \widetilde{x} < \Delta + c/\beta_2\,, \\ c + \beta_1(\widetilde{x} - \Delta - c/\beta_2) & \Delta + c/\beta_2 \le \widetilde{x}\,. \end{cases}$$

For exponentially distributed claims such a model was considered by Gerber [15]. A general formula ([15, p. 320]) was obtained for the probability of ruin. Unfortunately this formula cannot be directly applied to our model because $\int_0^y c(s)^{-1}(c(s)-1)\, ds = \infty$. But the method of [15, p. 319] also works for the above model. ■

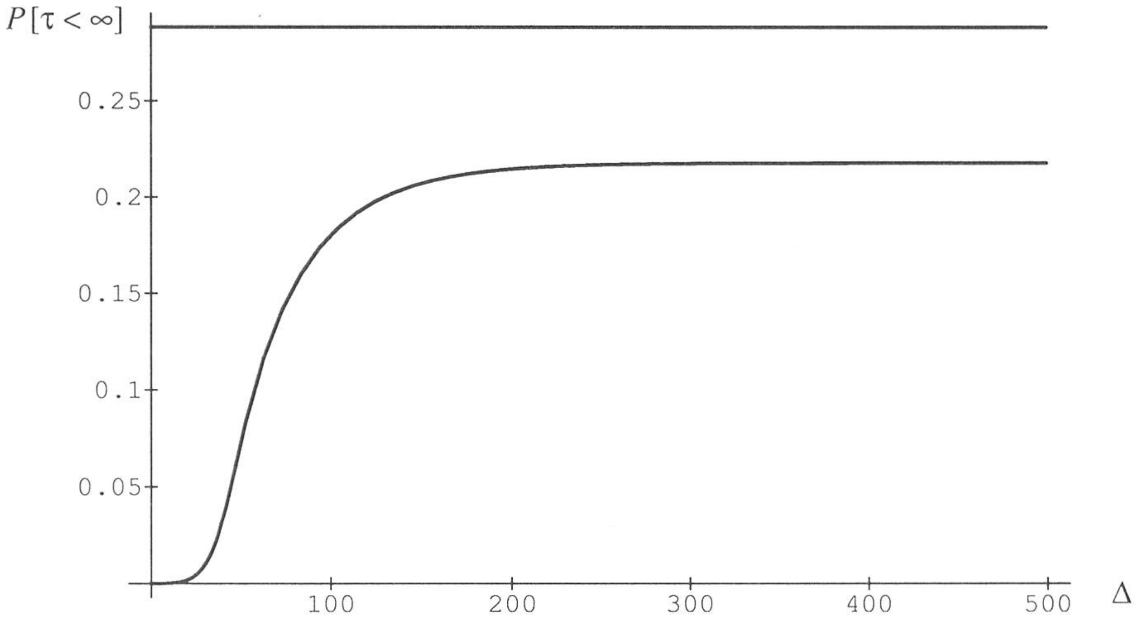

Figure 4: Ruin probabilities for an example of a risk process with exponential claims taking borrowing ($\beta_2 = 0.095$) and investment ($\beta_1 = 0.058$) into account. Initial capital 50.

In the Cramér-Lundberg model one can show that (in the small claim case) the ruin probability decreases exponentially fast. (See for instance [12, p. 378] or [16, p. 7]). The exponent is called Lundberg exponent. A similar theorem is also valid in our model. The exponent can be found from the Laplace transform of the ruin probability. For a proof see [11] or [19].

Theorem 1. *Define R by*

$$R := \sup\{r \in \mathbb{R} : \widehat{G}(-r) < \infty\}.$$

For $\Delta < \infty$, R is the Lundberg exponent for (X_t) in the sense that, for all $\varepsilon > 0$,

$$\lim_{u \to \infty} P[\tau < \infty \mid X_0 = u] e^{(R-\varepsilon)u} = 0,$$

and

$$\lim_{u \to \infty} P[\tau < \infty \mid X_0 = u] e^{(R+\varepsilon)u} = \infty. \qquad \square$$

The question arises, what happens in the case $\varepsilon = 0$.

Example 2 (continued). Using (10) in the case of exponentially distributed claims one obtains for the limit

$$\lim_{u\to\infty} P[\tau < \infty \mid X_0 = u]e^{ru} = \begin{cases} 0 & r < \frac{1}{\mu} \text{ or } \left(r = \frac{1}{\mu} \text{ and } \lambda < \beta_1\right), \\ C & r = \frac{1}{\mu} \text{ and } \lambda = \beta_1, \\ \infty & \text{otherwise,} \end{cases} \tag{11}$$

for $C = e^{\Delta/\mu}(\beta_1/c)^{(\lambda/\beta_1)-1}\mu f_2'(\Delta)/f_1(\infty)$. This example shows that in fact we cannot say anything about the case $\varepsilon = 0$. ■

Remarks.

i) Lundberg exponents in the sense above also appear in context of other generalizations of the classical risk model (see for instance [16]).

ii) It is not surprising that the right end point of the interval where the moment generating function exists takes the role of the Lundberg exponent. This could be also obtained by considering the Lundberg exponent of the classical model and then to let the premium rate c tend to infinity.

iii) The dependence of λ/β_1 of the limit in the example above is surprising. But it turns out, that this holds not in general. For Gamma distributed claims $\widehat{G}(s) = (R/(R+s))^\gamma$ one obtains (see [11])

$$\lim_{u\to\infty} P[\tau < \infty \mid X_0 = u]e^{ru} = \begin{cases} 0 & \text{if } \gamma < 1, \\ (11) & \text{if } \gamma = 1, \\ \infty & \text{if } \gamma > 1. \end{cases}$$

iv) The correct asymptotic behaviour of the ruin probability must entail another factor besides the exponential one. For the case of state dependent premium income the problem for the Lundberg inequality is partially solved by Søren Asmussen and Hanne Nielsen (private communication). Note that we can consider the risk process with the possibility of interest and borrowing as a risk process with state dependent premiums.

v) The question arises, how to estimate R. By restriction on a certain set, a solution was found in [17]. Recently it was shown [20] that the problem is equivalent to the problem of estimation of the coefficient of regular variation. But in any case the rate of convergence is slow. ■

4 An inversion formula for f_3

We consider in this section the case $\Delta = 0$ and $\beta_1 = \beta_2 = \beta$. This particular model was also considered by Gerber [13]. He obtained the relation between f_3 and S (see below) and computed the characteristic function of S. His results are crucial in this section, thus we derive these results again. Furthermore we compute an inversion formula for f_3.
Because all the time the same (constant) force of interest is valid, the process (X_t) can be expressed explicitly as

$$X_t = ue^{\beta t} + \frac{c}{\beta}(e^{\beta t} - 1) - \sum_{k=1}^{N_t} Y_k e^{\beta(t-T_k)}\,.$$

Therefore

$$\{\tau \le t\} = \left\{X_t < -\frac{c}{\beta}\right\} = \left\{\sum_{k=1}^{N_t} Y_k e^{-\beta T_k} > u + \frac{c}{\beta}\right\}$$

and

$$\{\tau = \infty\} = \left\{\sum_{k=1}^{\infty} Y_k e^{-\beta T_k} \le u + \frac{c}{\beta}\right\}. \tag{12}$$

Setting $S = \sum_{k=1}^{\infty} Y_k e^{-\beta T_k}$ we see that

$$f_3\left(x - \frac{c}{\beta}\right) = P[S \le x]\,.$$

It is well-known in shot-noise theory (see for instance [18, p. 46]) that the characteristic function of S is given by (see also Lemma 1)

$$\psi_S(\vartheta) = E[e^{i\vartheta S}] = \exp\left\{-\frac{\lambda}{\beta}\int_0^{\vartheta} \frac{1-\psi_G(\xi)}{\xi}\,d\xi\right\}$$

where ψ_G denotes the characteristic function of the claim sizes. Hence we get the following inversion formula for f_3.

Theorem 2. *Let f_3 be as above and define*

$$C(\vartheta) = \frac{\lambda}{\beta}\int_0^{\vartheta}\int_0^{\infty} \xi^{-1}(1-\cos\xi z)\,dG(z)d\xi$$

and

$$D(\vartheta) = \frac{\lambda}{\beta} \int_0^{\vartheta} \int_0^{\infty} \xi^{-1} (\sin \xi z) \, dG(z) \, d\xi \, .$$

Then

$$f_3\left(x - \frac{c}{\beta}\right) = \frac{1}{\pi} \int_0^{\infty} \left(\frac{1 - \cos \vartheta x}{\vartheta} \sin D(\vartheta) + \frac{\sin \vartheta x}{\vartheta} \cos D(\vartheta) \right)$$
$$\times \exp\{-C(\vartheta)\} \, d\vartheta \, . \tag{13}$$

Proof. We first want to show that $\int_\delta^\infty |\psi_S(\vartheta)/\vartheta| \, d\vartheta = \int_{-\infty}^{-\delta} |\psi_S(\vartheta)/\vartheta| \, d\vartheta < \infty$ for all $\delta > 0$. Because $|\psi_S(\vartheta)| \le 1$ it suffices to show the relation to be true for one value of δ. Denote by $\lceil x \rceil$ the largest integer smaller or equal to x. Choose $z_0 > 0$ such that $G(z_0) < 1$. Let $z \ge z_0$ and $\vartheta \ge 2\pi/z_0$. Then

$$\int\limits_0^{\vartheta} \frac{1 - \cos \xi z}{\xi} \, d\xi \ge \sum_{k=1}^{\lceil \vartheta z / 2\pi \rceil} \int\limits_{(k-1)2\pi/z}^{k2\pi/z} \frac{1 - \cos \xi z}{\xi} \, d\xi$$

$$\ge \frac{z}{2\pi} \sum_{k=1}^{\lceil \vartheta z / 2\pi \rceil} \frac{1}{k} \int\limits_{(k-1)2\pi/z}^{k2\pi/z} (1 - \cos \xi z) \, d\xi = \sum_{k=1}^{\lceil \vartheta z / 2\pi \rceil} \frac{1}{k}$$

$$\ge \log(\lceil \vartheta z / 2\pi \rceil + 1) \ge \log \frac{\vartheta z_0}{2\pi} \, .$$

We choose $\delta = 2\pi/z_0$. Then

$$\int\limits_{2\pi/z_0}^{\infty} |\psi_S(\vartheta)/\vartheta| \, d\vartheta = \int\limits_{2\pi/z_0}^{\infty} \vartheta^{-1} \exp\left\{ -\frac{\lambda}{\beta} \int\limits_0^{\infty} \int\limits_0^{\vartheta} \frac{1 - \cos \xi z}{\xi} \, d\xi \, dG(z) \right\} d\vartheta$$

$$\le \int\limits_{2\pi/z_0}^{\infty} \vartheta^{-1} \exp\left\{ -\frac{\lambda}{\beta} \int\limits_{z_0}^{\infty} \int\limits_0^{\vartheta} \frac{1 - \cos \xi z}{\xi} \, d\xi \, dG(z) \right\} d\vartheta$$

$$\le \int\limits_{2\pi/z_0}^{\infty} \vartheta^{-1} \exp\left\{ -\frac{\lambda}{\beta} \int\limits_{z_0}^{\infty} \log \frac{\vartheta z_0}{2\pi} \, dG(z) \right\} d\vartheta$$

$$= \left(\frac{2\pi}{z_0} \right)^{\lambda(1 - G(z_0))/\beta} \int\limits_{2\pi/z_0}^{\infty} \vartheta^{-1 - \lambda(1 - G(z_0))/\beta} d\vartheta = \frac{\beta}{\lambda(1 - G(z_0))} \, .$$

It follows (see for instance [12, p. 511]) that

$$f_3(x - c/\beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-i\vartheta x}}{i\vartheta} \psi_S(\vartheta)\, d\vartheta\,.$$

Note that the complex part of the integrand is an odd function in ϑ and vanishes therefore, while the real part of the integrand is an even function. □

Remark. In order to obtain formula (12) no assumptions on (N_t) and (Y_i) are necessary. In any case where the distribution of S can be computed we get the solution of the corresponding ruin problem. For further literature on S see [21]. ■

To compute the functions f_1, f_2 and f_3 one might solve equations (5), (6) and (7) numerically. But the question how to control the error and how to choose $f'(-c/\beta_2)$ arises. In fact, considering (10) shows that $f'(-c/\beta_2)$ may be 0, finite or infinite depending on the parameters. But it suffices to find an alternative method to compute f_3. We would propose the following algorithm.

- Compute f_3 using (13).
- Solve equations (6) and (5) numerically in order to get f_2 and f_1.

5 Final Comments

The method of piecewise deterministic Markov processes has turned out to be an ideal tool to model risk processes of bounded variation. The easy way one can construct martingales (see also [19], [5] or [10]) makes it easy to compute ruin probabilities or exponential bounds of the ruin probabilities. In more general models, like the one considered in this work, the corresponding integro differential equations are hard so solve. One has to resort to Laplace transforms and to use Tauberian theorems in order to get some properties of the process. The problem of inverting the obtained expressions is hard. The numerical procedures work slowly. The method proposed in the previous section only works well, if $C(\vartheta)$ and $D(\vartheta)$ can be computed explicitly (as it is the case for exponentially distributed claims). Therefore quicker methods to compute f_3 numerically are called for.

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Hanspeter Schmidli
Theoretical Statistics
University of Århus
Ny Munkegade
DK-8000 Århus C

References

[1] *Andersen, E.Sparre* (1957). On the collective theory of risk in the case of contagion between the claims. Transactions XVth International Congress of Actuaries, New York, II, 219–229.
[2] *Asmussen, S.* (1989). Risk theory in a Markovian environment. Scand. Actuarial J., 66–100.
[3] *Asmussen, S./Bladt, M.* (1993). Renewal theory and queueing algorithms for matrix-exponential distributions. Technical report, University of Aalborg.
[4] *Björk, T./Grandell, J.* (1988). Exponential Inequalities for Ruin Probabilities in the Cox Case. Scand. Actuarial J., 77–111.
[5] *Dassios, A./Embrechts, P.* (1989). Martingales and Insurance Risk. Commun. Statist. Stochastic Models 5, 181–217.
[6] *Davis, M.H.A.* (1984). Piecewise-deterministic Markov Processes: A General Class of Non-diffusion Stochastic Models. J. R. Statist. Soc. B 46, 353–388.
[7] *Davis, M.H.A.* (1993). Markov Models and Optimization. Chapman & Hall, London.
[8] *Delbaen, F./Haezendonck, J.* (1987). Classical risk theory in an economic environment. Insurance Math. Econom. 6, 85–116.
[9] *Dellacherie, C./Meyer, P.A.* (1980). Probabilités et potentiel. Ch. VI, Hermann, Paris.
[10] *Embrechts, P./Grandell, J./Schmidli, H.* (1993). Finite-Time Lundberg Inequalities in the Cox Case. Scand. Actuarial J., 17–41.
[11] *Embrechts, P./Schmidli, H.* (1994). Ruin Estimation for a General Insurance Risk Model. Adv. in Appl. Probab., to appear.
[12] *Feller, W.* (1971). An Introduction to Probability Theory and Its Applications. Volume II, Wiley, New York.
[13] *Gerber, H.U.* (1971). Der Einfluss von Zins auf die Ruinwahrscheinlichkeit. Schweiz. Verein. Versicherungsmath. Mitt. 71, 63–70.
[14] *Gerber, H.U.* (1973). Martingales in Risk Theory. Schweiz. Verein. Versicherungsmath. Mitt. 73, 205–216.
[15] *Gerber, H.U.* (1975). The surplus process as a fair game – utilitywise. ASTIN Bulletin 8, 307–322.
[16] *Grandell, J.* (1991). Aspects of Risk Theory. Springer-Verlag, New York.

[17] *Hall, P./Teugels, J.L./Vanmarcke, A.* (1992). The Abscissa of Convergence of the Laplace Transform. J. Appl. Probab. 29, 353–362.

[18] *Orsingher, E./Battaglia, F.* (1982). Probability Distributions and Level Crossings of Shot Noise Models. Stochastics 8, 45–61.

[19] *Schmidli, H.* (1992). A General Insurance Risk Model. Ph.D. Thesis, ETH Zürich.

[20] *Schmidli, H.* (1994). Estimation of the Abscissa of Convergence of the Moment Generating Function. Research Report No. 276, Dept. Theor. Statist., Aarhus University.

[21] *Todorovic, P./Gani, J.* (1987). Modeling the Effect of Erosion on Crop Production. J. Appl. Probab. 24, 787–797.

[22] *Williams, D.* (1979). Diffusions, Markov Processes and Martingales, Volume I, Wiley, New York.

Summary

Two models of the collective theory of risk, one introduced by Gerber [13] and the other by Dassios and Embrechts [5], where borrowing is allowed, are extended with the possibility of investment above a certain level. An application of Davis' method of piecewise deterministic Markov processes [6] yields the Laplace transform of the ruin probabilities as well as a 'Lundberg exponent' for the model. For the case of equal interest rates for invested and borrowed money an explicit inversion formula is given.

Zusammenfassung

Zwei Modelle der kollektiven Risikotheorie (das eine geht zurück auf Gerber [13], das andere auf Dassios und Embrechts [5]), in denen die Möglichkeit von Geldanleihe besteht, werden durch die Möglichkeit von Investition des Kapitals über einer gewissen Schranke erweitert. Durch Anwendung der Methode der von Davis [6] eingeführten stückweise deterministischen Markov Prozesse lässt sich die Laplace-Transformierte der Ruinwahrscheinlichkeiten sowie ein 'Lundberg Exponent' bestimmen. Für den Fall von gleichen Zinsraten für geliehenes und investiertes Kapital lässt sich eine explizite Umkehrformel herleiten.

Résumé

On propose une extension de deux modèles de la théorie du risque collectif (l'un introduit par Gerber [13], l'autre par Dassios et Embrechts [13]) incluant la possibilité d'emprunter, en prenant en compte la possibilité d'investir le capital supérieur à une certaine limite. La méthode des processus de Markov déterministes par morceaux, introduite par Davis [6], permet d'obtenir la transformée de Laplace de la probabilité de ruine ainsi qu'un "exposant de Lundberg". Dans le cas où les taux d'intérêt du capital emprunté et du capital investi sont égaux on donne une formule d'inversion explicite.