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Limiting Dynamics, KMS–States, and Macroscopic Phase Angle for Weakly Inhomogeneous BCS–Models

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Abstract. We study a class of inhomogeneous BCS–models (with complex momentum dependent interaction coefficients) in terms of a generalized perturbation theory with possibly singular perturbations in the thermodynamic limit. We start from the averaged homogeneous model, which we formulate by recent algebraic mean–field techniques. We arrive at a C^*–dynamical system over a classically extended observable algebra, the KMS–states of which are in a bi–unique correspondence to the unperturbed ones. For the momentum dependent gap parameters a rigorous form of the self–consistency equation is derived. The macroscopic phase is evaluated as the average of the momentum dependent gap parameter phases.

1 Introduction

In discussing many body systems there is a big difference between a strictly microscopic derivation of a collective phenomenon and its anticipation by making an intuitively motivated ansatz. In a strictly microscopic discussion the collective variables should be formulated in terms of (limits of) microscopic expressions, which often have the form of an averaging procedure. Only these are able to provide a unified, consistent theoretical description and give then also relevant information on the fluctuations, the microscopic quantities undergo around the average value. This may be important for a detailed analysis of (quantum) noise.

In superconductivity the ad hoc use of a macroscopic wave function (with a macroscopic phase) does not comply with the requirements of microscopic consistency (at least not in every aspect, as show the discrepancies of its dynamics). In order to obtain a more comprehensive microscopic understanding of the collective phenomenon in (traditional) superconductors we elaborate here a class of inhomogeneous BCS–models, where the kinetic energy and pairing interaction are momentum dependent. Since the coupling parameters of the latter are of dynamical origin it is not unreasonable to attribute to them a momentum dependent complex phase. More details of the models are described in Section 2.

The basic idea of our approach is to consider the inhomogeneities as perturbations from the homogeneous BCS–model, which has averaged kinetic energy and interaction constants. In fact, the deviation from a homogeneous model in our model class may be so strong that it transcends the class treated in [1], the latter apparently comprising all previous BCS–models of mathematical physics beside the *macroscopically* inhomogeneous ones. (The macroscopically inhomogeneous ones employ a parameter dependent scaling in the thermodynamic limit which is more similar to hydrodynamics than the usual quantum field theory [2], [3], [4].) There are nowadays several developments to a perturbation and stability theory with strategies somehow related to ours. Let us mention only the path integral approach with quantum fluctuations about the most probable classical path [5], [6], and the idea of [7], [8], [9], to consider a quantum lattice system as a perturbation from a purely classical one.

Here we use rather recent tools of algebraic mean–field theory to introduce the homogeneous temperature GNS–representation [10], [11], with its β–dependent limiting dynamics [1]. This concerns especially the classical part of the dynamics, which is connected with a flow on a classical parameter space. In [12] it has been emphasized that such a classical part arises by extending the physical limiting dynamics from local observables to global ones if one uses the grand canonical GNS–representation space. This extension procedure is here replaced by the application of a general rigorously derived scheme [1], how to extract from the limiting Heisenberg generator a differential equation. The solution of this gives a multiplicative cocycle in the state space of the one lattice algebra. We carry this through for the homogeneous physical, reduced, and gauge dynamics. The classical dynamics takes place here on the small phase space E_β isomorphic to the one–dimensional torus. The used method is, however, generalizable to rather arbitrary parameter spaces. The form of the homogeneous limiting dynamics in Prop. 3.4 follows then from the mentioned general scheme. It transcends the original quasi–local electron algebra $\mathfrak{A}$ and constitutes a C^*–dynamical system in the C^*–algebra $\mathcal{C}_\beta = \mathcal{C}(E_\beta, \mathfrak{A})$ of continuous functions from the "phase space" E_β into the electron field algebra. Both the physical (non–reduced) dynamics and the gauge transformations (of the first kind) have a non–trivial classical part, that is a rotation on the torus.

We show that the gauge invariant KMS–states for the reduced dynamics, as well as their extremal pure phase components, minimize the free energy density. Instability or metastability does not arise in spite of the mean–field character of the model.

The main part of our investigation is the construction of the limiting dynamics of the inhomogeneous model. Again one starts with the limiting Heisenberg generator applied to local observables, where now only the commutators with the perturbations (relative to the homogeneous local Hamiltonians) have to be calculated. The thermodynamic limit of these applied to an observable in the finite (momentum) region Λ is the commutator with a local element $P_\Lambda^\beta \in \mathcal{C}_\beta$, that is a bounded local, effective perturbation. In our model class the limit of the P_Λ^β, with Λ tending to the infinite lattice, would in general be totally divergent. Nevertheless, the limit of the iterated commutators in the perturbation expansions of the finite time translations applied to local elements may be shown to converge (Appendix). This limiting dynamics is then extended to all of $\mathcal{C}_\beta$ and constitutes again a C^*-dynamical system with the same classical part as the homogeneous model.

In spite of the rather singular perturbations in our model class the inhomogeneous KMS–states are shown to correspond to the homogeneous counterparts in the analogous way as for bounded perturbations [13]. This connection is made manifest by the identical classical parametrization of the pure phase states in both cases. Thermodynamic stability expresses itself again by the minimalization of a free energy density. The latter has the same values as in the homogeneous case but is much harder to calculate.

The momentum dependent gap parameters with their (momentum dependent) complex phases are shown to satisfy as a necessary condition the gap equation in a precise version for the thermodynamic limit. There is a systematic degeneration for its solutions, which is parametrized by a momentum independent phase (from the mentioned torus). This macroscopic phase is here disclosed as the average value of the momentum dependent microscopic phases.

According to Gorkov [14] the position dependent gap parameters are proportional to the macroscopic wave function of the Cooper pair condensate. Since only the homogeneous part of our gap function is directly connected with the condensate, the Fourier transform of our momentum dependent gap parameter should not be identified with the macroscopic wave function. Our considerations suggest rather that the rigorous elaboration of Gorkov's idea requires a macroscopically inhomogeneous BCS–model, which at the present has not been studied from this point of view.

Altogether one may state that rigorous perturbation theory has a much wider range of applicability than using bounded perturbations only. This nourishes the hope that certain aspects of our treatment may be transferred to the perturbation of Green's functions, provided that the latter starts — not with a free but — with an interacting symmetrized model.

2 Introduction of the Model–Class

We consider the conduction electrons of a metallic superconductor in a sequence of increasing, finite volumina V_n. The effective interactions between the electrons are split into two parts: One part is subsumed into a lattice periodic external potential and gives rise to the Bloch wave functions with energies $\varepsilon_{\vec{k}}$, where the momenta $\vec{k}$ are taken from a V_n–dependent set. This set is finite, if the $\varepsilon_{\vec{k}}$ are restricted to a shell around the Fermi energy ε_F. In this momentum region one has as second part a pair–pair interaction which is in the average attractive.

The Bloch eigenstates are used to realize the electronic CAR–algebra as a tensor product. With increasing volumina V_n the set of considered Bloch momenta $\mathcal{K}$ becomes countably infinite. Considering a numbering $i : \mathcal{K} \longrightarrow \mathbb{N}$, we have for each $k = i(\vec{k})$ two spin values $\sigma \in \{\uparrow, \downarrow\}$ and the CAR–algebra $\mathfrak{A}$ for the considered set of states is

$$\mathfrak{A} \cong \bigotimes_{k\in\mathbb{N}} \mathfrak{B}, \tag{2.1}$$

with $\mathfrak{B} \cong \mathbb{M}_4 \cong \mathbb{M}_2 \otimes \mathbb{M}_2$. Here we have considered two spin values in the algebra $\mathfrak{B}$ leading by a generalized Jordan–Wigner representation to (2.1). (Comp. Eq. (2.2) below, [13, Chap. 5.2.2], and [15]).

We introduce a *quasi–local* structure in momentum space by associating the local algebra $\mathfrak{A}_\Lambda := \otimes_{k\in\Lambda}\mathfrak{B}$ with each finite subset $\Lambda \in \mathfrak{L} := \{\Lambda \subset \mathbb{N} \mid |\Lambda| < \infty\}$, even if there is no corresponding volume in position space for Λ. Dropping the embedding operators we have $\mathfrak{A}_0 := \bigcup_{\Lambda\in\mathfrak{L}} \mathfrak{A}_\Lambda$ as a norm dense sub–algebra of $\mathfrak{A}$.

According to our numbering i we take into account pairs of electrons in the Jordan–Wigner representation for annihilation operators $c_{\vec{k},\sigma}$, $\vec{k} \in \mathcal{K}$, $\sigma \in \{\uparrow, \downarrow\}$

$$\begin{aligned} c_{\vec{k}\uparrow} &= \Big(\bigotimes_{j=1}^{i(\vec{k})-1} (\sigma_z \otimes \sigma_z)\Big) \otimes (\sigma_z \otimes \sigma_-) \otimes \Big(\bigotimes_{j=i(\vec{k})+1}^{\infty} 1\!\!1_4\Big), \\ c_{\vec{k}\downarrow} &= \Big(\bigotimes_{j=1}^{i(-\vec{k})-1} (\sigma_z \otimes \sigma_z)\Big) \otimes (\sigma_- \otimes 1\!\!1_2) \otimes \Big(\bigotimes_{j=i(-\vec{k})+1}^{\infty} 1\!\!1_4\Big), \end{aligned} \tag{2.2}$$

where σ_x, σ_y, σ_z are the Pauli matrices and $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. If $i(\vec{k}) = k \in \mathbb{N}$ we write $c_{k\uparrow} := c_{\vec{k}\uparrow}$ and $c_{-k\downarrow} := c_{-\vec{k}\downarrow}$.

The *local Hamiltonian* for a finite set Λ of Bloch modes is obtained by adding to the Bloch energy the pair–pair interaction. Since the latter is due to a complicated mechanism involving infinitely many phonon exchanges, the exact values of the effective pair coupling energies are not known. They are usually calculated up to second order perturbation theory [16], [17], and in the original BCS–paper [18] they are assumed momentum independent. We allow rather arbitrary complex values with non–trivial dynamical phases for them and obtain (see e.g. [19], [20], [21])

$$H_\Lambda := \sum_{k\in\Lambda} \varepsilon_k \left(c^*_{k\uparrow}c_{k\uparrow} + c^*_{-k\downarrow}c_{-k\downarrow}\right) - \sum_{k,k'\in\Lambda} \frac{g_{kk'}}{|\Lambda|}\, c^*_{k\uparrow}c^*_{-k\downarrow}c_{-k'\downarrow}c_{k'\uparrow}, \tag{2.3}$$

with $g_{kk'} = \overline{g_{k'k}}$ for all $k, k' \in \mathbb{N}$. Introducing the pair annihilation and number operators

$$b_k = c_{-k\downarrow}c_{k\uparrow}, \qquad n_k = c^*_{k\uparrow}c_{k\uparrow} + c^*_{-k\downarrow}c_{-k\downarrow}, \tag{2.4}$$

we write

$$H_\Lambda = \sum_{k\in\Lambda} \varepsilon_k\, n_k - \sum_{k,k'\in\Lambda} \frac{g_{kk'}}{|\Lambda|}\, b^*_k b_{k'}. \tag{2.5}$$

Note that b_k and n_k are embeddings of the same operators $b = \sigma_- \otimes \sigma_-$ and $n = \frac{1}{2}(\sigma_z \otimes 1\!\!1_2 + 1\!\!1_2 \otimes \sigma_z) + 1\!\!1_2 \otimes 1\!\!1_2$ at the lattice site $k \in \mathbb{N}$.

As mentioned in the Introduction the basic idea behind our approach is to consider a given inhomogeneous BCS–model as a perturbation of a homogeneous one. The latter is obtained uniquely by averaging the given model data

$$\varepsilon := \lim_{\Lambda\in\mathfrak{L}} \frac{1}{|\Lambda|} \sum_{k\in\Lambda} \varepsilon_k, \qquad 0 < g := \lim_{\Lambda\in\mathfrak{L}} \frac{1}{|\Lambda|^2} \sum_{k,k'\in\Lambda} g_{kk'}, \tag{2.6}$$

and has the local Hamiltonians

$$H^0_\Lambda := \sum_{k\in\Lambda} \varepsilon\, n_k - \sum_{k,k'\in\Lambda} \frac{g}{|\Lambda|} b^*_k b_{k'}, \qquad \Lambda \in \mathfrak{L}. \tag{2.7}$$

As a general assumption of our investigation we assume the validity of Eq. (2.6).

In order to arrive at a well behaved perturbation theory one has to require that the perturbations

$$P_\Lambda := H_\Lambda - H^0_\Lambda = \sum_{k\in\Lambda} \delta\varepsilon_k n_k - \frac{1}{|\Lambda|} \sum_{k,k'\in\Lambda} \delta g_{kk'} b^*_k b_{k'} \tag{2.8}$$

with

$$\delta\varepsilon_k := \varepsilon_k - \varepsilon, \qquad \delta g_{kk'} := g_{kk'} - g \tag{2.9}$$

be "small" in some sense. In mathematical physics the most common assumptions imply that $\{\|P_\Lambda\| \mid \Lambda \in \mathfrak{L}\}$ be a bounded net. This allows still for the interesting case, that the $\{P_\Lambda \mid \Lambda \in \mathfrak{L}\}$ do not converge in norm in $\mathfrak{A}$ but in a weaker sense in certain representations or as a so-called "quasi-symmetric" net [1], [22]. We found, however, that a much weaker postulate allows for a reasonable dynamical perturbation expansion.

2.1 Model Assumption

We say that the BCS–model is in the allowed model class, if the constants (2.9) satisfy the following relations:

$$\lim_{k\to\infty} \delta\varepsilon_k = 0, \qquad \lim_{k'\to\infty} \delta g_{kk'} =: \delta g_k \quad \text{exists with} \quad \lim_{k\to\infty} \delta g_k = 0 \tag{2.10}$$

and

$$\lim_{\Lambda\in\mathfrak{L}} \frac{1}{|\Lambda|} \sum_{k,k'\in\Lambda} \left| \delta g_{kk'} - \delta g_k - \overline{\delta g_{k'}} \right| = 0. \tag{2.11}$$

Here $\lim_{\Lambda\in\mathfrak{L}}$ *denotes the net limit over the index set* $\mathfrak{L}$.

Observe that no summability assumption for the $\delta\varepsilon_k$ and δg_k or their squares has been formulated, so that $\|P_\Lambda\|$ may tend to infinity in a rather strong sense.

Concerning the symmetries of our model class we introduce first the internal symmetries (in reference to the pair structure of the Hamiltonians). Let be $\mathsf{V}(\mathfrak{B})$ the group of all unitary and anti–unitary operators in $\mathbb{C}^4$. For each $v \in \mathsf{V}(\mathfrak{B})$ we define an (anti–) automorphism by

$$\alpha_v(\bigotimes_{k\in\mathbb{N}} a_k) := \begin{cases} \bigotimes_{k\in\mathbb{N}} v a_k v^* & v \text{ unitary,} \\ \bigotimes_{k\in\mathbb{N}} v a_k^* v^* & v \text{ anti–unitary,} \end{cases} \tag{2.12}$$

and by linear and norm continuous extension (where $a_k \in \mathfrak{B}$ for all $k \in \mathbb{N}$). If $\alpha_v(H_\Lambda) = H_\Lambda$ for all $\Lambda \in \mathfrak{L}$, α_v is called a strict internal symmetry of the model [23].

The gauge group (of the first kind) $\mathsf{U}(1)$ acts as a strict internal symmetry group, where $v_\vartheta = e^{i\,\sigma_z\,\vartheta/2} \otimes e^{i\,\sigma_z\,\vartheta/2} \in \mathbb{M}_2 \otimes \mathbb{M}_2$, $\vartheta \in [0, 2\pi[$. We write for convenience

$$\kappa_\vartheta := \alpha_{v_\vartheta}, \qquad \vartheta \in [0, 2\pi[. \tag{2.13}$$

For later use we introduce the gauge group for pairs

$$\widetilde{\mathsf{U}}(1) = \mathsf{U}(1)/\{\mathbb{1}, -\mathbb{1}\}.$$

An example for an anti–automorphic internal symmetry is the time reversal transformation, which, however, is not spontaneously broken in the considered models.

The *spatial symmetry* in our model class is an approximate invariance against permutations of the k–indices. The group P of all finite k–permutations acts in $\mathfrak{A}$ via $*$–automorphisms [24]. Most of our considered states are in the folium* $\mathcal{F}^{\mathsf{P}}(\mathfrak{A})$ which is generated by the Bauer simplex $\mathfrak{S}^{\mathsf{P}}(\mathfrak{A})$ of all permutation invariant states. It is remarkable that our introduced model class has equilibrium states, which transcend this quasi–permutation invariance.

3 Equilibrium Properties of the Homogeneous BCS–Mod

The homogeneous model, defined in terms of the net $(H_\Lambda^0)_{\Lambda\in\mathfrak{L}}$ of local Hamiltonians (2.7), is a (permutation symmetric) mean–field model, for which there exists a well elaborated, operator algebraic strategy of treating its dynamics [1], [26], [27] and its equilibrium states [10], [11], [23], [28], [29], [30]. Let us reproduce the basic steps, supplementing some new features needed in the subsequent discussion.

We discuss the limiting Gibbs states of the homogeneous model which constitute the starting point for the further investigations. The unique equilibrium state of a system with

*The notion of a folium is introduced in [25]: A folium $\mathcal{F}(\mathfrak{A})$ of a C^*–algebra $\mathfrak{A}$ is a norm-closed, convex subset of the state space $\mathfrak{S}(\mathfrak{A})$ with $\varphi_A \in \mathcal{F}(\mathfrak{A})$ for all $\varphi \in \mathcal{F}(\mathfrak{A})$ and $A \in \mathfrak{A}$. φ_A is the state $\langle\varphi_A\,;\cdot\rangle = \langle\varphi\,;A^*\cdot A\rangle\,/\,\langle\varphi\,;A^*A\rangle$, for $A \in \mathfrak{A}$ with $\langle\varphi\,;A^*A\rangle \neq 0$ and $\varphi_A = \varphi$ otherwise (i.e. $\mathcal{F}(\mathfrak{A})$ is closed under perturbations from $\mathfrak{A}$). There is a one–to–one order preserving correspondence between folia in $\mathfrak{S}(\mathfrak{A})$, (quasi–equivalence classes of) representations of $\mathfrak{A}$, and central projections in the universal von Neumann algebra $\mathfrak{M}_u$ of $\mathfrak{A}$. If $\mathcal{F} \equiv \mathcal{F}(\mathfrak{A})$, $\Pi_\mathcal{F}$, and $c_\mathcal{F} \in \mathfrak{M}_u \cap \mathfrak{M}_u'$ are in correspondence, then $\mathcal{F}$ consists just of the $\Pi_\mathcal{F}$-normal states on $\mathfrak{A}$, that is $\mathrm{linh}(\mathcal{F})^* \cong \mathfrak{M}_{\Pi_\mathcal{F}} \cong \Pi_\mathcal{F}(\mathfrak{A})'' \cong c_\mathcal{F}\mathfrak{M}_u$.

local Hamiltonian H_Λ at inverse temperature $\beta = \frac{1}{k_B T} > 0$ (with absolute temperature T and Boltzmann constant k_B), is given by the Gibbs state ω^{β,H_Λ} as an element in the state space $\mathfrak{S}(\mathfrak{A}_\Lambda)$ of the local algebra $\mathfrak{A}_\Lambda$

$$\omega^{\beta,H_\Lambda} : \mathfrak{A}_\Lambda \longrightarrow \mathbb{C},\ A \longmapsto \langle \omega^{\beta,H_\Lambda} ; A \rangle := \frac{\mathrm{tr}_\Lambda(\exp\{-\beta H_\Lambda\} A)}{\mathrm{tr}_\Lambda(\exp\{-\beta H_\Lambda\})}.$$

Without changing the notation we extend ω^{β,H_Λ} to a state on $\mathfrak{A}$ by continuation with the trace state. Each w^*–accumulation point ω of the net $(\omega^{\beta,H_\Lambda})_{\Lambda\in\mathfrak{L}}$ is called a limiting Gibbs state. The state space $\mathfrak{S}(\mathfrak{A})$ of $\mathfrak{A}$ is w^*–compact and thus at least one accumulation point exists.

In order to fix a given particle density, we introduce the chemical potential $\mu \in \mathbb{R}$ and the *reduced local Hamiltonians* H_Λ^r by

$$H_\Lambda^r := H_\Lambda - \mu N_\Lambda, \qquad \Lambda \in \mathfrak{L}, \tag{3.1}$$

with the local number number operator

$$N_\Lambda := \sum_{k\in\Lambda} n_k, \qquad \Lambda \in \mathfrak{L}, \tag{3.2}$$

which counts electrons in the lattice region Λ. Correspondingly, the homogenized reduced local Hamiltonian H_Λ^{0r} is obtained from H_Λ^0 by replacing ε with $\varepsilon - \mu$ in Eq. (2.7). $\omega^{\beta,H_\Lambda^{0r}}$ has two external parameters, the chemical potential $\mu \in \mathbb{R}$ and the inverse temperature $\beta > 0$, which are fixed in the following†. The P_Λ, Eqns. (2.8) and (2.9), do not depend on the chemical potential.

We determine the limiting Gibbs states of the homogenized model by using the symmetries of the model and the minimum principle of the free energy density for limiting Gibbs states:

3.1 Proposition

Let be $\beta > 0$ and H_Λ^{0r} as introduced above. Every limiting Gibbs state ω_0^β of the net of local Gibbs states $\omega^{\beta,H_\Lambda^{0r}}$ minimizes the functional $f_0(\beta,\cdot)$ of the free energy density on $\mathfrak{S}^{\mathsf{P}}(\mathfrak{A})$:

$$f_0(\beta,\cdot) : \mathfrak{S}^{\mathsf{P}}(\mathfrak{A}) \longrightarrow \mathbb{R},\ \omega \longrightarrow f_0(\beta,\omega) := \lim_{\Lambda\in\mathfrak{L}} \frac{1}{|\Lambda|} \left(\langle \omega ; H_\Lambda^{0r} \rangle + \frac{1}{\beta} \mathrm{tr}_\Lambda \left(\varrho_\Lambda^\omega \ln(\varrho_\Lambda^\omega) \right) \right),$$

where $\varrho_\Lambda^\omega \in \mathfrak{A}_\Lambda$ is the density matrix of $\omega|_{\mathfrak{A}_\Lambda}$. $f_0(\beta,\cdot)$ is a w^*–continuous affine functional on the Bauer simplex $\mathfrak{S}^{\mathsf{P}}(\mathfrak{A})$.

The set $\mathfrak{S}_\beta^0(\mathfrak{A}) \subset \mathfrak{S}^{\mathsf{P}}(\mathfrak{A})$ of states with minimal free energy density

$$f_0(\beta) := \inf\{f_0(\beta,\omega) \mid \omega \in \mathfrak{S}^{\mathsf{P}}(\mathfrak{A})\} = \inf\{f_0(\beta,\omega) \mid \omega \in \partial_e \mathfrak{S}^{\mathsf{P}}(\mathfrak{A})\}$$

†If the thermodynamic limit of the local Gibbs states is determined at fixed particle density, the chemical potential μ will vary with the local region Λ. Nevertheless these local chemical potentials converge in the thermodynamic limit [31]. Here we use this limiting value.

is a Bauer simplex with extremal boundary $\partial_e \mathfrak{S}^0_\beta(\mathfrak{A}) = \mathfrak{S}^0_\beta(\mathfrak{A}) \cap \partial_e \mathfrak{S}^{\mathsf{P}}(\mathfrak{A}) \neq \emptyset$.

All states $\varphi \in \partial_e \mathfrak{S}^0_\beta(\mathfrak{A})$ *are product states* $\bigotimes_{k\in\mathbb{N}} \varrho$ *determined by solutions* $\varrho \in \mathfrak{S}(\mathfrak{B})$ *of the self-consistency equation*‡

$$\varrho = \frac{\exp\{-\beta\, h^{0r}(\varrho)\}}{\mathrm{tr}_{\mathfrak{B}}\left(\exp\{-\beta\, h^{0r}(\varrho)\}\right)}, \tag{3.3}$$

with the effective Hamiltonian $h^{0r} := (\varepsilon - \mu)\, n - g\, (\langle \varrho\,; b\rangle\, b^* + \langle \varrho\,; b^*\rangle\, b)$. *We note that there are also solutions* ϱ *of Eq. (3.3) such that* $\bigotimes_{k\in\mathbb{N}} \varrho \notin \mathfrak{S}^0_\beta(\mathfrak{A})$.

PROOF: The convergence, the w^*–continuity, and the affinity of $f_0(\beta,\cdot)$ is proved in [11, Proposition 3.9] and the minimum principle for limiting Gibbs states in [10, Theorem 2.3], or [11, Section 4]. $\mathfrak{S}^{\mathsf{P}}(\mathfrak{A})$ and $\mathfrak{S}^0_\beta(\mathfrak{A})$ are Bauer simplices with the stated extremal boundary according to [24, Theorem 2.8], [11, Theorem 4.4]. The self–consistency condition (3.3) is shown in [28, Theorem II.4], [32, Satz 1.7.5], [10, Proposition II.5], and [11, Theorem 5.4]. □

If we use the parametrization of product states $\bigotimes_{k\in\mathbb{N}} \varrho \in \partial_e \mathfrak{S}^{\mathsf{P}}(\mathfrak{A})$ in terms of $\varrho \in \mathfrak{S}(\mathfrak{B})$, we find for the free energy density:

$$f_0(\beta, \bigotimes_{k\in\mathbb{N}} \varrho) = (\varepsilon - \mu)\,\langle \varrho\,; n\rangle - g\,\langle \varrho\,; b^*\rangle\,\langle \varrho\,; b\rangle + \frac{1}{\beta}\,\mathrm{tr}_{\mathfrak{B}}(\varrho \ln(\varrho)).$$

For $\omega \in \mathfrak{S}^{\mathsf{P}}(\mathfrak{A})$, $f_0(\beta,\omega)$ is obtained by integration of $f_0(\beta, \bigotimes_{k\in\mathbb{N}} \varrho)$ with the corresponding decomposition measure of ω (see Prop. 3.3 below).

The solutions of Eq. (3.3), the minimum principle of the free energy density, and the symmetries of the local Hamiltonians determine the unique limiting Gibbs state:

3.2 Proposition

For $\beta > 0$ *and* H^{0r}_Λ *as above, there exists a unique limiting Gibbs state* $\omega^\beta_0 = \mathrm{w}^*\text{-}\lim_{\Lambda\in\mathfrak{L}} \omega^{\beta, H^{0r}_\Lambda}$. *With the critical (inverse) temperature* $\beta_c(\mu)$,

$$\beta_c(\mu) = \begin{cases} \frac{4}{g} & |\varepsilon - \mu| = 0, \\ \frac{2}{\varepsilon-\mu}\,\mathrm{artanh}\left(\frac{2(\varepsilon-\mu)}{g}\right) & 0 < |\varepsilon - \mu| < \frac{g}{2}, \\ \infty & |\varepsilon - \mu| \geq \frac{g}{2}, \end{cases} \tag{3.4}$$

this state is given

(i) for $\beta \leq \beta_c$ *(resp.* $\beta < \beta_c$ *if* $\beta_c = \infty$*) by the product state* $\omega^\beta_0 = \bigotimes_{k\in\mathbb{N}} \varrho_0$, *with*

$$\varrho_0 = \frac{\exp(-\beta\,(\varepsilon-\mu)\,n)}{\mathrm{tr}_{\mathfrak{B}}\left(\exp(-\beta\,(\varepsilon-\mu)\,n)\right)}; \tag{3.5}$$

‡In the following we identify the states on $\mathfrak{B}$ with the corresponding density matrices.

(ii) for $\beta > \beta_c$ by its extremal decomposition into elements of $\partial_e \mathfrak{S}^0_\beta(\mathfrak{A})$

$$\omega_0^\beta = \int_0^{2\pi} \omega_0^\vartheta \frac{\mathrm{d}\,\vartheta}{2\pi}, \tag{3.6}$$

with $\omega_0^\vartheta = \bigotimes_{k\in\mathbb{N}} \varrho_0^\vartheta$ and

$$\varrho_0^\vartheta = \frac{\exp(-\beta\left\{(\varepsilon-\mu)\,n - \Delta_0(e^{-i\vartheta}b^* + e^{i\vartheta}b)\right\})}{\mathrm{tr}_{\mathfrak{B}}\left(\exp(-\beta\left\{(\varepsilon-\mu)\,n - \Delta_0(e^{-i\vartheta}b^* + e^{i\vartheta}b)\right\})\right)}. \tag{3.7}$$

Δ_0 is the positive solution of

$$\frac{2\sqrt{(\varepsilon-\mu)^2+\Delta_0^2}}{g} = \tanh\left(\frac{\beta\sqrt{(\varepsilon-\mu)^2+\Delta_0^2}}{2}\right). \tag{3.8}$$

PROOF: For $\beta \le \beta_c$, ϱ_0 in (3.5) is the unique solution of Eq. (3.3) and thus the limiting Gibbs state ω_0^β is unique. Due to the gauge symmetry of H_Λ^{0r}, each limiting Gibbs state ω_0^β has to be gauge invariant, i.e. $\omega_0^\beta = \omega_0^\beta \circ \kappa_\theta$ with $\theta \in [0, 2\pi[$ and κ_θ from Eq. (2.13). For $\beta > \beta_c$ there are solutions ϱ_0^ϑ, $\vartheta \in [0, 2\pi[$ of Eq. (3.3) (with $\Delta_0 > 0$ in (3.8)) such that $\omega_0^\vartheta \circ \kappa_\theta = \omega_0^{\vartheta+2\theta}$ for $\theta \in [0,\pi[$. All states in $\partial_e\mathfrak{S}^0_\beta(\mathfrak{A})$ are elements of exactly one orbit of $\widetilde{\mathsf{U}}(1)$ in $\partial_e\mathfrak{S}^{\mathsf{P}}(\mathfrak{A})$. The decomposition measure of the unique invariant state ω_0^β in $\mathfrak{S}^0_\beta(\mathfrak{A})$ into extremal states ω_0^ϑ is given by the Haar–measure of $\widetilde{\mathsf{U}}(1)$ [32], cf. also [23], [30]. □

We discuss explicitly only the case $\beta_c < \beta < +\infty$, but the case $0 \le \beta \le \beta_c$ may be obtained therefrom by continuously deforming the quantities (like $\Delta_0(\beta)$) into the region $\beta \le \beta_c$ (which here gives $\Delta_0(\beta) = 0$). It has some advantages to parametrize the pure phase states ω_0^ϑ in (3.6) directly in terms of the density matrices ϱ from

$$E_\beta := \{\varrho_0^\vartheta \mid \vartheta \in [0, 2\pi[\} \subset \mathfrak{S}(\mathfrak{B}), \tag{3.9}$$

where ϱ_0^ϑ is from Eq. (3.7). The Lebesgue measure $\mathrm{d}\,\vartheta/2\pi$ (Haar–measure of $\widetilde{\mathsf{U}}(1)$) induces then the measure $\mathrm{d}\,\overline{\mu}(\varrho)$ in $M_+^1(E_\beta)$.

3.3 Proposition

(i) The decomposition (3.6) is the (unique) central decomposition of ω_0^β. Especially we find by the spatial decomposition theory for the GNS–representations $(\Pi^0_\beta, \mathcal{H}^0_\beta, \Omega^0_\beta)$ of ω_0^β and $(\Pi^0_\varrho, \mathcal{H}^0_\varrho, \Omega^0_\varrho)$ of $\bigotimes_{k\in\mathbb{N}} \varrho$:

$$(\Pi^0_\beta, \mathcal{H}^0_\beta, \Omega^0_\beta) = \int_{E_\beta}^{\oplus} (\Pi^0_\varrho, \mathcal{H}^0_\varrho, \Omega^0_\varrho)\,\mathrm{d}\,\overline{\mu}(\varrho)$$

with the corresponding decomposition of the associated von Neumann algebra

$$\mathfrak{M}^0_\beta := \Pi^0_\beta(\mathfrak{A})'' = \int_{E_\beta}^{\oplus} \mathfrak{M}^0_\varrho\,\mathrm{d}\,\overline{\mu}(\varrho), \qquad \mathfrak{M}^0_\varrho := \Pi^0_\varrho(\mathfrak{A})''.$$

(ii) The smallest C^*-algebra $\mathcal{C}_\beta \subset \mathfrak{M}_\beta$ which contains all "mean-field operators"

$$a_\beta := \operatorname*{s-lim}_{\Lambda\in\mathfrak{L}} \Pi^0_\beta(\frac{1}{|\Lambda|}\sum_{k\in\Lambda} a_k), \qquad a\in\mathfrak{B}, \tag{3.10}$$

has the $*$-isomorphic realizations

$$\mathcal{C}_\beta \cong \mathfrak{A}\otimes\mathcal{C}(E_\beta) \cong \mathcal{C}(E_\beta,\mathfrak{A}) \tag{3.11}$$

where $\otimes$ denotes an arbitrary C^*-tensor-product, $\mathcal{C}(E_\beta)$ the continuous complex functions and $\mathcal{C}(E_\beta,\mathfrak{A})$ the continuous $\mathfrak{A}$-valued functions on E_β. $a_k \in \mathfrak{A}_{\{k\}}$ is the embedded $a\in\mathfrak{B}$ and $\text{s-lim}_{\Lambda\in\mathfrak{L}}$ denotes the limit in the strong operator topology on $\mathfrak{B}(\mathcal{H}^0_\beta)$.

Each $\varphi\in\mathcal{F}^\beta_0$, the smallest folium containing ω^β_0, has a unique $\sigma(\mathcal{F}^\beta_0,\mathcal{C}_\beta)$-continuous extension to $\mathcal{C}_\beta$, which is also denoted by φ. For this φ we have the following relations: Setting $\mu_\varphi(f) := \langle\varphi\,;\mathbb{1}\otimes f\rangle$, $f\in\mathcal{C}(E_\beta)$, there is a μ_φ-a.e. unique measurable family $E_\beta \ni \varrho \longrightarrow \varphi_\varrho \in \mathfrak{S}(\mathfrak{A})$ with: For $A\in\mathcal{C}(E_\beta,\mathfrak{A})\cong\mathcal{C}_\beta$ define $\varphi_\varrho\in\mathfrak{S}(\mathcal{C}_\beta)$ by $\langle\varphi_\varrho\,;A\rangle := \langle\varphi_\varrho\,;A(\varrho)\rangle$. Then

$$\langle\varphi\,;A\rangle = \int_{E_\beta}\langle\varphi_\varrho\,;A\rangle\,\mathrm{d}\mu_\varphi(\varrho) = \int_{E_\beta}\langle\varphi_\varrho\,;A(\varrho)\rangle\,\mathrm{d}\mu_\varphi(\varrho), \quad A\in\mathcal{C}(E_\beta,\mathfrak{A})\cong\mathcal{C}_\beta \tag{3.12}$$

is the central decomposition of $\varphi\in\mathfrak{S}(\mathcal{C}_\beta)$ in $\mathcal{C}_\beta$, and

$$\langle\varphi\,;A\rangle = \int_{E_\beta}\langle\varphi_\varrho\,;A\rangle\,\mathrm{d}\,\mu_\varphi(\varrho), \qquad A\in\mathfrak{A} \tag{3.13}$$

is the central decompositions of φ in $\mathfrak{A}$.

PROOF: (i) Obviously (3.6) is the decomposition into permutation invariant product states, which coincides with the central decomposition according to [33]. The rest follows from the spatial decomposition theory on the standard Borel space $\mathfrak{S}(\mathfrak{A})$ (cf. e.g. [34]).

(ii) (3.11) may be derived as in [27]. For $A\in\mathcal{C}_\beta$ there is a net $(A_\Lambda)_{\Lambda\in\mathfrak{L}}$, $A_\Lambda\in\mathfrak{A}_\Lambda$, such that $\sigma(\mathfrak{M}^0_\beta,\mathfrak{M}^0_{\beta*})\text{-lim}_{\Lambda\in\mathfrak{L}} A_\Lambda = A$. Then for $\varphi\in\mathcal{F}^0_\beta\subset\mathfrak{M}^0_{\beta*}$, $\lim_{\Lambda\in\mathfrak{L}}\langle\varphi\,;A_\Lambda\rangle$ exists and defines $\langle\varphi\,;A\rangle$. (3.12) and (3.13) are restrictions of the central decomposition on $\mathfrak{M}^0_\beta$. cf. also [33]. □

In the sense of [33], Eq. (3.13) constitutes a parametrization of the central decomposition in a uniform way for all $\varphi\in\mathcal{F}^0_\beta$.

We study the limiting dynamics (limiting gauge transformations) which are induced by the local Hamiltonians (local particle number operators) $\widetilde{H}^0_\Lambda$, where

$$\widetilde{H}^0_\Lambda \text{ stands for } H^0_\Lambda,\ H^{0r}_\Lambda,\ N_\Lambda \quad \in\mathfrak{A}_\Lambda. \tag{3.14}$$

We use here and in the following the sign $\widetilde{\ }$ to indicate a variable symbol. The $\widetilde{H}^0_\Lambda$ are connected with ϱ-dependent one-particle Hamiltonians $\widetilde{h}^0\in\mathcal{C}(E_\beta,\mathfrak{B})$

$$\widetilde{h}^0(\varrho) \text{ for } h^0(\varrho),\ h^{0r}(\varrho),\ n \quad \in\mathfrak{B}, \tag{3.15}$$

and unitaries

$$\widetilde{u}_t^0(\varrho) := \exp(i\,t\,\widetilde{h}^0(\varrho)).$$

The mentioned connection between $\widetilde{H}_\Lambda^0$ and $\widetilde{h}^0(\varrho)$ is most easily obtained by means of the limiting Heisenberg generator, the first quantity to be determined in a systematic discussion of a mean–field dynamics in a prescribed representation.

We stipulate for the following that $\mathfrak{A}$ be identified with the isomorphic sub-algebra $\Pi_\beta^0(\mathfrak{A}) \subset \mathcal{C}_\beta \subset \mathfrak{B}(\mathcal{H}_\beta^0)$. Then one obtains

$$\operatorname*{s-lim}_{\Lambda' \in \mathfrak{L}} [\widetilde{H}_{\Lambda'}^0, A] = [\widetilde{H}_\Lambda^{\beta 0}, A], \qquad A \in \mathfrak{A}_\Lambda,$$

with $\widetilde{H}_\Lambda^{\beta 0} \in \mathcal{C}(E_\beta, \mathfrak{A})$ and $\widetilde{H}_\Lambda^{\beta 0}(\varrho) = \sum_{k\in\Lambda} \widetilde{h}_k^0(\varrho)$ for $\varrho \in E_\beta$. $\widetilde{h}_k^0(\varrho)$ is the embedding of $\widetilde{h}^0(\varrho)$ at site k in $\mathfrak{A}$. For $h^{0r}(\varrho)$ we find the same one–particle Hamiltonian as introduced in Prop. 3.1 and $h^0(\varrho)$ becomes $h^0(\varrho) := h^{0r}(\varrho) + \mu\, n$.

Each $\widetilde{h}^0 = h^0$, h^{0r}, n gives rise to a flow $\widetilde{\gamma}_t^0 = \gamma_t^0$, γ_t^{0r}, γ_t^κ on E_β, which is the solution of

$$-i\,\frac{\mathrm{d}}{\mathrm{d}\,t}\,\langle \widetilde{\gamma}_t^0 \varrho\,; a\rangle = \left\langle \widetilde{\gamma}_t^0 \varrho\,; [\widetilde{h}^0(\widetilde{\gamma}_t^0 \varrho), a]\right\rangle .$$

E_β parametrizes grand canonical equilibrium states, so that $\gamma_t^{0r}\varrho$ commutes with $h^{0r}(\gamma_t^{0r}\varrho)$. Thus $\gamma_t^{0r} = \mathrm{id}$. Further $\gamma_t^0\varrho = \exp(it\mu\, n)\,\varrho\,\exp(-it\mu\, n)$ since $h^{0r}(\varrho) = h^0(\varrho) - \mu\, n$, and $\gamma_t^\kappa\varrho = \exp(-it\, n)\,\varrho\,\exp(it\, n)$.

3.4 Proposition

For each net of local Hamiltonians (gauge transformations) $\widetilde{H}_\Lambda^0$, $\Lambda \in \mathfrak{L}$, there is a unique C^*-dynamical system $(\mathcal{C}_\beta, \mathbb{R}, \widetilde{\tau}^{\beta 0})$, $\widetilde{\tau}^{\beta 0}$ for $\tau^{\beta 0}$, $\tau^{\beta 0r}$, κ^β such that for $A \in \mathcal{C}(E_\beta, \mathfrak{A}) \cong \mathcal{C}_\beta$:

$$\widetilde{\tau}_t^{\beta 0}(A) = \operatorname*{s-lim}_{\Lambda\in\mathfrak{L}} e^{it\,\widetilde{H}_\Lambda^0} A_\Lambda e^{-it\,\widetilde{H}_\Lambda^0}, \quad \textit{for all } t \in \mathbb{R}, \tag{3.16}$$

where $\Lambda \longrightarrow A_\Lambda \in \mathfrak{A}_\Lambda$ is a quasi–symmetric net with $\operatorname*{s-lim}_{\Lambda\in\mathfrak{L}} A_\Lambda = A$ [1]. $\widetilde{\tau}_t^{\beta 0}$ is given by

$$\widetilde{\tau}_t^{\beta 0}(A)(\varrho) = \alpha_{\widetilde{u}_t^0(\varrho)} A(\widetilde{\gamma}_t^0 \varrho), \qquad \varrho \in E_\beta, \tag{3.17}$$

where α_u, $u \in \mathsf{U}(\mathfrak{B})$, is from (2.12). It holds

$$\tau_t^{\beta 0} = \tau_t^{\beta 0r} \circ \kappa_{\mu t}^\beta = \kappa_{\mu t}^\beta \circ \tau_t^{\beta 0r}. \tag{3.18}$$

Proof: (3.16) and (3.17) follow from a combination of [1] with [22, Prop. 4.2]. See also [26], [27]. (3.18) is a consequence of (3.17) and the remarks before Prop. 3.4. □

The considered flows $\widetilde{\gamma}_t^0$ on the differentiable manifold E_β, which is homeomorphic to the torus $\mathbb{T}$, are obviously differentiable

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\widetilde{\gamma}_t^0\varrho = -i[\widetilde{h}^0(\widetilde{\gamma}_t^0\varrho), \widetilde{\gamma}_t^0\varrho] \ \in \mathfrak{B}^*, \quad \text{for all } \varrho \in E_\beta, \tag{3.19}$$

where $\mathfrak{B}^*$ contains the tangent spaces of E_β. Eq. (3.19) gives rise to the vector field

$$\widetilde{\lambda}^0\varrho := -i[\widetilde{h}^0(\varrho), \varrho], \quad \text{for all } \varrho \in E_\beta. \tag{3.20}$$

Using the phase angle parametrization of (3.6), (3.7), we have e.g.

$$\gamma^\kappa_{\vartheta'}\varrho^\vartheta = \varrho^{\vartheta+2\vartheta'} = e^{-i\vartheta' n}\varrho^\vartheta e^{i\vartheta' n} \tag{3.21}$$

and

$$\lambda^\kappa\varrho^\vartheta = -i[n, \varrho] = 2\,\frac{\mathrm{d}}{\mathrm{d}\vartheta}\varrho^\vartheta. \tag{3.22}$$

For a one-times differentiable function $f \in \mathcal{C}^1(E_\beta)$ we introduce

$$(\widetilde{\lambda}^{0*}f)(\varrho) := \frac{d}{dt}\widetilde{\gamma}^{0*}_t f(\varrho)|_{t=0} = \frac{d}{dt}f(\widetilde{\gamma}^0_t\varrho)|_{t=0} =: \left\langle \widetilde{\lambda}^0\varrho\,; \mathrm{d}f(\varrho)\right\rangle \tag{3.23}$$

where the total differential $\mathrm{d}f(\varrho)$ may be realized by an element in $\mathfrak{B}^{**} = \mathfrak{B}$ (which is not unique). $\widetilde{\lambda}^{0*}$ extends to $\mathcal{C}^1_\beta := \mathfrak{A} \otimes \mathcal{C}^1(E_\beta) \cong \mathcal{C}^1(E_\beta, \mathfrak{A})$ by

$$[\widetilde{\lambda}^{0*}A](\varrho) := \frac{d}{dt}A(\widetilde{\gamma}^0_t\varrho)|_{t=0}, \quad \text{for all } \varrho \in E_\beta. \tag{3.24}$$

Let us further introduce $\mathcal{C}^1_{\beta,\Lambda} := \mathfrak{A}_\Lambda \otimes \mathcal{C}^1(E_\beta) \cong \mathcal{C}^1(E_\beta, \mathfrak{A}_\Lambda)$ and $\mathcal{C}^1_{\beta,0} := \bigcup_{\Lambda\in\mathfrak{L}} \mathcal{C}^1_{\beta,\Lambda}$.[§]

3.5 Proposition
Let be $\widetilde{\tau}^{\beta 0}_t = \exp(it\widetilde{L}^{\beta 0})$ the $*$-automorphisms of Prop. 3.4. Then $\mathcal{C}^1_{\beta,0}$ is a core for the corresponding generators $\widetilde{L}^{\beta 0}$ and for $A \in \mathcal{C}^1(E_\beta, \mathfrak{A}_\Lambda) \cong \mathcal{C}^1_{\beta,\Lambda}$ one has

$$[\widetilde{L}^{\beta 0}A](\varrho) = [\widetilde{H}^{\beta 0}_\Lambda(\varrho), A(\varrho)] - i\,[\widetilde{\lambda}^{0*}A](\varrho). \tag{3.25}$$

PROOF: The form of (3.25) follows from Eq. (3.17) by differentiation. Since the $\widetilde{h}^0$ and $\widetilde{u}^0_t$ are in $\mathcal{C}^1_\beta$, $\widetilde{\tau}^{\beta 0}_t$ leaves $\mathcal{C}^1_\beta$ invariant. By the product structure of α_u, each $\mathcal{C}^1_{\beta,\Lambda}$ is left invariant. Thus $\mathcal{C}^1_{\beta,0}$ is a core for $\widetilde{L}^{\beta 0}$ [34]. □

The minimizing set $\mathfrak{S}^0_\beta(\mathfrak{A})$ in Prop. 3.1 is in $\mathfrak{S}(\mathfrak{A})$ whereas the dynamics $(\mathcal{C}_\beta, \mathbb{R}, \widetilde{\tau}^{\beta 0}_t)$ acts in $\mathcal{C}_\beta$. By means of Eq. (3.12) we extend the states in $\mathfrak{S}^0_\beta(\mathfrak{A})$ to states on $\mathcal{C}_\beta$ and denote the corresponding minimizing set by $\mathfrak{S}^0_\beta(\mathcal{C}_\beta)$, in spite of the elements in $\mathfrak{S}^0_\beta(\mathfrak{A})$ not all being in $\mathcal{F}^0_\beta \subset \mathcal{F}^{\mathsf{P}}(\mathfrak{A})$.

3.6 Proposition
The set $\mathfrak{S}^0_\beta(\mathcal{C}_\beta)$ (affine homeomorphic to $\mathfrak{S}^0_\beta(\mathfrak{A})$) is homeomorphic to $M^1_+(E_\beta)$ by means of

$$\omega_\mu := \int_{E_\beta} \omega^\varrho_0\,\mathrm{d}\,\mu(\varrho), \qquad \mu \in M^+_1(E_\beta),$$

[§]We renounce in our simple cases to introduce total differentials and Poisson brackets for functions in $\mathcal{C}^1_{\beta,0}$.

which is the central decomposition on $\mathcal{C}_\beta$. $\mathfrak{S}^0_\beta(\mathcal{C}_\beta)$ is (therefore) a Bauer simplex with compact extremal boundary $\partial_e\mathfrak{S}^0_\beta(\mathcal{C}_\beta) = \{\omega^\varrho_0 \mid \varrho \in E_\beta\}$.

$\mathfrak{S}^0_\beta(\mathcal{C}_\beta)$ consists of all β–KMS–states of $(\mathcal{C}_\beta, \mathbb{R}, \tau^{\beta 0r})$ (which minimize $f_0(\beta, \cdot)$ of Proposition 3.1) and is a face in $\mathfrak{S}(\mathcal{C}_\beta)$. Its elements are called "stable thermal phases at β".

PROOF: There is a bi–unique reduction of $\omega^\beta_0 \in \mathfrak{S}^0_\beta(\mathcal{C}_\beta)$ to a state on $\mathfrak{A}$ which is affine homeomorphic to $M^1_+(E_\beta)$ according to Prop. 3.3 (ii). By direct calculation every $\omega^\varrho_0 \in \partial_e\mathfrak{S}^0_\beta(\mathcal{C}_\beta)$ satisfies the β–KMS–condition for $\tau^{\beta 0r}$ [29] and by convex superposition so do all ω_μ, $\mu \in M^1_+(E_\beta)$. Let be ω a β–KMS–state to $(\mathcal{C}_\beta, \mathbb{R}, \tau^{\beta 0r})$. Then it has the central decomposition $\omega = \int_{E_\beta} \omega^\varrho \,\mathrm{d}\,\mu(\varrho)$ [35], [36], where $\omega^\varrho \in \mathfrak{S}(\mathfrak{A})$ and $\omega^\varrho|_{\mathfrak{A}_{\{k\}}}$ sees the dynamics with $h^{0r}_k(\varrho) \cong h^{0r}(\varrho)$. But $\omega^\varrho|_{\mathfrak{A}_{\{k\}}}$ is then the unique KMS–state to this dynamics for all $k \in \mathbb{N}$. It has the form (3.3) and minimizes the free energy. □

4 Equilibrium Dynamics and KMS–States of the Inhomogeneous Model

Similar to the homogeneous case we extract the first information on the inhomogeneous equilibrium dynamics from the limiting Heisenberg generator acting on local observables. In the spirit of a perturbation theory we employ the homogeneous representation space $\mathcal{H}^0_\beta$ resp. the "homogeneous C^*–algebra" $\mathcal{C}_\beta \subset \mathfrak{B}(\mathcal{H}^0_\beta)$. The weak topologies w–, s–, σ-w– refer to this representation. Thus we study for $A \in \mathfrak{A}_\Lambda$

$$\operatorname*{s-lim}_{\Lambda' \in \mathfrak{L}} [H^{(r)}_{\Lambda'}, A] = \operatorname*{s-lim}_{\Lambda' \in \mathfrak{L}} [H^{0(r)}_{\Lambda'}, A] + \operatorname*{s-lim}_{\Lambda' \in \mathfrak{L}} [P_{\Lambda'}, A]$$

where $H^{(r)}_\Lambda$, $H^{0(r)}_\Lambda$, and P_Λ are from Eqns. (2.5), (2.7), and (2.8), respectively, using (3.1), (3.2).

4.1 Lemma
Under the model Assumption 2.1 it holds for $A \in \mathfrak{A}_\Lambda$, $\Lambda \in \mathfrak{L}$,

$$\operatorname*{s-lim}_{\Lambda' \in \mathfrak{L}} [P_{\Lambda'}, A] = [P^\beta_\Lambda, A], \tag{4.1}$$

where

$$P^\beta_\Lambda = \sum_{k \in \Lambda} \delta h_k \in \mathcal{C}(E_\beta, \mathfrak{A}_\Lambda) \subset \mathcal{C}(E_\beta, \mathfrak{A}) \cong \mathcal{C}_\beta \tag{4.2}$$

with

$$\delta h_k(\varrho) = \delta\varepsilon_k n_k - \delta g_k \langle \varrho\,; b \rangle\, b^*_k - \overline{\delta g_k} \langle \varrho\,; b^* \rangle\, b_k. \tag{4.3}$$

PROOF: First we consider the case $\delta g_{kk'} = \delta g_k + \overline{\delta g_{k'}}$ for the inhomogeneities in (2.9). δg_k has to be chosen according to the model Assumption 2.1. For $A \in \mathfrak{A}_\Lambda$ and $\Lambda' \supseteq \Lambda$ we have

$$[P_{\Lambda'}, A] = \Big[\sum_{k \in \Lambda} \delta\varepsilon_k\, n_k - \frac{1}{|\Lambda'|}(\sum_{k \in \Lambda'} \delta g_k\, b^*_k)\,(\sum_{k' \in \Lambda'} b_{k'}) - \frac{1}{|\Lambda'|}(\sum_{k \in \Lambda'} b^*_k)\,(\sum_{k' \in \Lambda'} \overline{\delta g_{k'}}\, b_{k'}), A\Big]$$

$$
\begin{aligned}
= & \Big[\sum_{k\in\Lambda} \delta\varepsilon_k\, n_k, A\Big] \\
& - \Big[\sum_{k\in\Lambda} \delta g_k\, b_k^*, A\Big] \Big(\frac{1}{|\Lambda'|}\sum_{k'\in\Lambda'} b_{k'}\Big) - \Big(\frac{1}{|\Lambda'|}\sum_{k\in\Lambda'} \delta g_k\, b_k^*\Big) \Big[\sum_{k'\in\Lambda} b_{k'}, A\Big] \\
& - \Big[\sum_{k\in\Lambda} b_k^*, A\Big] \Big(\frac{1}{|\Lambda'|}\sum_{k'\in\Lambda'} \overline{\delta g_{k'}}\, b_{k'}\Big) - \Big(\frac{1}{|\Lambda'|}\sum_{k\in\Lambda'} b_k^*\Big) \Big[\sum_{k'\in\Lambda} \overline{\delta g_{k'}}\, b_{k'}, A\Big] .
\end{aligned}
$$

Now use $\|\cdot\|$-$\lim_{\Lambda'\in\mathfrak{L}} \frac{1}{|\Lambda'|}\sum_{k\in\Lambda'} \delta g_k\, b_k^* = 0$ (since $\delta g_k \overset{k\to\infty}{\longrightarrow} 0$) and s-$\lim_{\Lambda'\in\mathfrak{L}} \frac{1}{|\Lambda'|}\sum_{k\in\Lambda'} b_k^* =: b_\beta^*$ with $b_\beta^*(\varrho) = \langle\varrho\,;b^*\rangle\,\mathbb{1}$, comp. (3.10). Thus we have

$$
\begin{aligned}
\text{s-}\lim_{\Lambda'\in\mathfrak{L}}[P_{\Lambda'}, A] &= \Big[\sum_{k\in\Lambda} \delta\varepsilon_k\, n_k, A\Big] - \Big[\sum_{k\in\Lambda} \delta g_k\, b_k^*, A\Big]\, b_\beta - b_\beta^* \Big[\sum_{k'\in\Lambda} \overline{\delta g_{k'}}\, b_{k'}, A\Big] \\
&= \Big[\sum_{k\in\Lambda} \big(\delta\varepsilon_k\, n_k - \delta g_k\, b_k^*\, b_\beta - \overline{\delta g_k}\, b_k\, b_\beta^*\big)\, , A\Big] .
\end{aligned}
$$

This proves Eqns. (4.1), (4.2), and with $b_\beta^*(\varrho) = \langle\varrho\,;b^*\rangle\,\mathbb{1}$ it follows Eq. (4.3).

Now we consider the case of P_Λ with arbitrary $g_{kk'}$ according to the model Assumption 2.1 and choose δg_k as $\lim_{k'\to\infty} g_{kk'}$. Then we have

$$
\begin{aligned}
0 &\leq \lim_{\Lambda'\in\mathfrak{L}} \Big\|[P_{\Lambda'}, A] - \Big[\sum_{k\in\Lambda} \delta\varepsilon_k\, n_k - \frac{1}{|\Lambda'|}\sum_{k,k'\in\Lambda'} \big(\delta g_k + \overline{\delta g_{k'}}\big)\, b_k^* b_{k'}, A\Big]\Big\| \\
&= \lim_{\Lambda'\in\mathfrak{L}} \Big\|\Big[\frac{1}{|\Lambda'|}\sum_{k,k'\in\Lambda'} \big(\delta g_k + \overline{\delta g_{k'}} - g_{kk'}\big)\, b_k^* b_{k'}, A\Big]\Big\| \\
&\leq \lim_{\Lambda'\in\mathfrak{L}} \frac{2}{|\Lambda'|}\sum_{k,k'\in\Lambda'} \big|\delta g_k + \overline{\delta g_{k'}} - g_{kk'}\big|\, \|b^*\|\, \|b\|\, \|A\| = 0
\end{aligned}
$$

with Assumption (2.11). Eqns. (4.1)–(4.3) then follow immediately. □

The inhomogeneous coefficients $\delta\varepsilon_k$ and $\delta g_{kk'}$ in $P_{\Lambda'}$ make the handling of iterated commutators incomparably harder than for homogeneous commutators. Nevertheless we may announce a structure similar to the homogeneous model, the proof of which we indicate in the Appendix.

For this we introduce the one-particle Hamiltonians $h_k^{(r)} \in \mathcal{C}(E_\beta, \mathfrak{A}_{\{k\}})$ by

$$
h_k^{(r)} := h_k^{0(r)} + \delta h_k \tag{4.4}
$$

with $h_k^{0(r)}$ from (3.15) and δh_k from (4.3).

For an automorphism group τ_t in $\mathcal{C}_\beta$ and a bounded selfadjoint operator $P \in \mathcal{C}_\beta$ we denote by $(\tau_t)^P$ the perturbed automorphism group (cf. e.g. [13]).

4.2 Theorem

(i) For each BCS–model satisfying Assumptions 2.1 and for each $\beta > 0$ ($\beta > \beta_c(\mu)$, Eq. (3.4), and $\mu \in \mathbb{R}$ fixed) there is a unique C^*–dynamical system $(\mathcal{C}_\beta, \mathbb{R}, \tau^{\beta(r)})$ such that for each $A \in \mathcal{C}_{\beta,\Lambda}$ and $\Lambda \in \mathfrak{L}$ there is a $t_0 > 0$ with

$$\tau_t^{\beta(r)}(A) = \sigma\text{-w-}\lim_{\Lambda' \in \mathfrak{L}}(\tau_t^{\beta 0(r)})^{P_{\Lambda'}}(A) = (\tau_t^{\beta 0(r)})^{P_\Lambda^\beta}(A) \qquad \textit{for } |t| < t_0.$$

For arbitrary $A \in \mathcal{C}(E_\beta, \mathfrak{A}) \cong \mathcal{C}_\beta$, $\tau_t^{\beta(r)}(A)$ writes as

$$[\tau_t^{\beta(r)}(A)](\varrho) = (\bigotimes_{k\in\mathbb{N}} e^{ith_k^{(r)}(\varrho)})\, A(\gamma_t^{0(r)}\varrho)\, (\bigotimes_{k\in\mathbb{N}} e^{-ith_k^{(r)}(\varrho)}) \tag{4.5}$$

with

$$\varrho \to h_k(\varrho) = \varepsilon_k\, n_k - (g + \delta g_k)\, \langle \varrho\,; b\rangle\, b_k^* - (g + \overline{\delta g_k})\, \langle \varrho\,; b^*\rangle\, b_k \quad \in \mathcal{C}(E_\beta, \mathfrak{A}_{\{k\}}),$$

and $h_k^r = h_k - \mu\, n_k$.

(ii) It holds

$$\tau_t^\beta = \tau_t^{\beta r} \circ \kappa_{\mu t}^\beta = \kappa_{\mu t}^\beta \circ \tau_t^{\beta r}.$$

(iii) Denoting

$$H_\Lambda^{\beta(r)} := \sum_{k\in\Lambda} h_k^{(r)} \quad \in \mathcal{C}(E_\beta, \mathfrak{A}_\Lambda) \cong \mathcal{C}_{\beta,\Lambda},$$

the generator $L^{\beta(r)}$ of $\tau_t^{\beta(r)}$ has on the core $\mathcal{C}_{\beta,0}^1$ the form

$$L^{\beta(r)}A(\varrho) = [H_\Lambda^{\beta(r)}(\varrho), A(\varrho)] - i\lambda^{0(r)*}A(\varrho), \quad A \in \mathcal{C}^1(E_\beta, \mathfrak{A}_\Lambda) \cong \mathcal{C}_{\beta,\Lambda}^1.$$

PROOF: Appendix. □

If the asymptotic behaviour of $\Lambda \to P_\Lambda$ is more restrictive (P_Λ could be a quasi–symmetric net), the C^*–dynamical system $\tau_t^{\beta(r)}$ of the inhomogeneous model can be obtained as the thermodynamic limit of the local dynamics as in Prop. 3.4 for the homogeneous model. For this treatment and more technical details we refer to a future work.

Let us here remember the numerical parametrization of E_β in terms of $\vartheta(\varrho) = \mathrm{Arg}\,\langle \varrho\,; b^*\rangle$. Together with

$$\Delta_k := \left|(1 + \frac{\delta g_k}{g})\, g\, \langle \varrho\,; b\rangle\right| = \left|1 + \frac{\delta g_k}{g}\right| \Delta_0,$$

and Δ_0 from Eq. (2.8), we obtain

$$h_k(\varrho) = h_k^\vartheta = \varepsilon_k n_k - \left[\Delta_k e^{-i(\vartheta+\delta\vartheta_k)} b_k^* + \Delta_k e^{i(\vartheta+\delta\vartheta_k)} b_k\right]$$

where

$$\delta\vartheta_k := -\mathrm{Arg}\left(1 + \frac{\delta g_k}{g}\right)$$

is a microscopic fluctuation around the macroscopic phase ϑ.

Since we are looking for the grand canonical equilibrium states, we have a more detailed look at $H_\Lambda^{\beta r} \in \mathcal{C}(E_\beta, \mathfrak{A}_\Lambda) \cong \mathcal{C}_{\beta,\Lambda}$ with

$$H_\Lambda^{\beta r}(\varrho) = H_\Lambda^{\beta r \vartheta(\varrho)} = \sum_{k\in\Lambda} \left\{ (\varepsilon_k - \mu) n_k - \Delta_k [e^{-i(\vartheta+\delta\vartheta_k)} b_k^* + e^{i(\vartheta+\delta\vartheta_k)} b_k] \right\}. \tag{4.6}$$

The corresponding automorphisms $\tau_t^{\beta r}$ arise with the flow $\gamma_t^{0r*} = \mathrm{id}$ and leave the center of $\mathcal{C}_\beta$ invariant.

4.3 Theorem

(i) The extremal β–KMS–states ω^ϱ for the C^*–dynamical system $(\mathcal{C}_\beta, \mathbb{R}, \tau^{\beta r})$ are indexed with $\varrho \in E_\beta$ and are locally given as $\omega_\Lambda^\varrho := \omega^\varrho|_{\mathcal{C}_{\beta,\Lambda}}$

$$\langle \omega_\Lambda^\varrho ; A \rangle = \mathrm{tr}_\Lambda \left(e^{-\xi_\Lambda - \beta H_\Lambda^{\beta r}(\varrho)} A(\varrho) \right), \quad A \in \mathcal{C}(E_\beta, \mathfrak{A}_\Lambda) \cong \mathcal{C}_{\beta,\Lambda}. \tag{4.7}$$

$\omega^\beta = \int_{E_\beta} \omega^\varrho \,\mathrm{d}\,\overline{\mu}(\varrho)$ is the unique gauge invariant β–KMS state.

(ii) Replace in ω_0^ϑ of (3.7) ϑ by $\varrho \in E_\beta$ with $\vartheta = \mathrm{Arg}\,\langle \varrho ; b^* \rangle$ and extend this state ω_0^ϱ to $\mathcal{C}_\beta$ in the way of Prop. 3.3 (ii) leading to the restrictions $\omega_{0\Lambda}^\varrho$ on $\mathcal{C}_{\beta,\Lambda}$ for all $\Lambda \in \mathfrak{L}$. Then it holds

$$\omega_\Lambda^\varrho = (\omega_{0\Lambda}^\varrho)^{P_\Lambda^\beta(\varrho)} = (\omega_{0\Lambda}^\varrho)^{P_\Lambda^\beta}, \qquad \text{for all } \Lambda \in \mathfrak{L}, \tag{4.8}$$

where the perturbed KMS–state is defined in the sense of [13, Corollary 5.4.5]. It holds $(\omega_{0\Lambda}^\varrho)^{P_\Lambda^\beta} = (\omega_{0\Lambda}^\varrho)^{P_\Lambda^\beta(\varrho)}$, with $\omega_{0\Lambda}^\varrho$ on the right hand side as state on $\mathfrak{A}$ and $(\omega_{0\Lambda}^\varrho)^{P_\Lambda^\beta(\varrho)}$ extended to $\mathcal{C}_{\beta,\Lambda}$. Varying $\varrho \in E_\beta$ one obtains a homeomorphism between $\partial_e \mathfrak{S}_\beta^0(\mathcal{C}_\beta)$ and $\partial_e \mathfrak{S}_\beta(\mathcal{C}_\beta)$, which expresses a stability of the (pure) phase structure against the considered singular perturbations.

(iii) The set of all β–KMS–states for $(\mathcal{C}_\beta, \mathbb{R}, \tau^{\beta r})$ constitutes a Bauer simplex $\mathfrak{S}_\beta(\mathcal{C}_\beta)$ which is affine homeomorphic to $M_+^1(E_\beta)$.

(iv) Consider $\omega \in \partial_e \mathfrak{S}_\beta(\mathcal{C}_\beta)$ as state on $\mathfrak{A}$. Then the free energy density $f(\beta,\omega)$ exists and it holds $f(\beta,\omega) = f_0(\beta) = f_0(\beta,\omega)$ (see Prop. 3.1).

PROOF: (i) Clearly Eq. (4.7) defines extremal β–KMS–states. If ω is extremal β–KMS, then it is a factor state. But it must have the form

$$\omega = \int_{E_\beta} \varphi^\varrho \,\mathrm{d}\,\mu(\varrho), \qquad \varphi^\varrho \in \mathfrak{S}(\mathcal{C}_\beta),$$

cf. Eq. (3.12), and [35], [36]. A necessary condition is, that ω is pure on the center of $\mathcal{C}_\beta$ and thus $\mathrm{d}\,\mu(\varrho') = \delta(\varrho - \varrho')\mathrm{d}\,\varrho'$. It sees locally $H_\Lambda^{\beta r}(\varrho)$ and has necessarily the form (4.7).

Now use the parametrization $\vartheta \in [0, 2\pi[$ of E_β. For $\omega^{\varrho_0^\vartheta} \in \partial_e \mathfrak{S}_\beta(\beta)$ it holds $\omega^{\varrho_0^\vartheta} \circ \kappa_{\vartheta'}^\beta = \omega^{\varrho_0^{\vartheta+2\vartheta'}}$. This implies the gauge invariance and the uniqueness of ω^β.

(ii) Eq. (4.8) is calculated in terms of the density matrices. Since $P_{\Lambda'}|_{\mathcal{C}_{\beta,\Lambda}} = P_\Lambda$, $\Lambda \subseteq \Lambda'$, it defines a mapping of ω_0^ϱ onto ω^ϱ for all $\varrho \in E_\beta$, which is clearly bi–unique and w^*–continuous.

(iii) Obvious.

(iv) The restriction of a factor state $\omega^\varrho \in \partial_e\mathfrak{S}_\beta(\mathcal{C}_\beta)$ to $\mathfrak{A}$ is given by the product state (use (4.6) and (4.7))

$$\bigotimes_{k\in\mathbb{N}} \varrho_k \quad \text{with} \quad \varrho_k = \frac{e^{-\beta h_k^r(\varrho)}}{\mathrm{tr}_{\mathfrak{B}}(e^{-\beta h_k^r(\varrho)})}.$$

Then one calculates

$$\begin{aligned}
\left\langle \bigotimes_{k\in\mathbb{N}} \varrho_k \,;\, \frac{H_\Lambda^r}{|\Lambda|}\right\rangle &= \frac{1}{|\Lambda|}\sum_{k\in\Lambda}(\varepsilon_k-\mu)\left(1-\frac{\varepsilon_k-\mu}{E_k}\tanh\left(\frac{\beta E_k}{2}\right)\right) \\
&\quad -\frac{1}{|\Lambda|^2}\sum_{k\neq k'\in\Lambda} g_{kk'}\,\frac{\Delta_k}{2E_k}e^{i\delta\vartheta_k}\tanh\left(\frac{\beta E_k}{2}\right)\frac{\Delta_{k'}}{2E_{k'}}e^{-i\delta\vartheta_{k'}}\tanh\left(\frac{\beta E_{k'}}{2}\right) \\
&\quad -\frac{1}{|\Lambda|^2}\sum_{k\in\Lambda} g_{kk}\,\langle \varrho_k \,;\, b_k^* b_k\rangle\,,
\end{aligned} \tag{4.9}$$

with $E_k = \sqrt{(\varepsilon_k-\mu)^2+\Delta_k^2}$. Now use that $\langle \varrho_k\,;\, b_k^*b_k\rangle$ is uniformly bounded and condition (2.11) in our model assumption. Thus the last term in Eq. (4.9) vanishes in the thermodynamic limit. Again using (2.11) and the convergences $\lim_{k\to\infty}\Delta_k = \Delta_0$, $\lim_{k\to\infty}\varepsilon_k = \varepsilon$, $\lim_{k\to\infty}\delta g_k = 0$, $\lim_{k\to\infty}\delta\vartheta_k = 0$, and $\lim_{k\to\infty} E_k = E_0 = \sqrt{(\varepsilon-\mu)^2+\Delta_0^2}$, we find for the (net) limit:

$$\begin{aligned}
\lim_{\Lambda\in\mathfrak{L}}\left\langle \bigotimes_{k\in\mathbb{N}} \varrho_k \,;\, \frac{H_\Lambda^r}{|\Lambda|}\right\rangle &= (\varepsilon-\mu)\left(1-\frac{\varepsilon-\mu}{E_0}\tanh\left(\frac{\beta E_0}{2}\right)\right) - g\left(\frac{\Delta_0}{2E_0}\tanh\left(\frac{\beta E_k}{2}\right)\right)^2 \\
&= \lim_{\Lambda\in\mathfrak{L}}\left\langle \bigotimes_{k\in\mathbb{N}} \varrho \,;\, \frac{H_\Lambda^{0r}}{|\Lambda|}\right\rangle .
\end{aligned} \tag{4.10}$$

We find for the entropy density of the factor state $\bigotimes_{k\in\mathbb{N}} \varrho_k \in \partial_e\mathfrak{S}_\beta(\mathcal{C}_\beta)$ (as state on $\mathfrak{A}$):

$$\begin{aligned}
\lim_{\Lambda\in\mathfrak{L}}\frac{1}{|\Lambda|}\mathrm{tr}_\Lambda\left(\bigotimes_{k\in\Lambda}\varrho_k \ln \bigotimes_{k\in\Lambda}\varrho_k\right) &= \lim_{\Lambda\in\mathfrak{L}}\frac{1}{|\Lambda|}\sum_{k\in\mathfrak{L}}\mathrm{tr}_{\mathfrak{B}}(\varrho_k\ln\varrho_k) = \mathrm{tr}_{\mathfrak{B}}(\varrho\ln\varrho) \\
&= \lim_{\Lambda\in\mathfrak{L}}\frac{1}{|\Lambda|}\mathrm{tr}_\Lambda\left(\varrho_\Lambda^{\omega_0^\varrho}\ln\varrho_\Lambda^{\omega_0^\varrho}\right),
\end{aligned}$$

since $\lim_{k\to\infty}\varrho_k = \varrho$. Now the first equality in (iv) follows with Prop. 3.1. The second one follows in the same way by replacing H_Λ^r with H_Λ^{0r} in (4.9) and (4.10). □

In Propositions 3.1 and 3.2 we have determined the limiting Gibbs states of the homogeneous BCS–model with the help of the minimum principle for the free energy density as a functional on $\mathfrak{S}^{\mathsf{P}}(\mathfrak{A})$. (iv) demonstrates that the free energy alone is not suited to identify specific KMS–states (or even limiting Gibbs states), because there are arbitrarily many states $\omega \notin \mathfrak{S}^{\mathsf{P}}(\mathfrak{A})$ or $\omega \in \mathfrak{S}^{\mathsf{P}}(\mathfrak{A})$ with the same free energy density $f_0(\beta)$. For the inhomogeneous model the permutation symmetry of the limiting Gibbs states (KMS–states) is lost and there is no obvious domain where the free energy density has to be varied.

4.4 Theorem
Denoting

$$E_k := \sqrt{(\varepsilon_k - \mu)^2 + \Delta_k^2}, \quad \text{and } \vartheta_k = \vartheta + \delta\vartheta_k$$

for all $k \in \mathcal{K}$ the complex gap parameters $\Delta_k e^{-i\vartheta_k}$ of the extremal β–KMS–state states satisfy the "self–consistency equations"

$$\lim_{\Lambda \in \mathfrak{L}} \frac{1}{|\Lambda|} \sum_{k \in \Lambda} g_{lk} \frac{\Delta_k}{2E_k} e^{-i\vartheta_k} \tanh\left(\frac{\beta E_k}{2}\right) = \Delta_l \, e^{-i\vartheta_l} \tag{4.11}$$

PROOF: An elementary calculation shows for $\varrho_k^\vartheta = \exp(-\xi - \beta h_k^r(\varrho_0^\vartheta))$, $\varrho_0^\vartheta \in E_\beta$:

$$\begin{aligned}
\lim_{\Lambda \in \mathfrak{L}} \left\langle \otimes_{k \in \mathbb{N}} \varrho_k^\vartheta \, ; \sum_{k \in \Lambda} b_k \right\rangle &= \lim_{\Lambda \in \mathfrak{L}} \frac{1}{|\Lambda|} \sum_{k \in \Lambda} \frac{\Delta_k}{2\, E_k} e^{-i\vartheta_k} \tanh\left(\frac{\beta E_k}{2}\right) \\
&= \frac{\Delta_0}{2\, E_0} e^{-i\vartheta} \tanh\left(\frac{\beta E_0}{2}\right) = \frac{\Delta_0}{g} e^{-i\vartheta}.
\end{aligned}$$

Using $\lim_{l\to\infty} g_{lk} = g + \delta g_k$ and $(1 + \frac{\delta g_l}{g})\Delta_0 = \Delta_l e^{-i\delta\vartheta_l}$, it follows

$$\begin{aligned}
\Delta_l e^{-i\vartheta_l} &= \lim_{\Lambda \in \mathfrak{L}} \frac{1}{|\Lambda|} \sum_{k \in \Lambda} (g + \delta g_l) \frac{\Delta_k}{2E_k} e^{-i\vartheta_k} \tanh\left(\frac{\beta E_k}{2}\right) \\
&= \lim_{\Lambda \in \mathfrak{L}} \frac{1}{|\Lambda|} \sum_{k \in \Lambda} g_{lk} \frac{\Delta_k}{2E_k} e^{-i\vartheta_k} \tanh\left(\frac{\beta E_k}{2}\right).
\end{aligned}$$

□

Thus we have demonstrated that in our model class, which is rather large for a perturbation theory, the dynamics and KMS–states may be determined explicitly and have the expected shape. In the special case, that the perturbations constitute a quasi–symmetric net [1] one can employ [22] to derive that the unique gauge invariant β–KMS–state ω^β of Theorem 4.3 is the unique limiting Gibbs state. In our general case the corresponding statement is still unproven, in spite of having at hand the minimum principle of the free energy density for extremal KMS–states. The equality of the inhomogeneous and homogeneous free energy density demonstrates how coarse grained the thermodynamic level is in comparison to the quantum statistical equilibrium states.

The self–consistency equation (4.11) for the k–dependent complex gap parameters $\Delta_k e^{-i\vartheta_k}$ are here rigorously formulated and deduced in the thermodynamic limit. They have here solutions which are fixed up to a k–independent, global phase ϑ. The other way round ϑ is obtained as the average of those ϑ_k which correspond to a solution of the gap equation. This makes explicit the collective nature of the macroscopic phase angle.

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Appendix

A Proof of Theorem 4.2

(i) We consider only the special choice of the coupling constants

$$\delta g_{kk'} = \delta g_k + \overline{\delta g_{k'}}, \tag{A.1}$$

i.e. we use the asymptotical form of the $g_{kk'}$. The general case follows with

$$\begin{aligned}\lim_{\Lambda\in\mathfrak{L}} \Big\| P_\Lambda - \Big(\sum_{k\in\Lambda} \delta\varepsilon_k n_k - \frac{1}{|\Lambda|} \sum_{k,k'\in\Lambda} (\delta g_k + \overline{\delta g_{k'}})\, b_k^* b_{k'} \Big) \Big\| \\ = \lim_{\Lambda\in\mathfrak{L}} \Big\| \frac{1}{|\Lambda|} \sum_{k,k'\in\Lambda} (\delta g_{kk'} - \delta g_k - \overline{\delta g_{k'}})\, b_k^* b_{k'} \Big\| = 0\end{aligned} \tag{A.2}$$

according to the Assumption 2.1. Now use [13, Proposition 5.4.1]

$$\left\| \tau_t^P(A) - \tau_t(A) \right\| \leq (e^{|t|\|P\|} - 1)\, \|A\|\,, \qquad P \in \mathcal{C}_\beta$$

for the difference term. Thus it suffices to prove (i) for

$$P_\Lambda = \sum_{k\in\Lambda} \varepsilon_k\, n_k - \frac{1}{|\Lambda|} \sum_{k,k'\in\Lambda} (\delta g_k + \overline{\delta g_{k'}})\, b_k^* b_{k'}.$$

We estimate the perturbation series of $(\tau_t^{\beta 0(r)})^{P_\Lambda}$ for these inhomogeneities and show that all occurring limits can be interchanged. We consider only the case $(\tau_t^{\beta 0})^{P_\Lambda}$. The reduced dynamics $(\tau_t^{\beta 0r})^{P_\Lambda}$ can be treated in the same way.

To simplify the notation, we write:

$$A^t \;:=\; \tau_t^{\beta 0}(A), \quad \text{for } A \in \mathcal{C}_\beta.$$

We divide the rather lengthy proof in a number of Lemmata. Their proofs will be sketched while for details we refer to a future work in a more general context, which includes the present results.

A.1 Lemma
For $A_\Omega \in \mathfrak{A}_\Omega$, $n \in \mathbb{N}$, and t, $t_1,\ldots,t_n \in \mathbb{R}$ it holds

$$\operatorname*{s-lim}_{\Lambda\in\mathfrak{L}} \left[P_\Lambda^{t_n}, \left[\cdots, \left[P_\Lambda^{t_1}, A_\Omega^t\right]\cdots\right]\right] = \left[P_\Omega^{\beta t_n}, \left[\cdots, \left[P_\Omega^{\beta t_1}, A_\Omega^t\right]\cdots\right]\right] \tag{A.3}$$

with $P_\Omega^\beta = \sum_{k\in\Omega} \delta h_k \in \mathcal{C}(E_\beta, \mathfrak{A}_\Omega)$ from Eq. (4.2).

PROOF: One can show that $\left[P_\Lambda^{t_n}, \left[\cdots, \left[P_\Lambda^{t_1}, A_\Omega^t\right]\cdots\right]\right]$ is a fixed polynomial in the following kinds of operators:

$$\begin{aligned} &\frac{1}{|\Lambda|}\sum_{k\in\mathfrak{L}} x_k, && x\in\mathfrak{B}, \\ &\frac{1}{|\Lambda|}\sum_{k\in\mathfrak{L}} a_k\, x_k, && x\in\mathfrak{B} \text{ and } a_k\in\mathbb{C} \text{ with } \lim_{k\to\infty} a_k = 0, \\ &B_\Omega, && B_\Omega\in\mathcal{C}(E_\beta,\mathfrak{A}_\Omega). \end{aligned} \tag{A.4}$$

$\frac{1}{|\Lambda|}\sum_{k\in\mathfrak{L}} x_k$ is uniformly bounded and strongly convergent (Prop. 3.3 (ii)), $\frac{1}{|\Lambda|}\sum_{k\in\mathfrak{L}} a_k\, x_k$ converges in norm to 0 and B_Ω is fixed. Thus s-$\lim_{\Lambda\in\mathfrak{L}}\left[P_\Lambda^{t_n}, \left[\cdots, \left[P_\Lambda^{t_1}, A_\Omega^t\right]\cdots\right]\right]$ exists. The (norm–) limit of a monomial in operators as given in (A.4) vanishes if it contains a term $\frac{1}{|\Lambda|}\sum_{k\in\mathfrak{L}} a_k\, x_k$. Moreover, the commutator of such a monomial with P_Λ vanishes in the limit as well as the commutator of P_Λ with $\frac{1}{|\Lambda|}\sum_{k\in\mathfrak{L}} x_k$. This allows to write down the n–fold commutator $\left[P_\Lambda^{t_n}, \left[\cdots, \left[P_\Lambda^{t_1}, A_\Omega^t\right]\cdots\right]\right]$ up to terms vanishing in norm for large $\Lambda\in\mathfrak{L}$ and (A.3) follows. □

A.2 Lemma

For $A_\Omega\in\mathfrak{A}_\Omega$, $n\in\mathbb{N}$, and $t\in\mathbb{R}$ it holds

$$\sigma\text{-w-}\lim_{\Lambda\in\mathfrak{L}} \int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n \left[P_\Lambda^{t_n}, \left[\cdots, \left[P_\Lambda^{t_1}, A_\Omega^t\right]\cdots\right]\right] = \int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n \left[P_\Omega^{\beta t_n}, \left[\cdots, \left[P_\Omega^{\beta t_1}, A_\Omega^t\right]\cdots\right]\right]. \tag{A.5}$$

PROOF: For each $\omega\in\mathfrak{M}^0_{\beta*}$, we show $\lim_{\Lambda\in\mathfrak{L}}\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n \left\langle\omega\,; [P_\Lambda^{t_n}, [..., [P_\Lambda^{t_1}, A_\Omega^t]...]]\right\rangle = \int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n \left\langle\omega\,; [P_\Omega^{\beta t_n}, [..., [P_\Omega^{\beta t_1}, A_\Omega^t]...]]\right\rangle$. The n–fold commutators in (A.5) are uniformly bounded in norm for all $t, t_1, \ldots, t_n$ and sufficiently large $\Lambda\in\mathfrak{L}$. This allows to apply the dominated convergence Theorem (Lebesgue) which is still valid for the nets indexed by $\mathfrak{L}$ [32, Prop A2.2.3] and we find:

$$\lim_{\Lambda\in\mathfrak{L}}\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n \left\langle\omega\,; \left[P_\Lambda^{t_n}, \left[\cdots, \left[P_\Lambda^{t_1}, A_\Omega^t\right]\cdots\right]\right]\right\rangle = \int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n \lim_{\Lambda\in\mathfrak{L}} \left\langle\omega\,; \left[P_\Lambda^{t_n}, \left[\cdots, \left[P_\Lambda^{t_1}, A_\Omega^t\right]\cdots\right]\right]\right\rangle.$$

The strong and the σ–strong topology coincide on norm bounded sets [37, Lemma II.2.5] and the σ–strong topology is finer than the σ–weak topology. Thus Lemma A.1 implies $\lim_{\Lambda\in\mathfrak{L}}\left\langle\omega\,; [P_\Lambda^{t_n}, [\cdots, [P_\Lambda^{t_1}, A_\Omega^t]\cdots]]\right\rangle = \left\langle\omega\,; [P_\Omega^{\beta t_n}, [..., [P_\Omega^{\beta t_1}, A_\Omega^t]...]]\right\rangle$. □

A.3 Lemma

Let be $n\in\mathbb{N}$, $\Omega\in\mathfrak{L}$, and set

$$C_\Omega := \max\left\{\left\|\sum_{k\in\Omega} b_k^*\right\|, \left\|\sum_{k\in\Omega}\delta\varepsilon_k n_k\right\|, \left\|\sum_{k\in\Omega}\delta g_k b_k^*\right\|, 1, \|n\|, \|b^*\|\,\|\delta\varepsilon_k n\|, \|\delta g_k b^*\| \mid k\in\Omega\right\}.$$

Then it holds:

(i) $$\left\|\left[P_\Lambda^{t_n}, \left[\cdots, \left[P_\Lambda^{t_1}, A_\Omega^t\right]\cdots\right]\right]\right\| \le n!\,(12C_\Omega)^n\, C_\Omega^2\, \|A_\Omega\|,$$

(ii) $$\left\|\int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \left[P_\Lambda^{t_n}, \left[\cdots, \left[P_\Lambda^{t_1}, A_\Omega^t\right]\cdots\right]\right]\right\| \le (12C_\Omega\, t)^n\, C_\Omega^2\, \|A_\Omega\|,$$

(iii) $$\sum_{n=0}^{\infty}\left\|\int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \left[P_\Lambda^{t_n}, \left[\cdots, \left[P_\Lambda^{t_1}, A_\Omega^t\right]\cdots\right]\right]\right\| < \infty, \quad \textit{for } t < \frac{1}{12\,C_\Omega}.$$

PROOF: We only have to prove (i). Then (ii) and (iii) follow immediately.

At first we estimate the number of monomials of operators as in (A.4) in a commutator $[P_\Lambda^{t_n}, [\cdots, [P_\Lambda^{t_1}, A_\Omega^t]\cdots]]$. Using $[A, BC] = B[A, C] + [A, B]C$ to factorize the n-fold commutators, one finds that there are at most $6^n\, n!$ such monomials. Note, that this estimation also counts monomials which vanish in norm for large Λ.

Now we give a norm estimation of such a monomial. For this we have a look at the various terms in such a monomial:

I.) The operators B_Ω in (A.4) are of the form $[M_1, [\cdots, [M_{k_1}, A_\Omega^t]\cdots]]$, $k_1 \le n$, and M_i, $1 \le i \le k_1$, is the time evaluation of some $M \in \{\sum_{k\in\Omega} b_k, \sum_{k\in\Omega} b_k^*, \sum_{k\in\Omega} \overline{\delta g_k} b_k, \sum_{k\in\Omega} \delta g_k b_k^*, \sum_{k\in\Omega} \delta\varepsilon_k n_k\}$. This implies:

$$\|B_\Omega\| \le (2C_\Omega)^{k_1}\, \left\|A_\Omega^t\right\|. \tag{A.6}$$

II.) Now consider a term $\frac{1}{|\Lambda|}\sum_{k\in\mathfrak{L}} x_k$ as in (A.4). The operator $x \in \mathfrak{B}$ is given by $x = [N_1, [\cdots, [N_{k_2}, N_{k_2+1}]\cdots]]$, $k_2 \le n-1$, where $N_i \in \mathfrak{B}$, $1 \le i \le k_2+1$, is the time evaluation of some $N \in \{b, b^*\}$. This implies:

$$\left\|\frac{1}{|\Lambda|}\sum_{k\in\mathfrak{L}} x_k\right\| \le (2C_\Omega)^{k_2+1}. \tag{A.7}$$

III.) Finally we have to specify $\frac{1}{|\Lambda|}\sum_{k\in\mathfrak{L}} a_k\, x_k$, with $\lim_{k\to\infty} a_k = 0$ in (A.4). Here one finds that $\mathfrak{B} \ni x = [N_1, [\cdots, [N_{k_3}, N_{k_3+1}]\cdots]]$, $k_3 \le n-1$, with $N_i \in \mathfrak{B}$, $1 \le i \le k_3+1$, as the time evaluation of some $N \in \{b, b^*\}$ as above. a_k is given by $a_k = \prod_{i=1}^{k_3+1} c_k^i$ and $c_k^i \in \{1, \delta\varepsilon_k, \delta g_k, \overline{\delta g_k}\}$ for $1 \le i \le k_3+1$. This implies:

$$\left\|\frac{1}{|\Lambda|}\sum_{k\in\mathfrak{L}} a_k x_k\right\| \le (2C_\Omega)^{k_3+1}. \tag{A.8}$$

IV.) Each monomial contains exactly one term A_Ω^t and the number of commutators which can be found in the above terms is exactly n. Thus we have $k_1 + k_2 + k_3 = n$ and the norm of a certain monomial M is estimated with Eqns. (A.6)–(A.8) by

$$\|M\| \leq (2C_\Omega)^{k_1+k_2+1+k_3+1}\, \|A_\Omega\| = (2C_\Omega)^{n+2}\, \|A_\Omega\|.$$

Together with the number of monomial $6^n\, n!$, (i) follows. □

Now we are ready to prove the main result of Theorem 4.2:

$$\sigma\text{-w-}\lim_{\Lambda\in\mathfrak{L}}(\tau_t^{\beta 0})^{P_\Lambda}(A_\Omega) = (\tau_t^{\beta 0})^{P_\Omega^\beta}(A_\Omega).$$

We consider the perturbation series [13, Proposition 5.4.1] of $(\tau_t^{\beta 0})^{P_\Lambda}(A_\Omega)$ for $A_\Omega \in \mathfrak{A}_\Omega$, $\Omega \subseteq \Lambda$, and $t < \frac{1}{12\, C_\Omega}$:

$$(\tau_t^{\beta 0})^{P_\Lambda}(A_\Omega) = \tau_t^{\beta 0}(A_\Omega) + \sum_{n=0}^{\infty} i^n \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \left[P_\Lambda^{t_n}, \left[\cdots, \left[P_\Lambda^{t_1}, A_\Omega^t\right]\cdots\right]\right]$$

Using Lemmata A.2 and A.3 (iii), we can evaluate the σ–weak limit by the help of an $\varepsilon/3$–argument.

Finally, the C^*–dynamical system $(\mathcal{C}_\beta, \mathbb{R}, \tau^\beta)$ is obtained for arbitrary $A \in \mathcal{C}_\beta$ as

$$\tau^\beta(A) := \lim_{\Omega\in\mathfrak{L}}(\tau_t^{\beta 0})^{P_\Omega^\beta}(A).$$

The rest of Theorem 4.2 (i) follows by straight forward calculations.

(ii) Compare Eq. (4.5) for τ^β and $\tau^{\beta r}$ and use Prop. 3.4.

(iii) Differentiate Eq. (4.5) and observe that $\tau_t^{\beta(r)}$ leaves $\mathcal{C}_{\beta,0}^1$ invariant. □

References

[1] N. G. Duffield and R. F. Werner, Local Dynamics of Mean–Field Quantum Systems, *Helv. Phys. Acta*, **65**, 1016–1054 (1992).

[2] A. Blobel and J. Messer, Equilibrium states of inhomogeneous mean–field quantum spin systems with random sites, *J. Math. Phys.*, **26**, 1049–1056 (1985).

[3] N. G. Duffield and J. V. Pulé, A new method for the thermodynamics of the BCS model, *Commun. Math. Phys.*, **118**, 475–494 (1988).

[4] N. G. Duffield, H. Roos, and R. F. Werner, Macroscopic limiting dynamics of a class of inhomogeneous mean field quantum systems, *Ann. Inst. Henri Poincarè*, **56**, 143–186 (1992).

[5] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many–Particle Systems*, McGraw–Hill Book Company (1971).

[6] J. W. NEGELE AND H. ORLAND, *Quantum Many-Particle Systems*, Addison–Wesley Publishing Company (1988).

[7] N. DATTA, R. FERNÁNDEZ, AND J. FRÖHLICH, Low–Temperature Phase Diagrams Quantum Lattice Systems. I. Stability for Quantum Perturbations of Classical Systems with Finitely–Many Ground States, *J. Stat. Phys.*, **84**, 455–534 (1996).

[8] N. DATTA, R. FERNÁNDEZ, J. FRÖHLICH, AND L. REY-BELLET, Low-temperature phase diagrams of quantum lattice systems. II. Convergent perturbation expansions and stability insystems with infinite degeneracy, Preprint, mp-arc 96-226 (1996).

[9] C. BORGS, R. KOTECKÝ, AND D. UELTSCHI, Low Temperature Phase Diagrams for Quantum Perturbations of Classical Spin Systems, *Commun. math. Phys.*, **181**, 409–446 (1996).

[10] G. A. RAGGIO AND R. F. WERNER, Quantum statistical mechanics of general mean field systems, *Helv. Phys. Acta*, **62**, 980–1003 (1989).

[11] T. GERISCH AND A. RIECKERS, The Quantum Statistical Free Energy Minimum Principle for Multi–Lattice Mean Field Theories, *Z. Naturforsch.*, **45a**, 931–945 (1990).

[12] A. RIECKERS, On the classical part of the mean field dynamics for quantum lattice systems in grand canonical representations, *J. Math. Phys.*, **25**, 2593–2601 (1984).

[13] O. BRATTELI AND D. W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics*, volume II, Springer–Verlag (1981).

[14] L. P. GOR'KOV, Microscopic Derication of the Ginzburg–Landau Equations in the Theory of Superconductivity, *Soviet Phys.-JETP*, **9**, 1364–1367 (1959).

[15] T. GERISCH, R. HONEGGER, AND A. RIECKERS, Limiting dynamics of generalized Bardeen–Cooper–Schrieffer models beyond the pair algebra, *J. Math. Phys.*, **34**, 943–968 (1993).

[16] T. VAN DUZER AND C. W. TURNER, *Principles of Superconducting Devices and Circuits*, Elsevier, New York (1984).

[17] H. HAKEN, *Quantenfeldtheorie des Festkörpers*, Teubner, Stuttgart (1973).

[18] J. BARDEEN, L. N. COOPER, AND J. R. SCHRIEFFER, Theory of Superconductivity, *Phys. Rev.*, **108**, 1175–1204 (1957).

[19] D. J. THOULESS, *The Quantum Mechanics of Many Body Systems*, Academic Press, New York, London (1961).

[20] M. TINKHAM, *Introduction to Superconductivity*, Mc Graw–Hill, Tokyo (1975).

[21] G. RICKAYZEN, *Theory of Superconductivity*, J. Wiley, New York (1965).

[22] T. GERISCH, Local Perturbations and Limiting Gibbs States of Quantum Lattice Mean–Field Systems, *Helv. Phys. Acta*, **67**, 585–609 (1994).

[23] T. GERISCH, Internal Symmetries and Limiting Gibbs States in Quantum Lattice Mean Field Theories, *Physica A*, **197**, 284–300 (1993).

[24] E. STØRMER, Symmetric States of Infinite Tensor Products of C^*–Algebras, *J. Funct. Anal.*, **3**, 48–68 (1969).

[25] R. HAAG, R. V. KADISON, AND D. KASTLER, Nets of C^*-Algebras and Classification of States, *Commun. Math. Phys.*, **16**, 81 (1970).

[26] E. DUFFNER AND A. RIECKERS, On the Global Quantum Dynamics of Multi–Lattice Systems with Non–linear Classical Effects, *Z. Naturforsch.*, **43a**, 521 – 532 (1988).

[27] P. BÓNA, The dynamics of a class of quantum mean–field theories, *J. Math. Phys.*, **29**, 2223–2235 (1988).

[28] M. FANNES, H. SPOHN, AND A. VERBEURE, Equilibrium states for mean field models, *J. Math. Phys.*, **21**, 355–358 (1980).

[29] P. BÓNA, Equilibrium states of a class of quantum mean–field theories, *J. Math. Phys.*, **30**, 2994–3007 (1989).

[30] T. GERISCH, A. RIECKERS, AND H.-J. VOLKERT, Thermodynamic Formalism and Phase Transitions of Generalized Mean–Field Quantum Lattice Models, Preprint (1996).

[31] T. GERISCH AND A. RIECKERS, Comparsion of Weakly Inhomogeneous BCS– and Hubbard–Models, Preprint (1996).

[32] W. FLEIG, *Operatoralgebraische Quantenstatistik von zusammengesetzten langreichweitigen Gittersystemen*, PhD Thesis (Dissertation), Universität Tübingen (1985).

[33] T. GERISCH, A. RIECKERS, AND S. ZANZINGER, Operator Algebraic Transition Probabilities and Disjointness, Preprint (1996).

[34] O. BRATTELI AND D. W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics*, volume 1, Springer–Verlag (1987).

[35] G. A. RAGGIO AND R. F. WERNER, The Gibbs Variational Principle for Inhomogeneous Mean–Field Systems, *Helv. Phys. Acta*, **64**, 633–667 (1991).

[36] M. HÄBERLEN, *Gleichgewichts–Zustände makroskopisch inhomogener BCS–Modelle auf der hydrolokalen C^*–Algebra*, Masters Thesis (Diplomarbeit), Universität Tübingen (1996).

[37] M. TAKESAKI, *Theory of Operator Algebras I*, Springer, New York (1979).