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Quasi-Einstein metrics and their renormalizability properties

By Thierry CHAVE, Galliano VALENT

Laboratoire de Physique Théorique et des Hautes Energies,
Unité associée au CNRS URA 280, Université Paris 7,
2 Place Jussieu, 75251 Paris Cedex 05

Abstract. We construct a family of kählerian quasi-Einstein metrics with an isometry group $U(n)$ acting linearly on the holomorphic coordinates. Suitable restrictions on the parameters give rise to complete non-compact as well as compact metrics whose geometrical structure is studied in detail. And we discuss The two loop renormalizability properties of the bosonic σ-models.

1 Introduction

Quasi-Einstein metrics are defined by the constraints

$$Ric_{\mu\nu} = \lambda\, g_{\mu\nu} + \frac{1}{2}(D_\mu v_\nu + D_\nu v_\mu). \tag{1.1}$$

In section 2, using Kähler geometry and an isometry group $U(n)$, we reduce (1.1) to a non-linear differential equation. Since this equation cannot be integrated to get the Kähler potential we switch to non-holomorphic coordinates which are useful to display explicitly the distance.

In section 3 we give two classes of complete non-compact metrics and we describe the compact metrics of the family.

In section 4 we discuss the two-loop renormalizability of the bosonic σ-models built on kählerian quasi-Einstein metrics. Our main result is that in the chosen isometry class the requirement of two-loop renormalizability selects uniquely CP^n, which is not only Einstein but symmetric!

2 Quasi-Einstein Kähler metrics

2.1 Isometry group $U(n)$

The idea to obtain explicit quasi-Einstein metrics is to use holomorphic coordinates $\{z_i, i = 1, \ldots, n\}$ in $\mathbb{C}^n \backslash \{0\}$ and to choose for isometry group a $U(n)$ for which $\{z_i\}$ transform according to its fundamental representation. This implies that the Kähler potential K depends solely on $s = \overline{z} \cdot z = \sum_{i=1}^{n} \overline{z}_i z_i$. It follows that the metric is

$$\frac{1}{2}\, g = g_{i\overline{j}}\, dz^i d\overline{z}^j = A\, d\overline{z} \cdot dz + A' |\overline{z} \cdot dz|^2, \qquad A(s) = \frac{dK(s)}{ds}.$$

The Ricci tensor reads

$$Ric_{i\overline{j}} = -\partial^2_{i\overline{j}} \ln D, \qquad D = \det\,(g_{i\overline{j}}) = A^{n-1}(sA)'.$$

The restrictions put on the metric by the quasi-Einstein requirement (1.1) imply ([1])

$$v_i = c\, \partial_i (sA), \qquad A^{n-1}(sA)' = e^{-\lambda K - (Re\, c)sA}. \tag{2.1}$$

where c is an a priori complex integration constant.

2.2 The coordinates choice

In view of the complexity of (2.1), there is little hope to get the explicit Kähler potential $K(s)$ and the distance if we insist on using holomorphic coordinates. We define

$$z_i = \sqrt{s}\, \xi_i \qquad i = 1, \ldots, n$$

and

$$\xi = \frac{e^{i\tau}}{1 + \overline{u} \cdot u}\, (1, u), \qquad \tau \in [0, 2\pi], \qquad \{u_i, i = 1, \ldots, n-1\}.$$

This gives for the final form of our distance

$$\frac{1}{2}\, g = \frac{(dt)^2}{4\varrho} + \varrho\, (d\tau + \theta)^2 + t\, \frac{h}{2}\, (CP^{n-1})$$

with

$$\varrho = s\, \frac{dt}{ds}, \qquad t = s\, K', \qquad \theta = \frac{1}{1 + \overline{u} \cdot u}\, \frac{\overline{u} \cdot du - u \cdot d\overline{u}}{2i}.$$

The differential equation (2.1) becomes

$$\frac{d}{dt}\, (e^{ct}\, t^{n-1}\, \varrho(t)) = e^{ct}\, (n\, t^{n-1} - \lambda\, t^n). \tag{2.2}$$

The vector v is given by the holomorphic 1-form $v = c\, dt$ for some real constant c.

3 Complete and compact metrics

Integrating (2.2) gives

$$t^{n-1}\,\varrho = \frac{e^{-ct}\displaystyle\int_0^{ct} e^{u}\left(n\,u^{n-1} - \frac{\lambda}{c}\,u^{n}\right)du}{c^{n}} - B\,e^{-ct}, \qquad n = 1, 2, \ldots \tag{3.1}$$

3.1 Complete non-compact metrics

The first one appears for $B = 0, c > 0, \lambda < 0$.
At $t = 0$, setting $t = r^2$ the distance becomes

$$\frac{1}{2}\,g \approx dr^2 + r^2\left[(d\tau + \theta)^2 + \frac{h}{2}\,(CP^{n-1})\right] = dr^2 + r^2\,d\overline{\xi}\cdot d\xi$$

which shows that $r = 0$ is a " nut", [4].
For $t \to \infty$, setting $t = \dfrac{|\lambda|}{c}\,r^2$ the distance becomes

$$\frac{1}{2}\,g \approx dr^2 + r^2\left\{\left(\frac{|\lambda|}{c}\right)^2 |\overline{u}\cdot du|^2 + \frac{|\lambda|}{c}\,(d\overline{u}\cdot du - |\overline{u}\cdot du|^2)\right\}$$

showing that infinity is asymptotically flat (albeit not euclidean).
For $\lambda = 0$ the infinity is taubian, [2, p. 252].
The second one appears for $B > 0, c > 0, \lambda \leq 0$. But now t_n define by $\varrho(t_n) = 0$ is a " bolt" of twist $k = n+1, n+2, \ldots$, [4].

3.2 Compact metrics

Let us suppose $c < 0, \lambda > |c|$ and $n \geq 2$. From (3.1) we can deduce there does exist a finite interval $[t_n^{(1)}, t_n^{(2)}]$ on which $\varrho(t)$ is positive. The constraints for $t = t_n^{(1)}$ and $t = t_n^{(2)}$ to be bolts of twist k are [5]

$$\left\{\begin{array}{ll} \varrho(t_n^{(1)}) = 0, & \varrho'(t_n^{(1)}) = k, \\ \varrho(t_n^{(2)}) = 0, & \varrho'(t_n^{(2)}) = -k. \end{array}\right. \qquad k = 1, 2, 3, \ldots$$

which reduce, using the differential equation, $\varrho' + \left(c + \dfrac{n-1}{t}\right)\varrho = n - \lambda t$, to the transcendental equation

$$\int_{-k}^{+k} e^{-\xi\,u}\,u\,(n+u)^{n-1}\,du = 0, \qquad \xi = \frac{|c|}{\lambda}, \qquad k = 1, 2, \ldots, n-1.$$

And this equation has a unique solution $\xi \in]0, 1[$.

4 Bosonic σ-models at two loops

For a given bosonic σ-model with classical action

$$\frac{1}{g^2}\int d^2x\,\frac{1}{2}\,g_{ij}(\varphi)\,\partial_\mu\varphi^i\,\partial_\mu\varphi^j$$

the one and two-loop on-shell divergences follow from Friedan [3] and are given by

$$\left(\frac{\hbar}{4\pi}\right)\frac{1}{\epsilon}\int d^2x\,Ric_{ij}\,\partial_\mu\varphi^i\,\partial_\mu\varphi^j+\left(\frac{\hbar}{4\pi}\right)^2\frac{g^2}{\epsilon}\int d^2x\,\frac{1}{2}\,R_{is,tu}\,R_j{}^{s,tu}\,\partial_\mu\varphi^i\,\partial_\mu\varphi^j,\qquad \epsilon=d-2.$$

Two-loop renormalizability requires

$$X_{ij}=R_{is,tu}\,R_j{}^{s,tu}=0=D_iw_j+D_jw_i\qquad\Rightarrow\qquad w_i=c_2\,\partial_i(sA)$$

and

$$X_{i\bar{j}}=R_{i\bar{k},l\bar{n}}\,R_{\bar{j}}{}^{\bar{k},l\bar{n}}=\lambda_2\,g_{i\bar{j}}+D_iw_{\bar{j}}+D_{\bar{j}}w_i=\partial^2_{i\bar{j}}(\lambda_2\,K+(c_2+\bar{c}_2)\,sA).$$

For the divergence $X_{i\bar{j}}$ to be absorbable it is therefore necessary that $X_{i\bar{j}}$ be Kähler. In the class of metrics considered in this article, the large isometry group enables one to compute the curvature tensor and

$$X_{i\bar{j}}=\mu(s)\,\delta_{ij}+\nu(s)\,\bar{z}_i\,z_j.$$

The Kähler condition requires therefore $\nu=\mu'$ which is integrated to $K(s)=a\,\ln(s+b)$. Among all kählerian metrics of dimension $n\geq 2$, with an isometry group $U(n)$ acting linearly on the holomorphic coordinates, only CP^n is one-loop and two-loop renormalizable.

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