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Autor(en): **Shivamoggi, Bhimsen K.**

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Generalized Spectral Laws for the Energy and Enstrophy Cascades in a Two-Dimensional Turbulence

Bhimsen K. Shivamoggi
University of Central Florida
Orlando, FL 32816

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Abstract:

We consider generalized von Karman-Heisenberg-von Weizsacker type model for the inertial transfer to give generalized spectral laws for the energy and enstrophy cascades in a forced two-dimensional turbulence that provides a satisfactory unified description of the equipartition range and the inertial range for the energy cascade and the dissipation range and the inertial range for the enstrophy cascade. We will show that the equipartition range of the energy cascade and the dissipation range of the enstrophy cascade can be satisfactory modeled by a stationary continuous spectral cascading process. We will then discuss the intermittency aspects of the departures from the Batchelor-Kraichnan scaling laws and show that while the intermittency corrections within the framework of the β-model are in qualitative agreement with the predictions made by the generalized spectral laws given in this article, intermittency by itself is unable to account fully for the equipartition spectrum of the energy cascade observed in laboratory experiments and the dissipative spectrum of the enstrophy cascade observed in laboratory and numerical experiments. We will discuss further fractal aspects of the enstrophy cascades, and show that for the enstrophy cascade, the fractal dimension rules not only the manner in which the cascading proceeds but also the point where it stops, while for the energy cascade the fractal dimension rules only the manner in which the inverse cascade proceeds and not the point where it stops.

This article is dedicated to
Professor Mahinder S. Uberoi.

1. Introduction:

A principal reason for interest in two-dimensional turbulence is the possibility of applying the theory to planetary boundary layers (Rhines, 1979; Kraichnan and Montgomery, 1980). Strictly two-dimensional flow in a layer of fluid requires that the velocity vector everywhere lie in a given plane and that there be no variation of the velocity field perpendicular to that plane. On a global scale, the earth's atmosphere and oceans are a very thin layer so that it is reasonable to expect two-dimensional motion on scales large compared with the layer thickness. It may be noted that though several factors such as topographic surface variations and salinity variations in the oceans destroy the two-dimensional nature of the motion, the rotation of the earth plays a crucial role in preserving the latter. (This follows from Taylor-Proudman theorem (see Greenspan, 1968) which shows that uniform rotation of a plane layer of fluid about an axis, say z-axis, perpendicular to the plane tends to lock the fluid into two-dimensional motion independent of z.)

Kraichnan (1967) and Batchelor (1969) pointed out the possibility of two inertial ranges in a two-dimensional turbulence: the energy subrange in which energy propagates to larger scales, and the enstrophy subrange in which enstrophy cascades to smaller scales. Kraichnan (1967) and Batchelor (1969) invoked arguments similar to those used in Kolmogorov's (1941) theory of three-dimensional isotropic hydrodynamic turbulence to surmise that if the Reynolds number is sufficiently high the large-scale components are influenced only by the boundary conditions on the system. The statistical properties of the small-scale components of velocity and vorticity-fields in the inertial range were assumed to have some universality and are uniquely determined by the mean energy and enstrophy dissipation rates ε and τ, respectively, and the kinematic viscosity ν and depend only weakly on the large-scale features of these fields. By using dimensional arguments, they then derived $k^{-\frac{5}{3}}$ and k^{-3} power laws for the spectrum of kinetic energy density of the fluctuations in the stationary state for the energy subrange and enstrophy subrange, respectively. Kraichnan (1967) proposed that both inertial ranges would exist simultaneously in a continuously driven turbulence. Leith (1968) derived a diffusion approximation to inertial energy transfer in such a way that energy and enstrophy are conserved, and also predicted the $k^{-\frac{5}{3}}$ and k^{-3} inertial ranges. Numerical closures of Frisch et al. (1967) and Pouquet et al. (1975) and numerical simulations of Lilly (1969), Herring et al. (1974), Fornberg (1977), and Frisch and Sulem (1984) confirmed the conjecture of Kraichnan (1967) and Batchelor (1969) that there occurs a transfer of excitation to lower and higher wavenumbers in a manner qualitatively consistent with the simultaneous existence of both the energy and enstrophy inertial ranges. Lilly (1969) obtained an omnidirectional energy spectrum for a system driven continuously by a mode at wavenumber k_C. Figure 1 shows two inertial ranges—the energy cascade for $k > k_C$, developing from an initially peaked spectrum dominated by the source spectrum at $k = k_C$. Pouquet et al. (1975) used the stochastic models introduced by Kraichnan (1961) to numerically test a simultaneous direct enstrophy cascade and inverse energy cascade for two-dimensional turbulence. When enstrophy and energy are continuously injected at a fixed wavenumber, it was found that (see Figure 2) a quasi-steady regime is obtained where enstrophy cascades to large wavenumbers across k^{-3} inertial range with zero energy transfer while energy flows indefinitely to small wavenumbers across a $k^{-\frac{5}{3}}$ inertial range with zero enstrophy transfer. Atmospheric measurements have also revealed the existence of an energy cascade (Fjortoft, 1953) and an enstrophy cascade (Ogura, 1958; Wiin-Nielsen, 1967; Julian et al. 1970; Morel and Necco, 1973; Morel and Larcheveque, 1974; Desbois, 1975). Kraichnan (1971) proposed further that the k^{-3} spectrum for the enstrophy cascade should be modified by a logarithmic correction term to give $k^{-3}[\ln(k/k_c)]^{-\frac{1}{3}}$. However, the latter result does not extend to infinity, because it does not give the rapid decay of the spectrum prevalent at high wavenumbers. Kida (1981) applied the modified cumulant expansion and numerically calculated the equations for the energy spectrum, and confirmed a more rapid decay of the spectrum in the enstrophy cascade at very large wavenumbers. Numerical simulations of decaying flows (Basdevant and Sadourny, 1983; McWilliams, 1984; Brachet et al. 1988) and forced flows (Kida, 1985; Brachet et al. 1986, Basdevant et al. 1981) also gave for the enstrophy cascade energy spectra steeper than k^{-3} (see Figures 3 and 4).

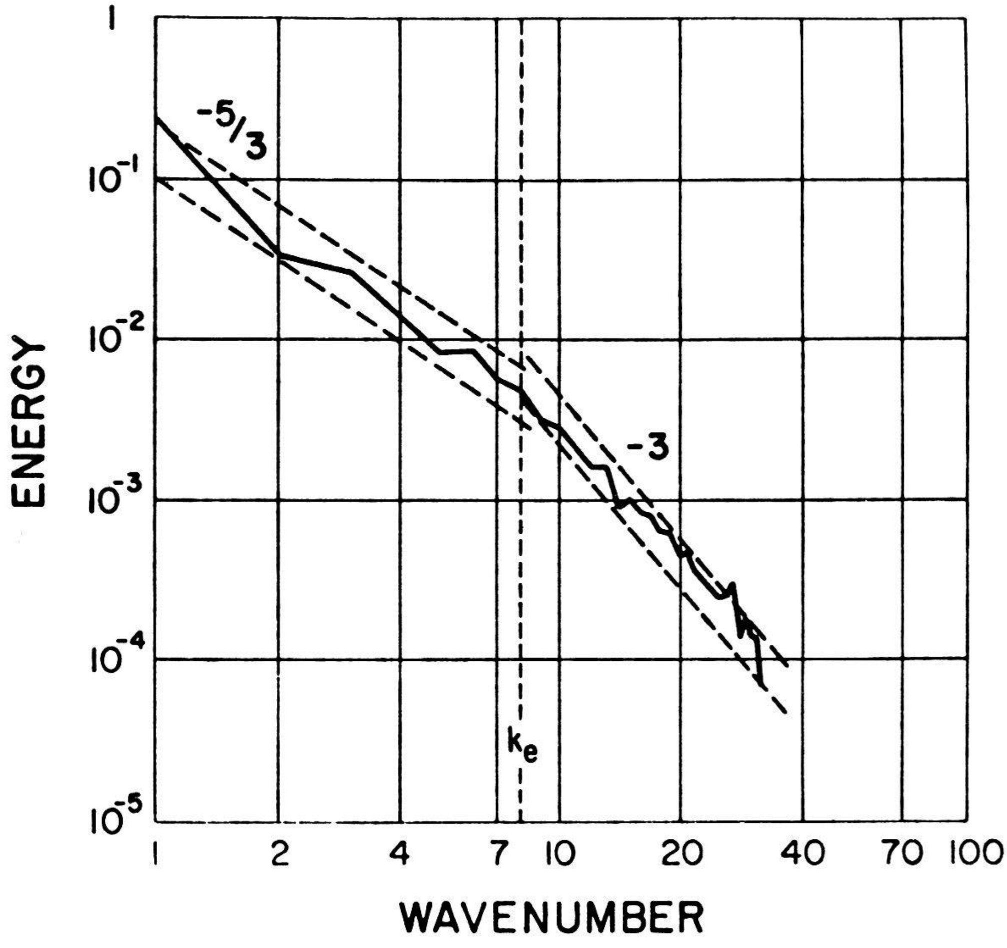


Figure 1. An omnidirectional energy spectrum of two-dimensional Navier-Stokes turbulence obtained numerically. The initial spectrum, which is dominated by the source spectrum at the source wave k_e, is shown to relax to the inertial range spectra for enstrophy at $k > k_e$ and energy at $k < k_e$.

(From D. K. Lilly, 1969)

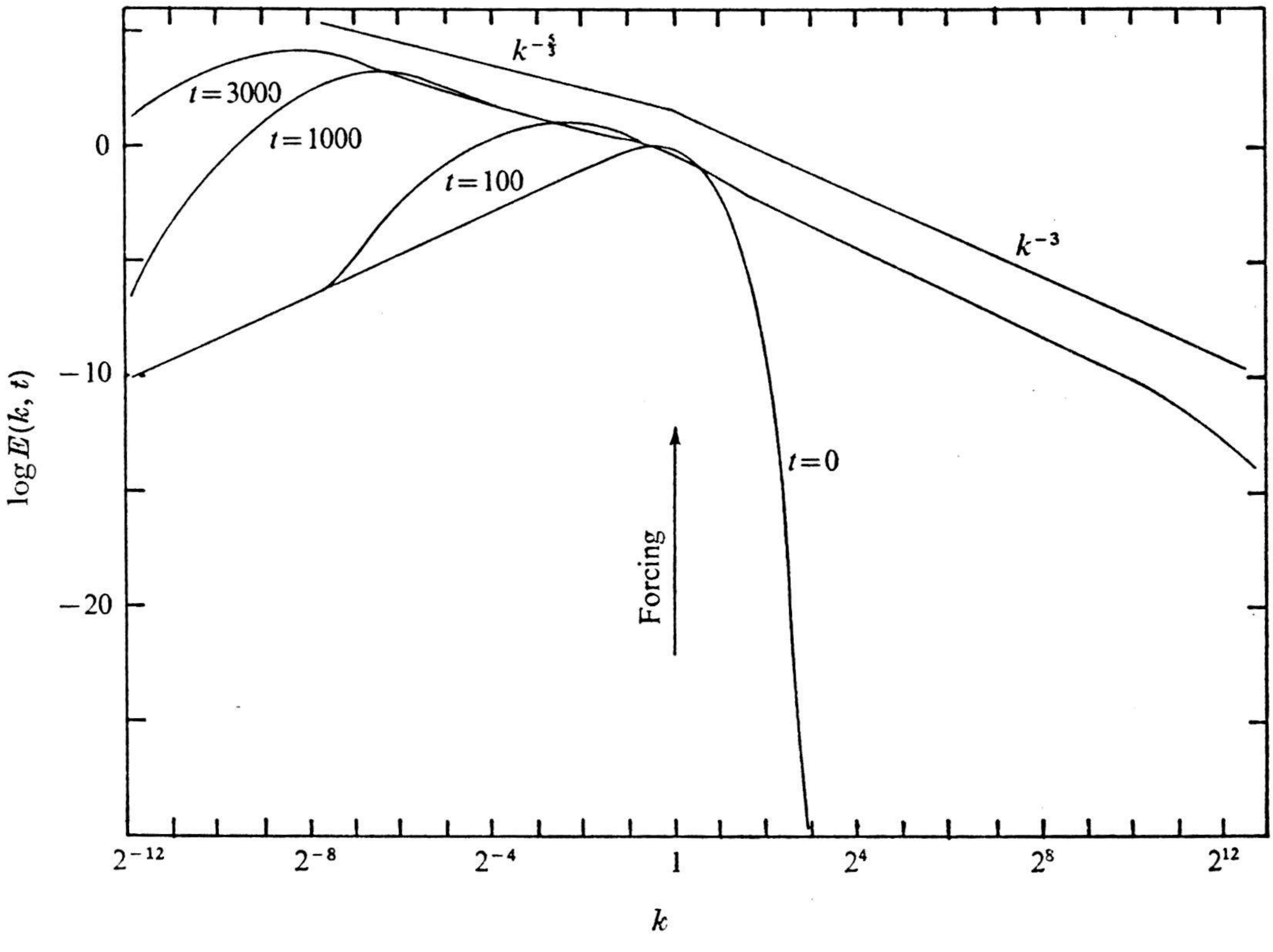


Figure 2. Quasi-steady energy spectrum $E(k,t)$ for $t = 100$, 1000 and 3000 corresponding to an injection spectrum constant in a half-octave band around $k_I = 1$ with injection rates $\varepsilon = 0.03$ and $\beta = 0.03$. Reynolds number $R = 2.4 \times 10^7$.

(From A. Pouquet, M. Lesieur, J. C. Andre and C. Basdevant, 1975)

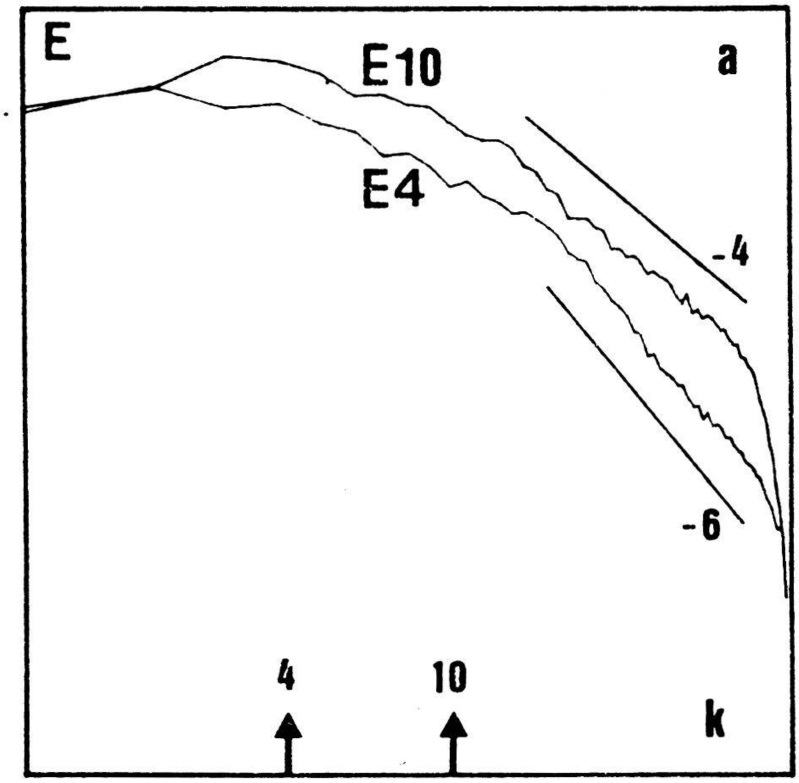


Figure 3. Energy spectra of experiments E4 and E10. Arrows indicate injection wavenumbers. Spectral slopes k^{-4} and k^{-6} are indicated (log-log scale).

(from A. Babiano, C. Basdevant and R. Sadourny, 1985)

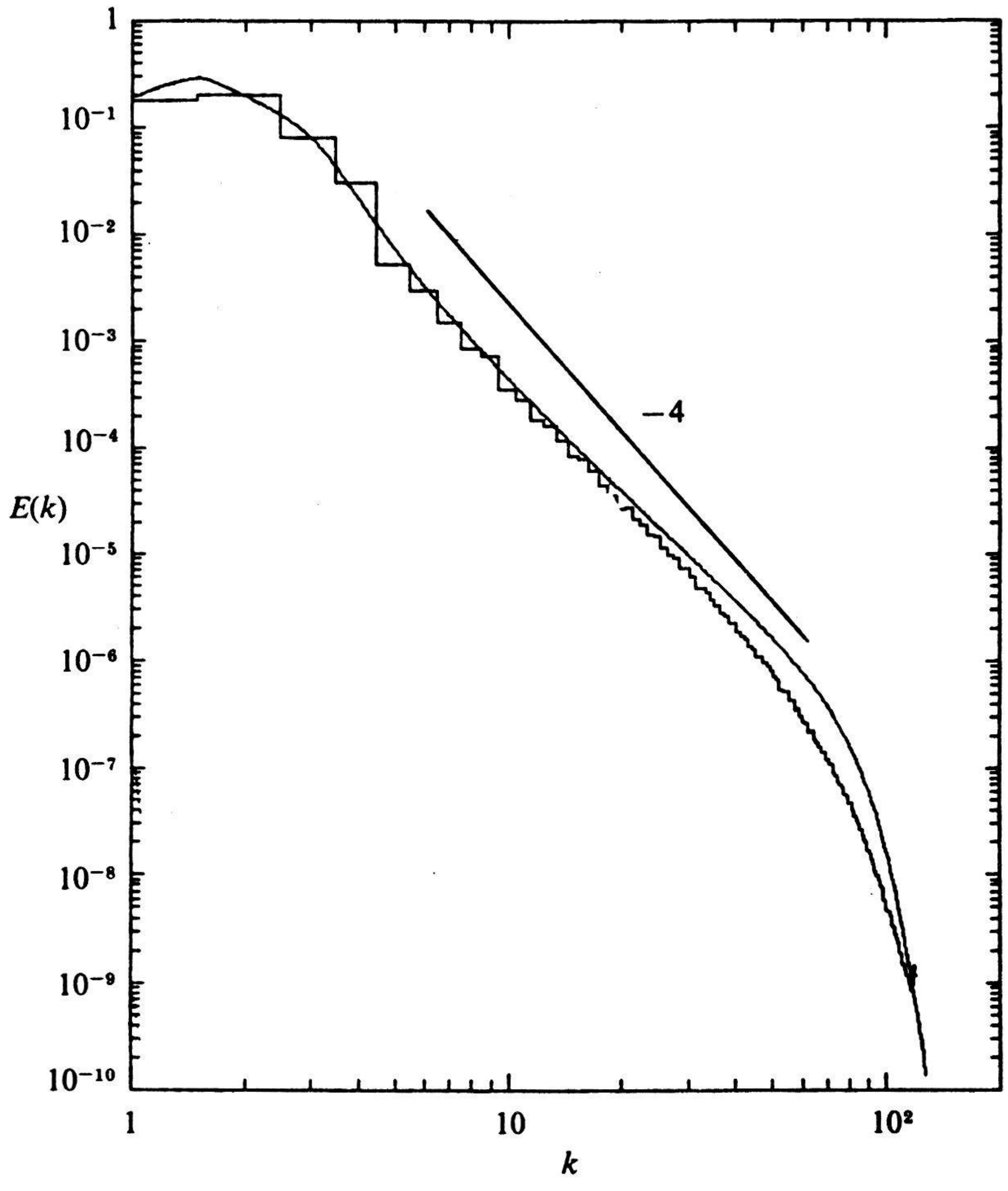


Figure 4. Energy spectrum $E(k)$ for case 1 (b) $(1 \leq k \leq 128)$ in the stationary state.

(From J. R. Herring and J. C. McWilliams, 1985)

The whole theory of two-dimensional turbulence had, until recently, remained almost an academic exercise, notwithstanding its possible connections with atmospheric and oceanic large-scale flows. Just recently, truly two-dimensional flows were produced to a close approximation in laboratory experiments. Experimental evidence of the existence of inverse energy cascade was first obtained by Couder (1984) on thin liquid soaps films, then by Sommeria (1986) in a shallow mercury layer immersed in a strong normal magnetic field.

The inverse energy cascade in a statistically steady forced two-dimensional turbulence (without forcing, the inverse cascade cannot develop) experimentally investigated by Sommeria (1986) showed a $k^{-\frac{5}{3}}$ behavior at large wavenumbers and a k^1 behavior corresponding to an equilibrium energy equipartition spectrum at small wavenumbers. Laboratory experiments in a cylindrical tank filled with a two-layer fluid system and driven by a surface stress of a forced, quasi-two-dimensional turbulence were performed by Narimousa et al. (1991) who obtained for large wavenumbers an energy spectrum steeper than k^{-3} (Figure 5).

Basdevant et al. (1981) argued that the steeper energy spectra at large wavenumbers is due to intermittency in the flow: an intermittent random variable is one which has a large probability of taking values both very large and very small compared with its standard deviation. Enstrophy dissipation is a highly-fluctuating quantity whose statistical properties significantly affect the energy spectrum at small scales. Due to intermittency, the small-scale structures are no longer uniformly distributed in space but show more and more spottiness, and their statistics are increasingly non-Gaussian. If intermittency increases as scale size decreases, and the Batchelor-Kraichnan theory is assumed to hold in local regions, then the enstrophy cascade would be expected to become more efficient as the scale size decreases. As a result, one would expect that the energy spectrum must fall off more rapidly than k^{-3} if, according to conservation of enstrophy, the overall enstrophy cascade rate is to be independent of the scale size. Earlier, Mandelbrot (1976) had argued that intermittency in the three-dimensional case is related to the fractal aspects of turbulence. In particular, Mandelbrot (1976) proposed that the dissipation is concentrated on a set

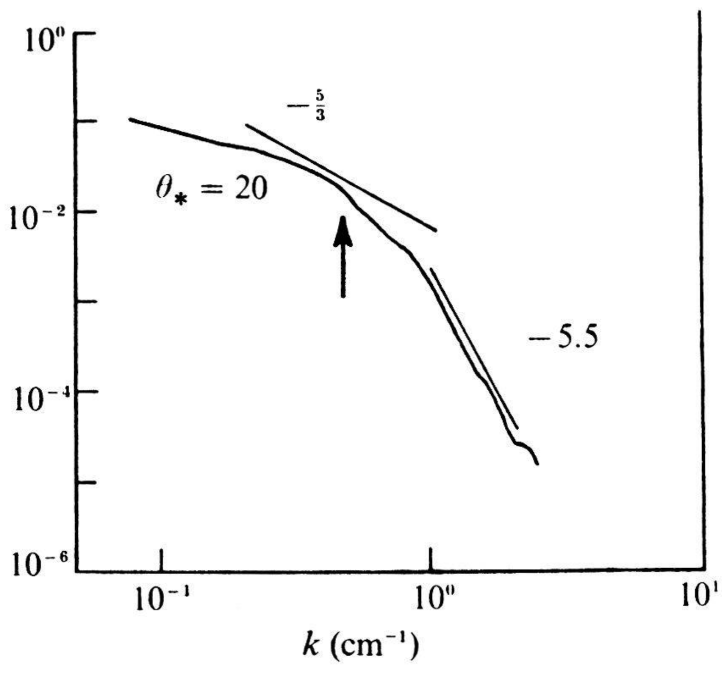


Figure 5. One-dimensional energy spectra (e) as a function of wavenumber (k), inferred from a direct two-dimensional, FFT of the u'^2 velocities deduced from the interpolated data. Rough estimates of spectral slopes are indicated for each experiment. The vertical arrows indicate the wavenumber of the frontal eddies.

(From S. Narimousa, T. Maxworthy and G. R. Spedding, 1991) with noninterger Hausdorff dimension. Mandelbrot's ideas were formulated for the three-dimensional case in a simpler way through a phenomenalogical model called the β-model (which was based on the ideas advanced by Kraichnan, 1972) by Frisch et al. (1978). For the two-dimensional case, one may also use the β-model to explore the fractal aspects of the departures from the Batchelor-Kraichnan scaling laws. The key assumption in this model is that the flux of energy is transferred to only a fixed fraction β of the eddies downstream in the cascade. A noteworthy feature of the β-model is that we do not have to assume the Batchelor-Kraichnan scaling laws initially and then derive their modified versions by somehow mysteriously incorporating the dissipation fluctuations. However, since the β-model requires that there is no mixing between the empty and nonempty regions it presupposes that the time-scale of spatial mixing is much larger than that associated with the aggregation/fragmentation processes in two-dimensional turbulence.

The application of the β-model to the inverse energy cascade was done by Frisch et al. (1978), who found that the intermittency corrections decrease the 5/3 exponent. Shivamoggi (1990_a) applied the β-model to the enstrophy cascade and confirmed that intermittency will steepen the energy spectrum, in qualitative agreement with the generalized spectral law for the enstrophy cascade given in this paper. However, we will show that intermittency by itself is unable to account fully for steeper spectra observed at large wavenumbers in laboratory and numerical experiments or flatter spectra observed at small wavenumbers in laboratory experiments.

The intermittency corrections mentioned above may also be too small to allow an experimental or numerical verification at the usual level of resolution of kinetic energy and enstrophy spectra. One may then take another approach and make a systematic analysis of the effect of nonlinear inertial and viscous effects on the kinetic energy and enstrophy spectra using generalized von Karman-Heisenberg-von Weizsacker type models for the inertial transfer (Shivamoggi, 1990_a and 1990_b). According to this model, the transfers of the kinetic energy from small to large wavelengths and the enstrophy from large to small wavelengths are described by gradient-diffusion type cascade processes characterized by "eddy viscosities" produced by small wavenumber modes acting to remove kinetic energy from large wavenumber modes and large wavenumber modes acting to remove enstrophy from small wavenumber modes, respectively. Using this model, one may deduce generalized spectral laws for the kinetic energy and enstrophy cascades that exhibit a flatter/steeper spectra for small/large wavenumbers and reduce to the well known inertial-range laws at the other ends of the spectra. This approach provides a unified framework for describing both the inertial and equipartition/dissipative ranges of the kinetic energy/enstrophy cascades observed in laboratory experiments/numerical calculations. The small-wavenumber limit (namely, the equipartition regime) of the energy cascade and the large-wavenumber limit (namely, the dissipative regime) of the enstrophy cascade can also be modeled in a satisfactory way as a stationary continuous spectral cascading process (Shivamoggi, 1987 and 1990_a).

2. Conserved Quantities for a Two-Dimensional Flow:

The Navier-Stokes equations for an incompressible fluid are

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \left(\underset{\sim}{v} \cdot \nabla\right)\underset{\sim}{v} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\underset{\sim}{v} \quad (1)$$

$$\nabla \cdot \underset{\sim}{v} = 0 \quad (2)$$

where $\underset{\sim}{v}$ is the fluid velocity, p the pressure, ρ the density, and ν is the kinetic viscosity.

Taking the curl of equation (1), and using equation (2), we find that the vorticity $\underset{\sim}{\Omega} = \nabla \times \underset{\sim}{v}$ obeys

$$\frac{\partial \underset{\sim}{\Omega}}{\partial t} + \left(\underset{\sim}{v} \cdot \nabla\right)\underset{\sim}{\Omega} = \left(\underset{\sim}{\Omega} \cdot \nabla\right)\underset{\sim}{v} + \nu\nabla^2\underset{\sim}{\Omega} \quad (3)$$

For a two-dimensional flow, taking the scalar product of equations (1) and (3) with $\underset{\sim}{v}$ and $\underset{\sim}{\Omega}$, respectively, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\underset{\sim}{v}^2}{2}\right) + \nabla \cdot \left(\underset{\sim}{v}\frac{v^2}{2} + \underset{\sim}{v}\frac{p}{\rho}\right) \equiv \nu\nabla \cdot \left(\underset{\sim}{v} \times \underset{\sim}{\Omega}\right) - \nu\underset{\sim}{\Omega}^2 \tag{4}$$

$$\frac{\partial}{\partial t}\left(\frac{\underset{\sim}{\Omega}^2}{2}\right) + \nabla \cdot \left(\frac{\underset{\sim}{\Omega}^2}{2}\underset{\sim}{v}\right) = \nu\nabla \cdot \left[\underset{\sim}{\Omega} \times \left(\nabla \times \underset{\sim}{\Omega}\right)\right] - \nu\left(\nabla \times \underset{\sim}{\Omega}\right)^2 \tag{5}$$

If the fluid is surrounded by a rigid boundary so that the normal component of velocity vanishes on the boundary, we have from equations (4) and (5),

$$\frac{\partial W}{\partial t} \equiv \frac{\partial}{\partial t}\int \frac{\underset{\sim}{v}^2}{2} d\underset{\sim}{x} = \oint \nu\left(\underset{\sim}{v} \times \underset{\sim}{\Omega}\right) \cdot d\underset{\sim}{s} - \int \nu\underset{\sim}{\Omega}^2 d\underset{\sim}{x} \tag{6}$$

$$\frac{\partial U}{\partial t} \equiv \frac{\partial}{\partial t}\int \frac{\underset{\sim}{\Omega}^2}{2} d\underset{\sim}{x} = \oint \nu\underset{\sim}{\Omega} \times \left(\nabla \times \underset{\sim}{\Omega}\right) \cdot d\underset{\sim}{s} - \int \nu\left(\nabla \times \underset{\sim}{\Omega}\right)^2 d\underset{\sim}{x} \tag{7}$$

In the absence of viscous dissipation ($\nu = 0$), equations (6) and (7) give the conservations of the total energy and the total enstrophy (which is the mean square vorticity) -

$$W = \text{const}, \quad U = \text{const} \tag{8}$$

Thus, in two-dimensional turbulence, there are two conserved quantities—the energy and the enstrophy. (Due to a finite viscosity, however, the enstrophy is dissipated at a nonnegligible rate; therefore the maintenance of a stationary state requires an external source since the vortex stretching which acts like a source of vorticity is inoperative here unlike the three-dimensional turbulence. However, energy dissipation will tend to zero as $\nu \rightarrow 0$ so that two-dimensional turbulence is almost nondissipative as $\nu \rightarrow 0$.) Therefore, there are two types of inertial ranges—one for energy and one for enstrophy.

If the enstrophy vanishes during the normal cascade, equation (6) shows that $\partial W/\partial t \Rightarrow 0$ even in the presence of a viscous dissipation. This implies that the system will evolve toward a state of minimum enstrophy with constant energy. Thus, there exists a selective dissipation process among the conserved quantities in a two-dimensional flow when dissipation is introduced—the enstrophy decays faster than the energy.

3. Fourier Analysis of the Turbulent Velocity Field:

Fourier analysis of the velocity field, when it is a stationary random function of position, affords a convenient identification of the scales of motion with Fourier modes and a view of the turbulent motion as comprised of the superposition of motions of a large number of components of different length scales. These Fourier components contribute additively to the total energy and total enstrophy and interact with each other according to the nonlinear inertial terms in the equations of flow. The observed properties of the turbulent field are thought of as being the statistical result of such interactions. It should be noted that Fourier representation is natural for infinitely extended homogeneous turbulent fields but not for inhomogeneous flows for which there is only a weak relation between the structure in real space and the Fourier modes. Certain spatially compact objects called wavelets (Argoul et al. 1989) have recently been advanced for an efficient decomposition of a turbulent field into various sizes.

Let us express the flow properties at any point $\underset{\sim}{x}$ at time t, as a superposition of plane waves of the form,

$$\underset{\sim}{\mathrm{v}}\left(\underset{\sim}{x},t\right) = \sum_{\underset{\sim}{k}} \underset{\sim}{V}\left(\underset{\sim}{k},t\right)e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}$$

$$\frac{1}{\rho}p\left(\underset{\sim}{x},t\right) = \sum_{\underset{\sim}{k}} P\left(\underset{\sim}{k},t\right)e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \tag{9}$$

Since $\underset{\sim}{V}$ and P are actually measurable, they must be real so that

$$\underset{\sim}{V}^{*}\left(\underset{\sim}{k}\right) = \underset{\sim}{V}\left(-\underset{\sim}{k}\right), P^{*}\left(\underset{\sim}{k}\right) = P\left(-\underset{\sim}{k}\right)$$

We have dropped the argument t for convenience. We then obtain from equations (1) and (2), the following equation -

$$\left(\frac{\partial}{\partial t}+\nu k^{2}\right)V_{j}\left(\underset{\sim}{k}\right) = -ik_{m}\left(\delta_{jl}-\frac{k_{j}k_{l}}{k^{2}}\right)$$

$$\times\sum_{k'} V_{m}\left(\underset{\sim}{k'}\right)V_{l}\left(\underset{\sim}{k}-\underset{\sim}{k'}\right) \tag{10}$$

which describes the mode coupling among different Fourier components.

In terms of the stream function ψ, defined by

$$\underset{\sim}{v} = \nabla \psi \times \underset{\sim}{\hat{i}_z} \tag{11a}$$

we have for the vorticity $\underset{\sim}{\Omega}$,

$$\underset{\sim}{\Omega} = -\nabla \times \underset{\sim}{v} = \nabla^2 \psi \, \underset{\sim}{\hat{i}_z} \tag{11b}$$

and equation (3) becomes

$$\frac{\partial}{\partial t} \nabla^2 \psi + \left(\nabla \psi \times \underset{\sim}{\hat{i}_z} \right) \cdot \nabla (\nabla^2 \psi) = \nu \nabla^4 \psi \tag{12}$$

Fourier analyzing $\psi\left(\underset{\sim}{x}, t\right)$, according to,

$$\psi\left(\underset{\sim}{x}, t\right) = \sum_{\underset{\sim}{k}} \Psi\left(\underset{\sim}{k}, t\right) e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}} \tag{13}$$

equation (12) becomes

$$\left(\frac{\partial}{\partial t} + k^2 \nu \right) \Psi\left(\underset{\sim}{k}\right) = \sum_{\underset{\sim}{k} = \underset{\sim}{k}' + \underset{\sim}{k}''} \Lambda^{k}_{k', k''} \Psi\left(\underset{\sim}{k}'\right) \Psi\left(\underset{\sim}{k}''\right) \tag{14}$$

where,

$$\Lambda^{k}_{k', k''} = \frac{1}{k^2} \left(\underset{\sim}{k}' \times \underset{\sim}{k}'' \right) \cdot \underset{\sim}{\hat{i}_z} (k'^2 - k''^2)$$

and we have again dropped the argument t for convenience. Λ becomes large when $\underset{\sim}{k}$, $\underset{\sim}{k}'$ and $\underset{\sim}{k}''$ have comparable magnitudes so that the modal cascade is dominated by local interactions in $\underset{\sim}{k}$-space.

4. <u>Energy and Enstrophy Cascades</u>:

Consider a source in the spectral space at $k = k_s$ with energy $W_s = W(k_s)$ (the omnidirectional energy spectrum $W(k)$ is defined such that $\int W(k) dk = \sum \left| \underset{\sim}{V}\left(\underset{\sim}{k}\right) \right|^2$ gives the total energy).

Through mode-mode coupling this source would decay into two modes with wavenumbers k_1 and k_2 with energies W_1 and W_2, ($k < k_s$ corresponds to the inertial range for energy and $k > k_s$ corresponds to the inertial range for enstrophy.) Since energy and enstrophy are conserved, we have

$$W_s = W_1 + W_2 \tag{15}$$

$$k_s^2 W_s = k_1^2 W_1 + k_2^2 W_2 \tag{16}$$

from which, the energy is partitioned as

$$\left.\begin{aligned} W_1 &= \frac{k_2^2 - k_s^2}{k_2^2 - k_1^2} W_s \\ W_2 &= \frac{k_s^2 - k_1^2}{k_2^2 - k_1^2} W_s \end{aligned}\right\} \tag{17}$$

This implies that

$$k_2^2 > k_s^2 > k_1^2 \tag{18}$$

so that the mode with wavenumber k_s decays into a mode with wavenumber $k_1 < k_s$ and to another mode with wavenumber $k_2 > k_s$.

Let us (following Hasegawa, 1985) assume that a mode k_s first decays into modes $k_1(k_1 = \sqrt{p}\, k_s)$ and $k_2(k_2 = \sqrt{1+p}\, k_s\,;\, p < 1)$, with corresponding energies $W_1 = pW_s$ and $W_2 = (1-p)W_s$, and enstrophies $U_1 = k_s^2 p^2 W_s$ and $U_2 = k_s^2(1-p^2)W_s$. In the next step of the cascade, the mode k_1 decays into a mode $\sqrt{p}\, k_1 = pk_s$ and another mode $\sqrt{1+p}\, k_1 = \sqrt{p(1+p)}\, k_s$, while the mode k_2 decays into a mode $\sqrt{pk_2} = \sqrt{p(1+p)}\, k_s$ and another mode $\sqrt{1+p}\, k_2 = (1+p)k_s$. The energies for the modes pk_s, $\sqrt{p(1+p)}\, k_s$ and $(1+p)k_s$ are $p^2 W_s$, $2p(1-p)W_s$ and $(1-p)^2 W_s$, respectively. Thus, at the *nth* step of the cascade, the energy is given by

$$\begin{aligned} W(k^2 &= p^{n-r}(1+p)^r k_s^2) \\ &= \binom{n}{r} p^{n-r}(1-p)^r W_s \end{aligned} \tag{19}$$

Now, by the de Moivre-Laplace approximation, we have for the binomial distribution, as $n \Rightarrow \infty$,

$$\binom{n}{r} p^{n-r}(1-p)^{n-r} \approx \frac{1}{\sqrt{2\pi np(1-p)}} e^{-\frac{(n-r-np)^2}{2\pi p(1-p)}} \tag{20}$$

so that the binomial distribution in (19) peaks at $r/n \sim 1-p$ as $n \Rightarrow \infty$. The corresponding wavenumber is

$$\begin{aligned} k_*^2 &= \underset{n \Rightarrow \infty}{Lt}\ p^{n-r}(1+p)^r k_s^2 \\ &= \underset{n \Rightarrow \infty}{Lt} \left[p^{1-\frac{r}{n}}(1+p)^{\frac{r}{n}}\right]^n k_s^2 \\ &= \underset{n \Rightarrow \infty}{Lt}\ [p^p(1+p)^{1-p}]^n k_s^2 \end{aligned} \tag{21}$$

Since, for $0 < p < 1$, $p^p(1+p)^{1-p} < 1$, we obtain $k_*^2 \sim 0$. This means that the peak of the energy distribution moves to $k \Rightarrow 0$ as $n \Rightarrow \infty$. Hence, the energy cascades inversely and condensates at $k \Rightarrow 0$ (or at the longest wavelength permissible for the system).

Next, the enstrophies for the modes $pk_s, \sqrt{p(1+p)}\,k_s$ and $(1+p)k_s$ are $k_s^4p^4W_s$, $2k_s^4p^2(1-p^2)W_s$ and $k_s^4(1-p^2)^2W_s$, respectively. Thus, at the *nth* step of the cascade, the enstrophy is given by

$$U(k^2 = p^{n-r}(1+p)^r k_s^2) = \binom{n}{r} p^{2(n-r)}(1-p^2)^r k_s^{2n} W_s \tag{22}$$

The binomial distribution in (22) peaks at $r/n \sim 1-p^2$ as $n \Rightarrow \infty$. The corresponding wavenumber is

$$\tilde{k}_*^2 = \underset{n \Rightarrow \infty}{Lt}\ p^{n-r}(1+p)^r k_s^2$$

$$= \underset{n \Rightarrow \infty}{Lt} \left[p^{1-\frac{r}{n}}(1+p)^{\frac{r}{n}}\right]^n k_s^2$$

$$= \underset{n \Rightarrow \infty}{Lt} \left[p^{p^2}(1+p)^{1-p^2}\right]^n k_s^2 \tag{23}$$

Since, for $0 < p < 1$, $p^{p^2}(1+p)^{1-p^2} > 1$, we obtain $\tilde{k}_*^2 \sim \infty$. This means that the peak of the enstrophy distribution moves to $k \Rightarrow \infty$ as $n \Rightarrow \infty$. Hence, the enstrophy cascades directly and condensates at $k \Rightarrow \infty$ (where strong viscous dissipation sets in).

5. Self-organization and Self-degradation in Two-dimensional Flows:

The energy cascade to lower wavenumbers has the result that random excitation at intermediate wavenumbers drives the (necessarily coherent) largest spatial scales of the system. Thus, two-dimensional flows seem to have a self-organizing character. Figures 6 and 7 show a numerical calculation (Lilly, 1969) of the evolution of the stream function and the vorticity. The smooth structure of the stream function is a consequence of the inverse cascade of the energy to large wavelengths, while the convoluted state of the vorticity is a result of the enstrophy cascading to smaller wavelengths.

In order to understand the self-degradation of vorticity, consider equations (11) and (12) in the inviscid limit,

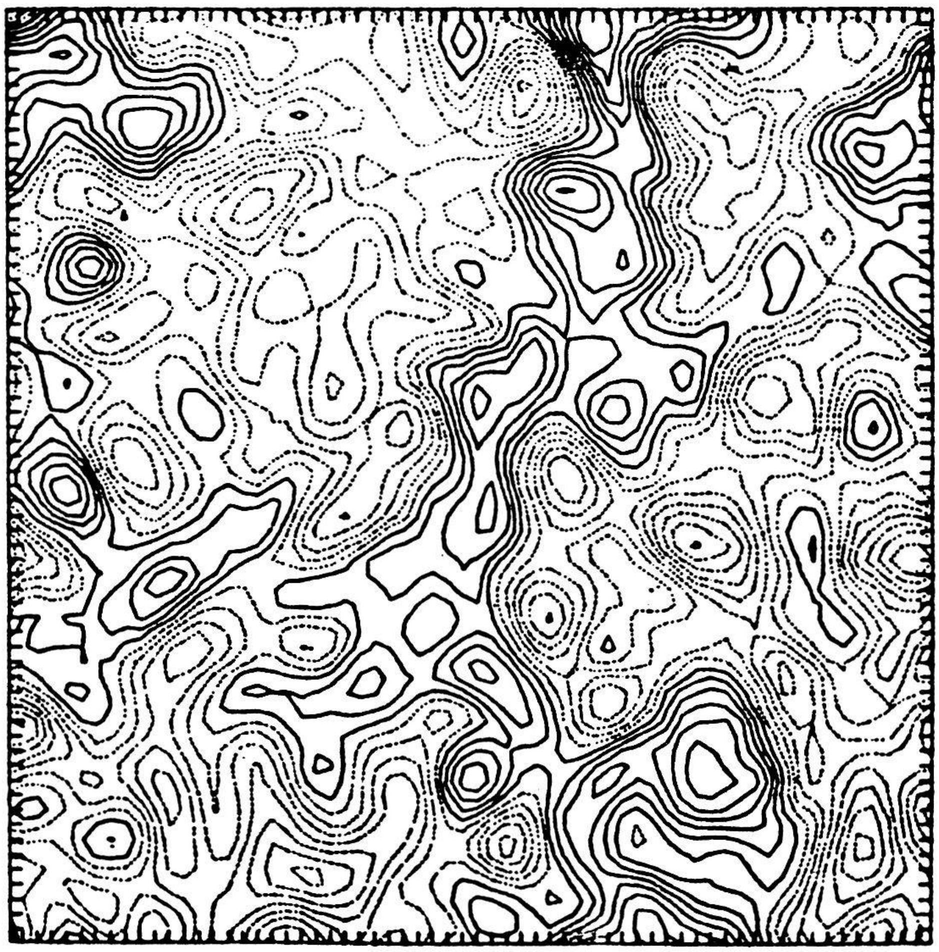
$$\left[\frac{\partial}{\partial t} + L(t)\right]\Omega(t) = 0 \tag{24}$$

$$\nabla^2\psi(t) = -\Omega(t) \tag{25}$$

where the operator $L(t)$ is given by

$$L(t) \equiv \nabla\psi(t) \times \hat{\underset{\sim}{i}}_z \cdot \nabla$$

and we have omitted showing the dependence on $\underset{\sim}{x}$ explicitly.



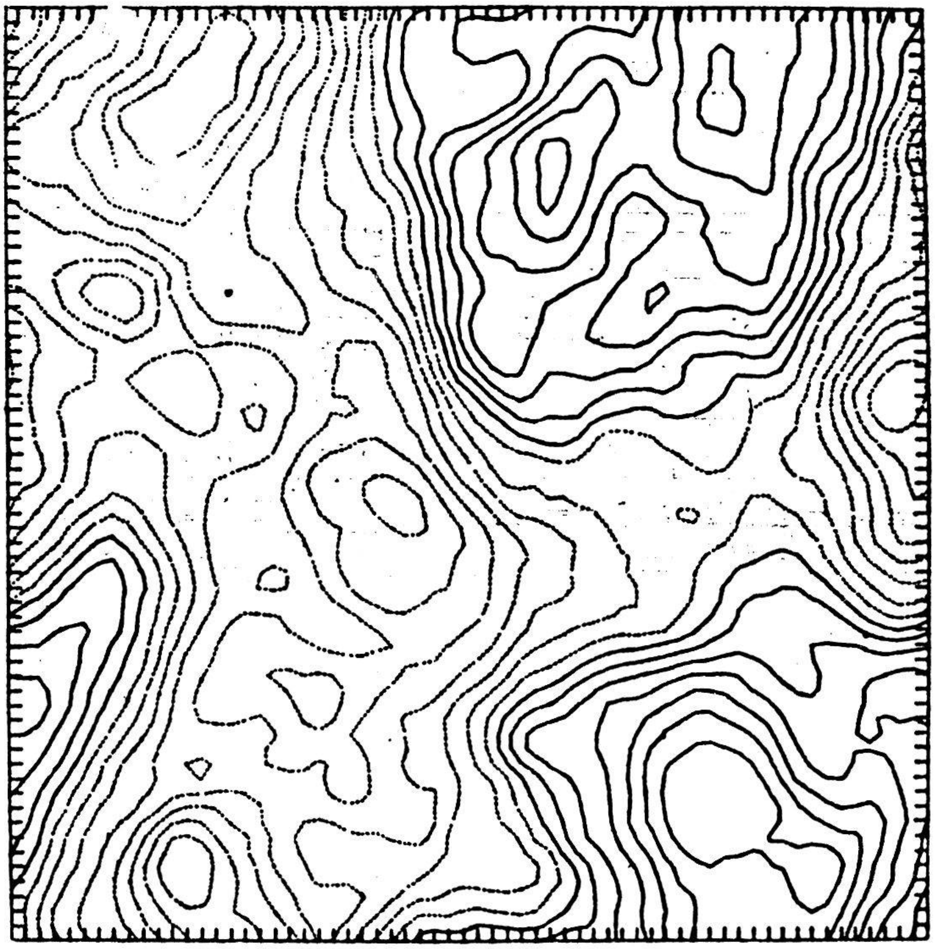
(a)

Figure 6. Stream function (a) and vorticity (b) at the 160th time step, $t = 10.87$, for the wavenumber 8 experiment with $\nu = 2.5 \times 10^{-4}$ and $\tau = 0.5$, all in dimensionless units.

(From D. K. Lilly, 1969)

(b)

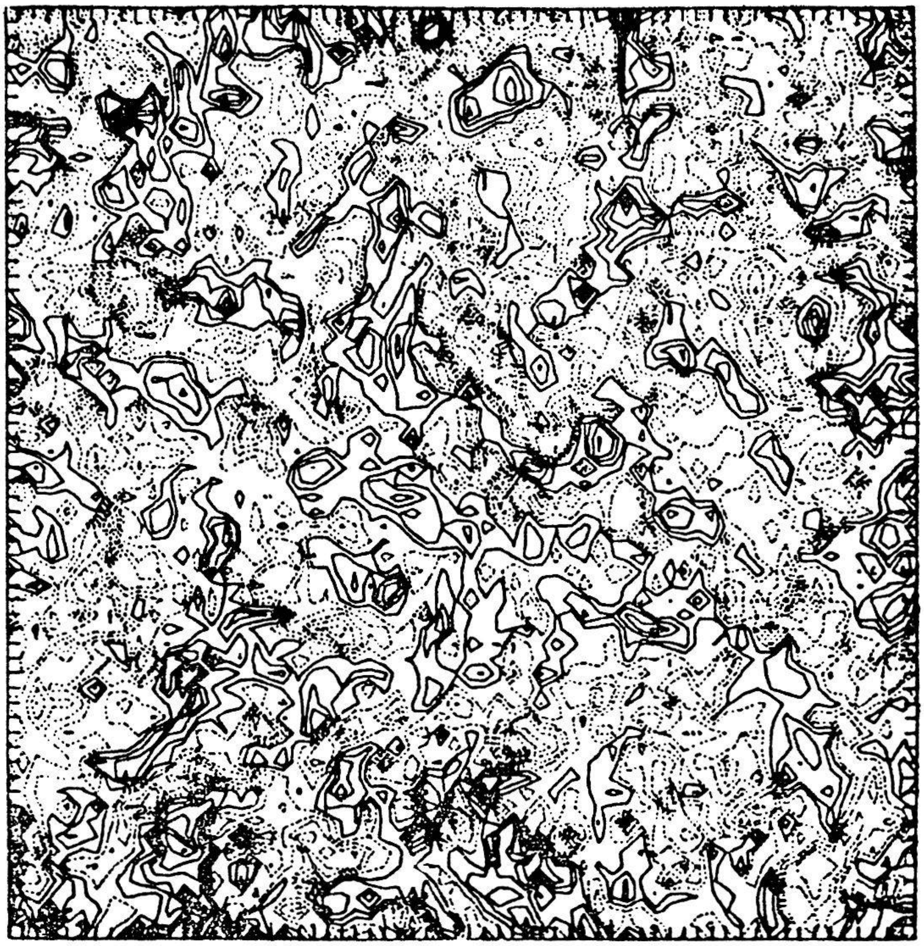
Figure 6(b)



(a)

Figure 7. Same as Figure 6 at the 2360th step, $t = 75.45$.

(From D. K. Lilly, 1969)



(b)

Figure 7(b)

Following Keller (1964), the operator $L(t)$ can be split into a mean part $\overline{L}(t)$ and a fluctuating part $L'(t)$ as follows -

$$L(t) = \overline{L}(t) + L'(t)$$

Upon averaging, equation (24) gives

$$\left[\frac{\partial}{\partial t} + \overline{L}(t)\right]\overline{\Omega}(t) = -\langle L'(t)\Omega'(t)\rangle \tag{26}$$

where the bars overhead (or $\langle\ \rangle$) refer to the average, and the primes refer to the fluctuation, and

$$\Omega(t) = \overline{\Omega}(t) + \Omega'(t)\,.$$

Upon subtracting equation (26) from equation (24), we obtain

$$\left[\frac{\partial}{\partial t} + L(t)\right]\Omega'(t) = -L'(t)\overline{\Omega}(t) + \langle L'(t)\Omega'(t)\rangle \tag{27}$$

If we introduce Green's function $G(t,t')$ for the operator $\left[\frac{\partial}{\partial t} + L(t)\right]$, defined by -

$$\left[\frac{\partial}{\partial t} + L(t)\right]^{-1} H(t) = \int G(t,t')H(t')dt' \tag{28}$$

the solution of equation (27) can be written formally as

$$\Omega'(t) = \int G(t,t')\,[\langle L'(t')\Omega'(t')\rangle - L'(t')\overline{\Omega}(t')]dt' \tag{29a}$$

and on iteration, as the Neumann series -

$$\Omega'(t) \approx -\int G(t,t')L'(t')\overline{\Omega}(t')dt' +$$

$$-\int G(t,t')\left\langle L'(t')\int G(t',t'')L'(t'')\overline{\Omega}(t'')dt''\right\rangle dt' + \ldots \tag{29b}$$

As a first approximation we will retain only the first term on the right-hand side of equation (29). Equation (26) and the fluctuating part of equation (25) then become

$$\left[\frac{\partial}{\partial t} + \overline{L}(t)\right]\overline{\Omega}(t) = \int dt'\langle L'(t)G(t,t')L'(t')\rangle\overline{\Omega}(t') \tag{30}$$

$$\nabla^2\psi'(t) = \int dt' G(t,t')L'(t')\overline{\Omega}(t') \tag{31}$$

If we now suppose that the integrand on the right in equation (30) peaks near $t' = t$ for a short period of time of τ_c, the correlation time of the fluctuations (or the eddy turnover time), and that $\overline{\Omega}(t)$ is sufficiently smooth and does not change significantly during this period, we may ignore the non-local character of the diffusion operator in equation (30). The latter then becomes the Fokker-Planck equation -

$$\left(\frac{\partial}{\partial t} + \overline{\underset{\sim}{v}}\cdot\nabla\right)\overline{\Omega}\left(\underset{\sim}{x},t\right) = \frac{\partial}{\partial \underset{\sim}{x}}\cdot\underset{\approx}{D}\left(\underset{\sim}{x},t\right)\cdot\frac{\partial\overline{\Omega}\left(\underset{\sim}{x},t\right)}{\partial \underset{\sim}{x}}$$

and equation (31) becomes

$$\left[\nabla\times\underset{\sim}{v}'\left(\underset{\sim}{x},t\right)\right]\cdot\underset{\sim}{\hat{i}}_z = -\frac{\partial\overline{\Omega}\left(\underset{\sim}{x},t\right)}{\partial \underset{\sim}{x}}\cdot\int dt' G(t,t')\underset{\sim}{v}'\left(\underset{\sim}{x},t'\right) \tag{33}$$

Here, assuming that the fluctuations are stationary, the diffusion coefficient $\underset{\approx}{D}$ is given by,

$$\underset{\approx}{D}(x,t) = \int dt'\left\langle \underset{\sim}{v}'\left(\underset{\sim}{x},t\right)G(t,t-t')\underset{\sim}{v}'\left(\underset{\sim}{x},t-t'\right)\right\rangle$$

which embodies the fluctuation-dissipation theorem (Kubo, 1957).

Equation (32) signifies the self-degradation of vorticity and implies that the evolution of vorticity in two-dimensional turbulence can be considered to be a Markov process. This is plausible if we note that when the vorticity has evolved for a time long compared with the correlation time of the enstrophy cascade the enstrophy-transfer process would have completed a large number of steps in the cascade, each of which produces a small random contribution.

6. Batchelor-Kraichnan Theory of the Inertial Ranges:

Kolmogorov's (1941) theory of the inertial range occupies a central place in the theory of three-dimensional turbulence. Kolmogorov (1941) argued that there exists a certain range, called the inertial range, in the wavenumber space which is in a state of statistical equilibrium in the sense that there is neither a source nor a sink of energy. The energy spectrum is assumed to cascade here smoothly through nonlinear processes in a stationary state. Furthermore, the energy spectrum $E(k)$ in the inertial range is assumed to depend only on the wavenumber k and on the rate ε at which energy is cascaded per unit mass. Dimensional arguments then imply that $E(k)$ has the form

$$E(k) = C\varepsilon^{2/3}k^{-5/3} \tag{34}$$

where C is a dimensionless constant.

In two-dimensional turbulence the existence of two conserved quantities, energy and enstrophy, imply the possibility of two cascades with inertial ranges of the Kolmogorov type. Using dimensional arguments, Kraichnan (1967) and Batchelor (1969) gave for the inertial-range energy spectrum $E(k)$ the following form in the energy cascade -

$$E(k) = C_1\varepsilon^{2/3}k^{-5/3} \tag{35}$$

and the following form in the enstrophy cascade -

$$E(k) = C_2\tau^{2/3}k^{-3} \tag{36}$$

where ε and τ is the rate of cascade of energy and enstrophy per unit area, respectively, and C_1, C_2 are dimensionless constants.

Kraichnan (1971) later gave a more refined analysis in which he suggested that the eddy-turnover time $\mathbf{T}$ (which was given by the local expression $\mathbf{T} \sim [k^3E(k)]^{-1/2}$ in deriving (25)) be given by the non-local expression $\mathbf{T}(k) \sim \left[\int_0^k p^2E(p)dp\right]^{-1/2}$. This would then lead to the log-corrected spectrum $E(k) \sim \tau^{2/3}k^{-3}[ln(k/k_c)]^{-1/3}$ for the enstrophy cascade.

7. <u>The Intermittency Corrections to the Batchelor-Kraichnan Scaling Laws</u>:

The Batchelor-Kraichnan study does not take into account the spatial intermittency in the flow that arises due to the non-Gaussian nature of the small-scale statistics and leads to the spatial randomness of kinetic energy and enstrophy dissipation rates. The latter would be expected to depend on the Reynolds number and to cause at the upper end of the energy subrange and the lower end of the enstrophy subrange systematic departures from the Batchelor-Kraichnan scaling laws which use mean dissipative values.

However, it is now well known that the intermittency effects associated with small scales are not accessible to traditional closure calculations (Kraichnan (1974), Nelkin (1975) and Frisch et al. (1978)). Therefore, one takes a heuristic approach to this problem whereby one makes ad hoc assumptions about the stochastic nature of the kinetic energy and enstrophy dissipation rates.

Alternatively, one could follow Mandelbrot (1976) and argue that the deviations from the Batchelor-Kraichnan scaling laws are related to fractal aspects of the geometry of two-dimensional turbulence. In particular, one may assume that the kinetic energy and enstrophy dissipations are concentrated on sets with noninteger Hausdorff dimensions. These ideas may be formulated in a simpler way through the so-called β-model (Frisch et al. (1978)). The key assumption in this model is that the kinetic energy and enstrophy are transferred to only a fixed fraction β of the eddies downstream in the cascade.

8. β-Model for the Intermittency Corrections to Inverse Energy Cascade:

Let us briefly review the β-model of Frisch et al. (1978) applied to the inverse energy cascade. One considers a discrete sequence of scales

$$l_n = l_0 p^n \ ; \quad n = 0, 1, 2, \dots \tag{37}$$

and a discrete sequence of wavenumbers $k_n = l_n^{-1}$. Here, p is the constant ratio of the cascade in sizes. The kinetic energy per unit mass in the *nth* scale is defined by

$$E_n = \int_{k_n}^{k_{n+1}} E(k)dk \tag{38}$$

One assumes a statistically stationary turbulence where energy is introduced into the fluid at scales $\sim l_0$ and is then transferred successively to scales $\sim l_1, l_2, \dots$, until some scale l^* is reached where l^* is the macroscopic size of the system. One now makes an assumption that at the *nth* step, only a fraction β^n of the total space has an appreciable excitation.

The kinetic energy per unit mass in the *nth* scale is then given by

$$E_n \sim \beta^n V_n^2 \tag{39}$$

where V_n is a characteristic velocity of the *nth* scale, and using (37),

$$\beta^n = (p^{D-2})^n \sim \left(\frac{l_n}{l_0}\right)^{D-2} \tag{40}$$

D is the fractal dimension of the region in which the energy dissipation is concentrated. (40) expresses the fact that intermittency increases with increase of size in the inverse cascade. If intermittency increases as scale size increases, and Kraichnan-Batchelor basic ideas hold in local regions, then the cascade becomes less and less efficient as l_n increases and $E(k)$ must fall off less rapidly than $k^{-5/3}$ if, according to conservation of energy, the overall energy cascade rate is independent of l_n.

The rate of transfer of energy per unit mass from the *nth* scale to the $(n+1)th$ scale is given by

$$\varepsilon_n \sim \frac{E_n}{t_n} \sim \frac{\beta^n V_n^3}{l_n} \tag{41}$$

where t_n is a characteristic time of the *nth* scale, $t_n = l_n/V_n$. In the energy inertial range, one assumes a stationary process in which energy is introduced at scales $\sim l_0$; conservation of energy requires

$$\varepsilon_n = \bar{\varepsilon}, \quad l^* \geq l_n \geq l_0 \tag{42}$$

It is convenient to think of $\bar{\varepsilon}$ also as the mean dissipation rate which is what it would be when the eddies are of the order of the Kolmogorov length scale η. (40)-(42) then give

$$V_n \sim \bar{\varepsilon}^{-1/3} l_n^{1/3} \left(\frac{l_n}{l_0}\right)^{-(D-2)/3} \tag{43}$$

$$t_n \sim \bar{\varepsilon}^{-1/3} l_n^{2/3} \left(\frac{l_n}{l_0}\right)^{(D-2)/3} \tag{44}$$

$$E_n \sim \bar{\varepsilon}^{2/3} l_n^{2/3} \left(\frac{l_n}{l_0}\right)^{(D-2)/3} \tag{45}$$

(45) leads to the energy spectrum (Frisch et al. 1978)

$$E(k) \sim \bar{\varepsilon}^{2/3} k^{-5/3} (kl_0)^{(2-D)/3} \tag{46}$$

(46) shows that the intermittency corrections to the inverse energy cascade decrease the 5/3 exponent. The general nature of this result was actually predicted by Kraichnan (1975). This result is also in qualitative agreement with the predictions for small wavenumbers of the generalized spectral law (70) for the inverse energy cascade (see below). Observe that according to (46), the inverse energy cascade cannot have a spectrum flatter than k^{-1}. Thus, intermittency by itself is unable to account fully for the energy equipartition spectrum.

Let us now discuss the manner in which the fractal dimension influences the development and termination of the inverse energy cascade. The first cascade stage leads to curds (to borrow Mandelbrot's terminology) of size $l_0 p$ in which energy dissipation is equal to either 0 or εp^{2-D}, and the Kolmogorov scale is $\eta p^{-(2-D)/4}$, where $\eta \equiv (\nu^3/\varepsilon)^{1/4}$. In the *nth* stage, the average dissipation is $\varepsilon p^{n(2-D)}$, the curd size is $l_0 p^n$, and the Kolmogorov scale is $\eta p^{-n(2-D)/4}$. Thus, the Kolmogorov scale decreases with increase in n, but the curd size increases with n. This means that the cascading will continue until $l_0 p^n \sim l^*$, where l^* is the macroscopic size of the system. Thus, the fractal dimension rules only the manner in which the inverse cascade proceeds and not the point where it stops, unlike the direct cascade.

9. β-Model for the Intermittency Corrections to the Enstrophy Cascade:

The transfer of enstrophy to the small scales of motion is less well understood and has been a subject of some controversy. Numerical simulations have not been able to clarify this process because of their limitations due to finite degree truncations and the use of eddy viscosities that distort the inviscid behavior at small scales. Weiss (1991), on the basis of numerical solution of Euler's equations, contends that enstrophy transfer is associated with the stretching and folding of fluid in the hyperbolic regions.

In view of the regularity of two-dimensional flows, the concomitant velocity field develops no singularities in the limit of infinite Reynolds numbers. Thus, it might appear that the enstrophy dissipation structures must be space filling and hence exhibit no intermittency. Also, Kraichnan (1971) argued that intermittency will not affect the small-scale energy spectrum because the enstrophy-cascade interaction is not local in wavenumber space. The nonlocality of the enstrophy cascade casts doubt on the universality of the scaling law of the energy spectrum, because small-scale motion cannot be independent of the large-scale forcing mechanism and/or boundary conditions. This nonlocality also means that it takes an infinitely long time to initiate a fully developed spectrum in a nearly inviscid flow driven by random forcing at a fixed wavenumber. However, Basdevant et al. (1981) and Benzi et al. (1986) have shown that in the absence of any organized large-scale motion, intermittency is able to steepen the energy spectrum by restoring the spectral localness of nonlinear interactions. This intermittency is the result of the formation of spatially organized vortices, found in the numerical simulations of McWilliams $(1984)_b$, Benzi et al. (1986), Brachet et al. (1988) and Santangelo et al. (1989) in decaying situations after long periods of time, and also in some stationary forced situations with a forcing spectrum at high wavenumbers (Basdevant et al. 1981, Herring and McWilliams, 1985). Coherent vortex structures were also found in the laboratory experiments of Narimousa et al. (1991), as seen on the contour maps of the vorticity field (Figure 8). These coherent structures inhibit the local inertial transfer of enstrophy and lead to fluctuations in the enstrophy dissipation and are believed to produce steeper energy spectra. This scenario has been confirmed by the recent numerical simulations of Ohkitani (1991). Thus, though the enstrophy cascades toward small scales through nonlinear interactions, the measure of the spatial domain in which such transfers are active decreases as the scale size decreases (Basdevant and Sadourny, 1983). This provides the rationale for the application of the β-model to the enstrophy cascade.

Consider now a discrete sequence of scales

$$l_n = l_0 p^{-n} \, ; \quad n = 0, 1, 2, \ldots \tag{47}$$

and a discrete sequence of wavenumbers $k_n = l_n^{-1}$. The kinetic energy per unit mass in the *nth* scale is defined, as before, by

$$E_n = \int_{k_n}^{k_{n+1}} E(k) dk \tag{48}$$

Let us assume that we have a statistically stationary turbulence where enstrophy is introduced into the fluid at scales $\sim l_0$ and is then transferred successively

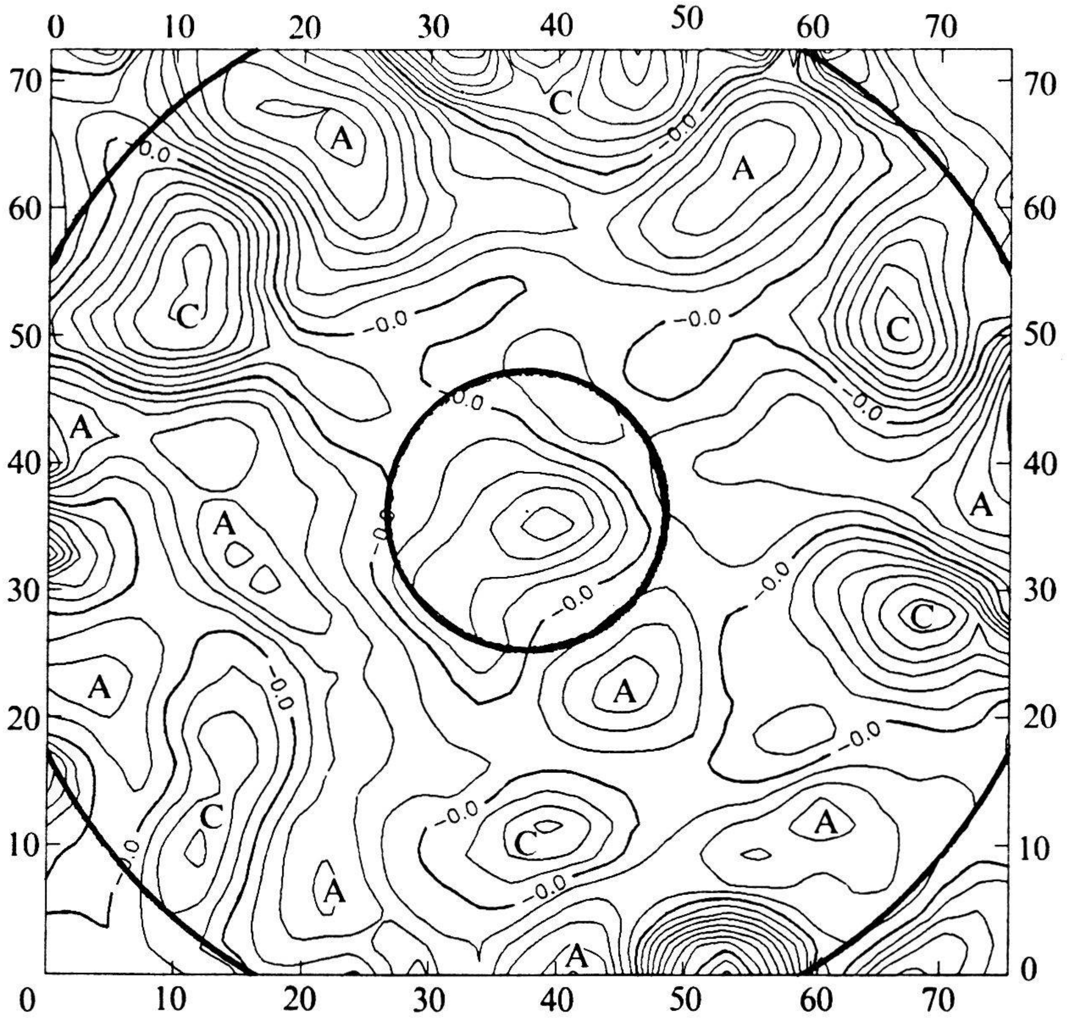


Figure 8. Isovorticity contours on a regular 32^2 grid. Contour levels drawn outside the boundaries of the tank (heavy lines) are artifacts of the contouring routine which insists on data on a rectangular grid. The cyclones are indicated by C, while the anticyclones are indicated by A.

(From S. Narimousa, T. Maxworthy and G. R. Spedding, 1991) to scales $\sim l_1, l_2, \ldots$, until some scale l_d is reached where viscous dissipation is able to compete with nonlinear transfer. We assume again that at the *nth* step, only a fraction β^n of the total space has an appreciable excitation.

The enstrophy per unit mass in the *nth* scale is then given by

$$D_n \sim \frac{\beta^n V_n^2}{l_n^2} \tag{49}$$

where,

$$\beta^n = (p^{\hat{D}-2})^n \sim \left(\frac{l_n}{l_0}\right)^{2-\hat{D}} \tag{50}$$

$\hat{D}$ is the fractal dimension of the region in which the enstrophy dissipation is concentrated. (34) expresses the fact that intermittency now increases with decrease of scale size. If intermittency increases as scale size decreases, and the Batchelor-Kraichnan theory is assumed to hold in local regions, then the enstrophy cascade would become more and more efficient as l_n decreases and $E(k)$ must fall off more rapidly than k^{-3} if, according to conservation of enstrophy, the overall enstrophy cascade rate is independent of l_n.

The rate of transfer of enstrophy per unit mass from the *nth* scale to $(n+1)th$ scale is given by

$$\tau_n \sim \frac{D_n}{t_n} \sim \frac{\beta^n \mathrm{V}_n^3}{l_n^3} \tag{51}$$

where t_n is a characteristic time of the *nth* scale, $t_n = l_n/\mathrm{V}_n$. In the enstrophy inertial range, we assume a stationary process in which enstrophy is introduced at scales $\sim l_0$ and removed at scales $\sim l_d$; conservation of enstrophy requires that

$$\tau_n = \bar{\tau}, l_d \leq l_n \leq l_0 \tag{52}$$

It is convenient to think of $\bar{\tau}$ also as the mean enstrophy dissipation rate which is what it would be when the eddies are of the order of the Kraichnan length scale ζ (see Sec. 11).

(50)-(52) then give

$$V_n \sim \bar{\tau}^{1/3} l_n \left(\frac{l_n}{l_0}\right)^{-(2-\hat{D})/3} \tag{53}$$

$$t_n \sim \bar{\tau}^{-1/3} \left(\frac{l_n}{l_0}\right)^{(2-\hat{D})/3} \tag{54}$$

$$E_n \sim \bar{\tau}^{2/3} l_n^2 \left(\frac{l_n}{l_0}\right)^{(2-\hat{D})/3} \tag{55}$$

(55) leads to the energy spectrum

$$E(k) \sim \bar{\tau}^{2/3} k^{-3} (kl_0)^{-(2-\hat{D})/3} \tag{56}$$

(56) shows that the intermittency corrections to the enstrophy cascade increase the 3 exponent. This is also in agreement with the predictions for large wavenumbers of the generalized spectral law (95) for the enstrophy cascade (see Sec. 11). Further, observe that according to (56), the enstrophy cascade cannot have a spectrum steeper than $k^{-11/3}$. The latter result has also been deduced directly from the Navier-Stokes equations (Sulem and Frisch, 1975; Pouquet, 1978). (The $k^{-11/3}$ spectrum was also shown by Gilbert, 1988, to correspond to the passive advection of spiral filaments which form around the coherent vortices observed in numerical simulation of decaying two-dimensional turbulence (McWilliams, 1984_a).) Thus, intermittency by itself is unable to account fully for steeper spectra observed in the numerical experiments.

Let us now discuss the lower bound for the fractal dimension $\hat{D}$ in the enstrophy cascade. Equating t_n to the viscous dissipation time, we have

$$\bar{\tau}^{-1/3} \left(\frac{l_n}{l_0}\right)^{(2-\hat{D})/3} \sim \frac{l_n^2}{\nu} \tag{57a}$$

or

$$l_n = l_0 R^{-\frac{3}{4+\hat{D}}} \tag{57b}$$

where,

$$R \equiv \frac{\bar{\tau}^{1/3} l_0^2}{\nu} \tag{58}$$

Now, from (50),

$$\hat{D} = \frac{\log \beta p^2}{\log p} = \frac{\log N}{\log p} \tag{59}$$

where N is the average number of offsprings, which can be less than unity, so that $\hat{D}$ can assume arbitrary negative values. However, according to (57), there is a dynamical reason to require $\hat{D} > -4$; otherwise, the enstrophy cascade will never be terminated by viscosity.

Let us next discuss further the manner in which the fractal dimension influences the development and termination of the enstrophy cascade. The first curdling stage leads to curds of size $l_0 p^{-1}$ in which enstrophy dissipation is equal to either 0 or $\tau p^{2-\hat{D}}$, and the Kraichnan scale is $\zeta p^{-(2-\hat{D})/6}$. In the *nth* stage, the average dissipation is $\tau p^{n(2-\hat{D})}$, the curd size is $l_0 p^{-n}$ and the Kraichnan scale is $\zeta p^{-n(2-D)/6.}$ Thus, both the Kraichnan scale and the curd size decrease with increase in n. However, curdling can continue only until the curd size is bigger than the Kraichnan scale and will stop thereafter. This occurs when

$$\zeta p^{-n(2-D)/6} \sim l_0 p^{-n}$$

or

$$\zeta / l_0 \sim p^{\left(1 - \frac{2-\hat{D}}{6}\right)n} \tag{60}$$

Hence, for the enstrophy cascade, the fractal dimension $\hat{D}$ rules not only the manner in which the curdling proceeds but also the point where it stops.

We have seen that the intermittency corrections mentioned above are inadequate and may actually be too small to allow an experimental or numerical verification at the usual level of resolution of kinetic energy and enstrophy spectra. However, it is possible to take a more general approach and to make a systematic analysis of the effect of nonlinear inertial and viscous effects on the kinetic energy and enstrophy spectra using the generalized von Karman-Heisenberg-von Weizsacker type model for the inertial transfer (Shivamoggi, 1990_a and 1990_b). For this purpose, first we need to start from a Fourier representation of the turbulent velocity and vorticity fields.

10. Generalized von Karman-Heisenberg-von Weizsacker Type Inertial-transfer Model for the Energy Cascade:

We have for the energy density $E(k)$ in the Fourier space

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)E\left(\underset{\sim}{k}\right)=\sum_{\underset{\sim}{k'}} W_{jml}\left(\underset{\sim}{k},\underset{\sim}{k'}\right) \tag{61}$$

where,

$$E\left(\underset{\sim}{k}\right)=\frac{1}{2}\left|V\left(\underset{\sim}{k}\right)\right|^2 \tag{62}$$

and

$$W_{jml}\left(\underset{\sim}{k},\underset{\sim}{k'}\right)=ik_m\left(\delta_{jl}-\frac{k_jk_l}{k^2}\right)V_j\left(\underset{\sim}{k}\right)V_m\left(\underset{\sim}{k'}\right)V_l\left(\underset{\sim}{k}-\underset{\sim}{k'}\right) \tag{63}$$

When the volume of the flow region becomes large, we may replace the Fourier sum in (61) by a Fourier integral

$$\sum_{\underset{\sim}{k'}} W\left(\underset{\sim}{k},\underset{\sim}{k'}\right)=\int Q(k,k')dk' \tag{64}$$

where $Q(k,k')$ is the net gain of energy by modes of wavenumber $\underset{\sim}{k}$ from all modes in the range k' to $k'+dk'$.

In order to write an expression for $Q(k,k')$, it is necessary to make some assumption about the nonlinear transfer of energy across the spectrum. We use a generalized von Karman-Heisenberg-von Weizsacker type model, according to which the transfer of energy from small to large wavelengths is described by a gradient-diffusion type cascade process (i.e., a large-scale rapidly adjusting motion superimposed on a large-scale slowly-adjusting motion) characterized by an eddy viscosity produced by small wavenumber modes acting to remove energy from large wavenumber modes. This idea is similar to the one proposed by von Karman (1948) for the transfer of turbulent kinetic energy in the three-dimensional case (which was a generalization of the idea proposed originally by Heisenberg (1948) and von Weizsacker (1948)). The present hypothesis of an eddy viscosity produced by small wavenumber modes is in accord with the conjecture of Kraichnan (1975a) that the eddy viscosity for the energy cascade in the two-dimensional case would be proportional to the total energy in the large scales and to a characteristic dynamical time of the latter. Kraichnan (1975b) also pointed out that the idea of a transport coefficient based on small-scale excitation in a two-dimensional turbulence is inapplicable.

If each mode in the range of wavenumbers from $k' = 0$ to $k' = k$ is to make a separate and similar contribution to the eddy viscosity $\tilde{\nu}(k)$ which depends on the energy density $E(k')$ and the wavenumber k' only, then by dimensional considerations, we may write

$$Q(k,k') = \begin{cases} 2A\,[E(k')]^{\frac{3}{2}-n} k'^{\frac{1}{2}-m} [E(k)]^n k^m, & k' > k \\ -2A\,[E(k)]^{\frac{3}{2}-n} k^{\frac{1}{2}-m} [E(k')]^n k'^m, & k' < k \end{cases} \tag{65}$$

where A is a universal constant and m and n are arbitrary constants.

The rate of loss of energy by modes with wavenumbers greater than some value k is given by

$$\int_k^\infty \frac{\partial E(k'')}{\partial t} dk'' = -2\nu \int_k^\infty E(k'')k''^2 dk'' - 2\tilde{\nu}(k) \int_k^\infty [E(k'')]^{\frac{3}{2}-n} k''^{\frac{1}{2}-m} dk'' \tag{66}$$

where

$$\tilde{\nu}(k) \equiv A \int_0^k [E(k')]^n k'^m dk' \tag{67}$$

Let us now replace the left-hand side in equation (66) by the total rate of decay of energy, ε. (This is valid for values of k such that

$$\int_k^\infty E(k'')dk'' \gg \int_0^k E(k'')dk''$$

which implies that only a negligible amount of energy is contained in wavenumbers less than k. This, in turn, requires the existence of a sink at the low-wavenumber end.) One then obtains from equation (66),

$$2\nu E(k)k^2 + [2A\,\{E(k)\}^n k^m] \times \frac{\left[-\varepsilon + 2\nu \int_k^\infty E(k'')k''dk''\right]}{2\tilde{\nu}(k)} + 2\tilde{\nu}(k)\,\{E(k)\}^{\frac{3}{2}-n} k^{\frac{1}{2}-m} = 0 \tag{68}$$

A solution of equation (68), for arbitrary values of m and n has not been obtained. However, it is possible to obtain the asymptotic forms of solution of equation (68), in the limit of small and large wavenumbers.

Thus, for large wavenumbers, which corresponds to $\nu \ll \tilde{\nu}(k)$, we obtain from equation (68),

$$E(k) \sim k^{-\frac{5}{3}} \tag{69}$$

which is the well-known result for the inertial range.

On the other hand, for small wavenumbers, which corresponds to $\nu \gg \tilde{\nu}(k)$, equation (68) gives the new branch

$$E(k) \sim k^{-\frac{m-2}{n-1}} \tag{70}$$

Now, for small wavenumbers, the two-dimensional turbulence tends to relax toward the equilibrium energy equipartition spectrum. Based on the work of Lee (1952) for three-dimensional turbulence, one may argue that, for a statistical equilibrium the canonical distribution for a given energy E will be $f(E) \sim e^{-(\sigma E + \mu k^2 E)}$ where σ is analogous to an inverse temperature (see Appendix). This yields, for a truncated system, an equipartition of energy among the various Fourier modes. Since the number of Fourier modes is, in two dimensions, proportional to $2\pi k$, the energy spectrum $E(k)$ goes like $E(k) \sim k$. (70) will agree with this result if we choose

$$m = \frac{5}{2}, \quad n = 1/2, \quad \text{say}. \tag{71}$$

It may be noted that there is, however, a radical difference between the character of the equipartition spectra in two-dimensional turbulence and three-dimensional turbulence. Whereas the equipartition spectrum in three-dimensional turbulence corresponds to the inviscid limit, viscosity plays an important role in the equipartition spectrum in two-dimensional spectrum. The numerical calculations of Pouquet et al. (1975) for two-dimensional turbulence have also shown that the dynamics of the large energetic scales is influenced by viscosity. Actually, this viscosity may be just "virtual" which simulates a sink at low wavenumbers required to sustain a stationary spectrum, as we discussed before. This would also enable the crossover from (69) to

(70) to occur at a wavenumber that lies inside the regime in question! Thus, the above inertial transfer model provides a satisfactory unified framework for describing both the inertial range and the equipartition range of energy cascade.

Now, in regard to the comparison of (69) and (70)/(71) with the experimental measurements, we first note that in experiments one measures the one-dimensional transverse spectrum $E_2(k)$ by the relation (Uberoi and Kovasznay, 1953)

$$\frac{E_2(k)}{k^2} = \frac{2}{\pi} \int_k^\infty \frac{E(p)/p^2}{\sqrt{p^2 - k^2}}\, dp \tag{72}$$

It is easy to see from (72), if $E(k) = Ck^n$, that

$$E_2(k) = \frac{2Ck^n}{\pi} \int_0^{\pi/2} \sec^{n-1} x dx \tag{73}$$

from which

$$\left. \begin{aligned} E_2(k) &\sim k^{-5/3} \quad \text{if} \quad E(k) \sim k^{-5/3} \\ E_2(k) &\sim k \quad \text{if} \quad E(k) \sim k \end{aligned} \right\} \tag{74}$$

(73) and (74) can now be compared with the experimentally measured spectral behavior (Sommeria, 1986) of the inverse energy cascade in a statistically steady two-dimensional turbulence. Figure 9 shows that the agreement is complete. (Nonetheless, it should be noted that there is some question about the significance of this agreement because in Sommeria's experiments dissipation is due to the interaction of the two-dimensional flow with the Hartmann boundary layers.) Physically, the inverse cascade is indicative of the formation of large-scale coherent structures like the pairing of large energetic scales of same vorticity sign.

It is of interest to note that the small wavenumber limit of (68), namely (60)/(61), can be recovered by a model based on a stationary continuous spectral cascading process for transfer of turbulent kinetic energy at small wavenumbers (Shivamoggi, 1987). (This idea is similar to the one proposed by Pao (1965) for transfer of turbulent kinetic energy at large wavenumbers in the three-dimensional case.)

In a stationary turbulence, Equation (61) can be written as

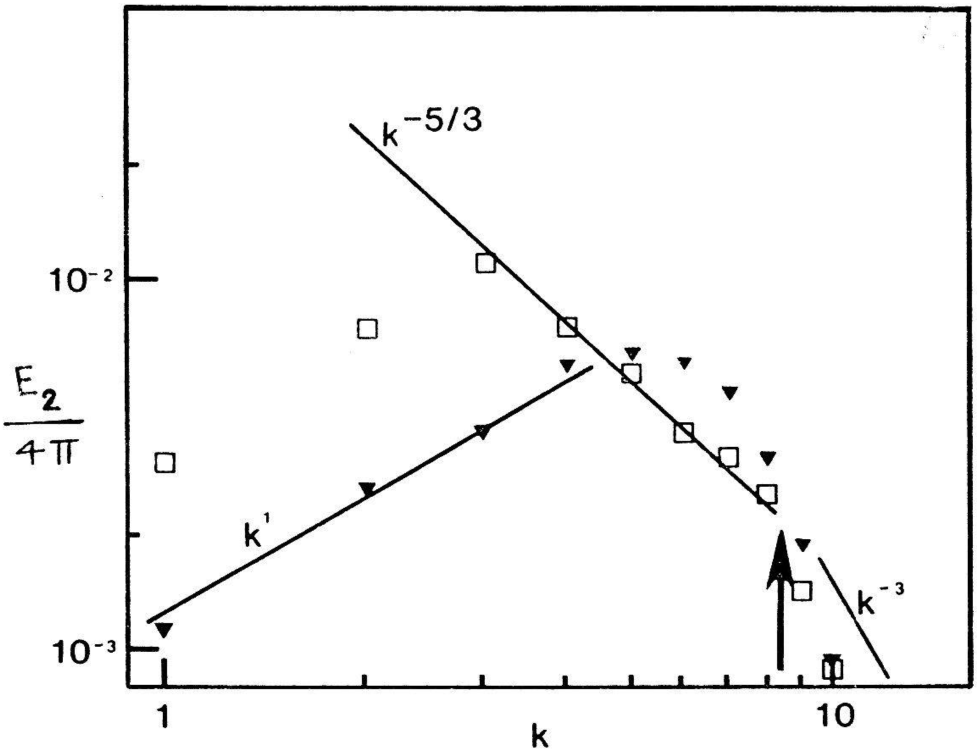


Figure 9. Experimentally observed energy spectra in a two-dimensional turbulence (J. Sommeria, 1986). Arrow indicates the injection wavenumber.

$$T(k) \equiv \int Q(k,k')dk' = \nu k^2 E(k) \tag{75}$$

$T(k)$ represents the contribution to the inertial transfer of energy to the mode of wavenumber k from all wavenumbers. Then, the energy flux from wavenumbers greater than k to the wavenumbers less than k is

$$S(k) = \int_0^k T(k)dk \tag{76}$$

or

$$\frac{dS}{dk} = T(k) \tag{77}$$

If we now visualize the transfer of turbulent energy as a cascading process in which the spectral energy is continuously transferred to ever smaller wavenumbers, the energy flux across k can then be written as

$$S(k) = E(k)\frac{dk}{dt} \tag{78}$$

where $\frac{dk}{dt}$ is the spectral cascading rate. Let us now assume that this process depends on ε (the rate at which the turbulent energy is fed to large eddies), on the viscosity ν (in accordance with (70)), and on the wavenumber k (or equivalently, the size of the large eddies). On dimensional grounds, we have then

$$\frac{dk}{dt} = -B\frac{\varepsilon}{k\nu^2} \tag{79}$$

where B is a positive constant. This reflects the fact that $dk/dt < 0$ for the inverse cascade. Using (77)-(79), (75) becomes

$$\frac{d}{dk}\left[-B\frac{\varepsilon}{k\nu^2}E(k)\right] = \nu k^2 E(k) \tag{80}$$

from which we have

$$E(k) = Cke^{-\frac{\nu^2}{16B\varepsilon}k^4} \tag{81}$$

where C is another constant. For $k \ll 2B^{1/4}\eta^{-1}$, η being the Kolmogorov length scale, $\eta \equiv (\nu^3/\varepsilon)^{1/4}$, (81) gives

$$E(k) \approx Ck \,. \tag{82}$$

This seems to show that the transfer of turbulent kinetic energy at small wavenumbers can be modeled in a satisfactory way as a stationary continuous spectral cascading process.

11. Generalized von Karman-Heisenberg-von Weizsacker Type Inertial-transfer Model for the Enstrophy Cascade:

We have from equation (10)

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)D\left(\underset{\sim}{k}\right) = \sum_{\underset{\sim}{k'}} U_{ml}\left(\underset{\sim}{k},\underset{\sim}{k'}\right) \tag{83}$$

where $D\left(\underset{\sim}{k}\right)$ is the enstrophy density in the Fourier space,

$$D\left(\underset{\sim}{k}\right) = \frac{1}{2}\left|\Omega\left(\underset{\sim}{k}\right)\right|^2 = \frac{1}{2}\left|\underset{\sim}{k}\times \underset{\sim}{V}\left(\underset{\sim}{k}\right)\right|^2 = k^2E\left(\underset{\sim}{k}\right) \tag{84}$$

and

$$U_{ml} = -ik_m\Omega_l\left(\underset{\sim}{k}\right)V_m\left(\underset{\sim}{k'}\right)\Omega_l\left(\underset{\sim}{k}-\underset{\sim}{k'}\right) \tag{85}$$

When the volume of the flow region becomes large, we may replace the Fourier sum in (83) by a Fourier integral

$$\sum_{\underset{\sim}{k'}} U\left(\underset{\sim}{k},\underset{\sim}{k'}\right) = \int G(k,k')dk' \tag{86}$$

where $G(k,k')$ is the net enstrophy again by modes of wavenumber k from all modes in the range k' to $k'+dk'$. In order to write an expression for this quantity, it is necessary to make some assumption about the nonlinear inertial transfer of enstrophy across the spectrum. We use a generalized von Karman-Heisenberg-von Weizsacker type model, according to which the process of transfer of enstrophy from large to small wavelengths is described by a gradient-diffusion type cascade process (i.e., a small-scale rapidly adjusting motion super-imposed on a large-scale slowly-adjusting motion) characterized by an eddy viscosity produced by large wavenumber modes acting to remove enstrophy from small wavenumber modes.

If each mode in the range of wavenumbers from $k' = k$ to $k' = \infty$ is to make a separate and similar contribution to the eddy viscosity $\tilde{\nu}(k)$ which depends on the energy density $E(k')$ and the wavenumber k' only, then by dimensional considerations, we may write

$$G(k,k') = \begin{cases} 2A\,[E(k')]^{\frac{3}{2}-n} k'^{\frac{1}{2}-m} [D(k)]^n k^m, & k' < k \\ -2A\,[D(k)]^{\frac{3}{2}-n} k^{\frac{1}{2}-m} [E(k')]^k k'^m, & k' > k \end{cases} \tag{87}$$

where A is a universal constant and m and n are arbitrary constants.

The rate of loss of enstrophy by modes with wavenumbers less than some value k is given by

$$\int_0^k \frac{\partial D(k'')}{\partial t} dk'' = -2\nu \int_0^k D(k'')k''^2 dk'' - 2\tilde{\nu}(k) \int_0^k [D(k'')]^{\frac{3}{2}-n} k''^{\frac{1}{2}-m} dk'' \tag{88}$$

where

$$\tilde{\nu}(k) \equiv A \int_k^\infty [E(k')]^n k'^m dk' \tag{89}$$

Let us now replace the left hand side in equation (88) by the total rate of decay of enstrophy, τ. (This is valid for values of k such that

$$\int_0^k D(k'')dk'' \gg \int_k^\infty D(k'')dk''$$

which implies that only a negligible amount of enstrophy is contained in wavenumbers greater than k.) One then obtains from equation (88),

$$2\nu D(k)k^2 + [2A\,\{E(k)\}^n k^m] \frac{\left[-\tau + 2\nu \int_0^k D(k'')k''^2 dk''\right]}{2\tilde{\nu}(k)}$$

$$+2\tilde{\nu}(k)\,[D(k)]^{\frac{3}{2}-n} k^{\frac{1}{2}-m} = 0 \tag{90}$$

A solution of equation (90), for arbitrary values of m and n has not been obtained. However, it is possible to obtain the asymptotic forms of solution of equation (90) in the limit of small and large values of the wavenumber k.

Thus, for small wavenumbers, which corresponds to $\nu \ll \tilde{\nu}(k)$, we obtain from equation (80),

$$E(k) \sim k^{\frac{4n}{3}-\frac{11}{3}} \tag{91}$$

(91) agrees with the well-known inertial-range result

$$E(k) \sim k^{-3} \tag{92}$$

if we choose

$$n = \frac{1}{2} \tag{93}$$

which also corresponds to the choice for n one has to make to reduce (87) to a Heisenberg-von Weizsacker type model. Thus, the present model has only one free parameter m and reduces completely to the Heisenberg-von Weizsacker type model by taking $m = -\frac{3}{2}$.

On the other hand, for large wavenumbers, which corresponds to $\nu \gg \tilde{\nu}(k)$, equation (90) gives the new dissipative branch

$$E(k) \sim k^{-\frac{m-4}{n-1}} \tag{94}$$

or on using (93),

$$E(k) \sim k^{2(m-4)} \tag{95}$$

For a Heisenberg-von Weizsacker type model, for which $m = -3/2$, an explicit solution of equation (90) can be obtained:

$$E(k) = \left(\frac{2\tau}{A}\right)^{2/3} k^{-3} \left[1 + \frac{4\nu^3}{A^2\tau} k^6\right]^{-4/3} \tag{96}$$

(96) shows that there is a new length scale ζ,

$$\zeta \equiv \left(\frac{\nu^3}{\tau}\right)^{1/6} \tag{97}$$

that characterizes the enstrophy cascade, just as the Kolmogorov scale characterizes the energy cascade. Let us call ζ the Kraichnan scale. (96) gives for $k \ll (A^2/4)^{1/6}\zeta^{-1}$,

$$E(k) \sim A^{-2/3}\tau^{2/3}k^{-3} \tag{98}$$

in agreement with (92). While (96) gives for $k \gg (A^2/4)^{1/6}\zeta^{-1}$,

$$E(k) \sim \frac{A^2\tau^2}{\nu^4}k^{-11} \tag{99}$$

in agreement with (95) when one puts $m = -3/2$.

(95) exhibits a more rapid decay of the spectrum for large wavenumbers. The spectrum in this range, according to (95), is in fact an arbitrarily steep power law. Nonetheless, it is possible to give an even more rapidly decaying exponential type spectrum, using a stationary continuous spectral cascading model.

A stationary continuous spectral cascading model gives a satisfactory description of the transfer of turbulent enstrophy at large wavenumbers because the later stages in the cascade tend toward a stationary process in the wavenumber space.

In stationary turbulence, Equation (83) can be written as

$$N(k) \equiv \int G(k,k')dk' = \nu k^2 D(k) \tag{100}$$

$N(k)$ represents the contribution to the inertial transfer of enstrophy to the mode of wavenumber k from all wavenumbers. Then, the enstrophy flux from wavenumbers less than k to wavenumbers greater than k is

$$R(k) = \int_k^\infty N(k)dk \tag{101}$$

or

$$\frac{dR}{dk} = -N(k) \tag{102}$$

If we now visualize the transfer of turbulent enstrophy as a cascading process in which the enstrophy is continuously transferred in the spectral space to ever larger wavenumbers, we can write

$$R(k) = D(K)\frac{dk}{dt} \tag{103}$$

where $\frac{dk}{dt}$ is the spectral cascading rate. Let us now assume that this process depends on τ (the rate at which the turbulent enstrophy is fed to small eddies), on the viscosity ν (in accordance with (99)), and on the wavenumber k (or equivalently, the size of the small eddies). On dimensional grounds, we have then

$$\frac{dk}{dt} = F\tau^{1/3}k \tag{104}$$

where F is a positive constant. This reflects the fact that $\frac{dk}{dt} > 0$ for the enstrophy cascade.

Using (102)-(104), (101) becomes

$$\frac{d}{dk}[F\tau^{1/3}k^3E(k)] = -\nu k^4 E(k) \tag{105}$$

from which we have

$$E(k) = Hk^{-3}e^{-\frac{\nu}{2F\tau^{1/3}}k^2} \tag{106}$$

where H is another arbitrary constant.

For $k \ll (8F^3)^{1/6}\zeta^{-1}$, (106) gives

$$E(k) \sim Hk^{-3} \tag{107}$$

in agreement with (92). (106) gives an exponential decay at very large wavenumbers.

12. Summary:

We have considered a generalized von Karman-Heisenberg-von Weizsacker type inertial transfer model for the energy and enstrophy cascades in a two-dimensional turbulence. This model gives spectra that are arbitrarily steep power laws for very high wavenumbers so that this model may be able to provide a satisfactory unified framework for describing both the inertial range and the strongly viscous range of the enstrophy cascade, like the case with three-dimensional turbulence. (This aspect is not conclusive yet, since the complete enstrophy cascade is still to be obtained in the laboratory.) This model also provides a satisfactory unified framework for describing both the inertial range and the equipartition range of the energy cascade observed in laboratory experiments. The small-wavenumber limit (namely, the equipartition regime) of the energy cascade and the large-wavenumber limit (namely, the dissipative regime) of the enstrophy cascade can also be modeled in a satisfactory way as stationary continuous spectral cascading processes.

The departures from the Batchelor-Kraichnan scaling laws can be described also in terms of intermittency corrections through the β-model which are found to be in qualitative agreement with the predictions made by the above generalized spectral laws. However, intermittency by itself has been shown to be unable to account fully for either the steeper spectra of the enstrophy cascade observed at large wavenumbers in numerical and laboratory experiments or the flatter spectra of the energy cascade observed at small wavenumbers in laboratory experiments. One may generalize the β-model to admit the possibility that the region containing in the energy or enstrophy dissipation is instead a non-homogeneous or a multi-fractal. Multi-fractal formalism is known (Stanley and Meakin, 1988) to be applicable to all systems where the underlying physics is governed by self-similar multiplicative processes like the aggregation/fragmentation processes in two-dimensional turbulence. Thus, in the spirit of Mandelbrot's (1976) weighted-curdling model, the contraction factors β's may be considered as independent random variables (Benzi et al. 1984) which can take different values in each scale i at the nth step of the cascade. It is to be noted that though much work has been done to account for the intermittency corrections, no definite theoretical framework toward this goal exists at the present time. A deductive theory, based directly on the Navier-Stokes equations, is what is really needed. But this has proved elusive as yet.

13. Appendix: Equilibrium Statistical Mechanics of Two-dimensional Turbulence

Let us consider a two-dimensional turbulence within a square which can be expanded into an infinite series of discrete wave vectors $\underset{\sim}{k}_n$ with velocity amplitudes $\underset{\sim}{V}\left(\underset{\sim}{k}_n, t\right)$ related by Euler's equations in Fourier space. These equations are truncated by retaining the modes lower than a cut-off wavenumber k_{max} (so as to preserve the validity of the inviscid model) and are suitably normalized to give for the stream function ψ (recall equation (14)):

$$\frac{d\Psi\left(\underset{\sim}{k}\right)}{dt}\left(\underset{\sim}{k}\right)=\frac{1}{2}\sum_{\underset{\sim}{k}=\underset{\sim}{k}'+\underset{\sim}{k}''}\Lambda^{k}_{k',k''}\Psi\left(\underset{\sim}{k}'\right)\Psi\left(\underset{\sim}{k}''\right) \tag{A.1}$$

Let $y_{n1}(t)$ and $y_{n2}(t)$ be the real and imaginary parts of each mode $\Psi\left(\underset{\sim}{k_n}\right)$. Then, if N wave vectors are retained in the truncation, the system can be represented by a point of $m = 2N$ coordinates $y_{n_i}(t)$ (i from 1 to 2) in a phase space determined by $y_\alpha(t)$ (α going from 1 to $2N$). Equation (A.1) conserves the kinetic energy

$$\frac{1}{2}\sum_{\underset{\sim}{k_n}} k_n^2\left|\Psi\left(\underset{\sim}{k_n}\right)\right|^2=\frac{1}{2}\sum_{\alpha=1}^{m} k_\alpha^2 y_\alpha^2(t) \tag{A.2}$$

and the enstrophy

$$\frac{1}{2}\sum_{\underset{\sim}{k_n}} k_n^4\left|\Psi\left(\underset{\sim}{k_n}\right)\right|^2=\frac{1}{2}\sum_{\alpha=1}^{m} k_\alpha^4 y_\alpha^2(t) \tag{A.3}$$

which implies that the system evolves in the phase space on the intersection of the kinetic energy sphere and the enstrophy ellipsoid. Let us consider in the phase space a collection of systems of density $\rho(\eta_1, .., \eta_m, t)$. Since the total number of such systems and hence the volumes are preserved in the phase space, we have the Liouville Theorem:

$$\frac{\partial\rho}{\partial t}+\sum_{\alpha=1}^{m}\frac{dy_\alpha}{dt}\frac{\partial\rho}{\partial y_\alpha}=0 \tag{A.4}$$

The typical approach of statistical mechanics is to explain the statistical behavior of a system in terms of its structural properties, like the conservation of energy. This would allow one to study the equilibrium spectra of two-dimensional turbulence from the viewpoint of microcanonical ensemble averages.

By the elementary Gibbsian methods of statistical mechanics, equilibrium solutions of equation (A.4) are constructed as functions of the conserved quantities, and are given by the Boltzmann type distribution

$$P(y_1, \ldots, y_m) = \frac{1}{Z} e^{-\frac{1}{2}\sum_{\alpha=1}^{m}\left(\sigma k_\alpha^2 y_\alpha^2 + \mu k_\alpha^4 y_\alpha^2\right)} \qquad (A.5)$$

where σ and μ are two constants and Z is the partition function of the system

$$Z = \int\int\cdots\int e^{-\frac{1}{2}\sum_{\alpha=1}^{m}\left(\sigma k_\alpha^2 y_\alpha^2 + \mu k_\alpha^4 y_\alpha^2\right)} dy_1 dy_2 \ldots dy_m \qquad (A.6)$$

One then assumes that the microcanonical ensemble average $\langle P(y_1, \ldots, y_m, t)\rangle$ of an ensemble of given system $P(y_1, \ldots, y_m, t)$ obeying equations (A.1) and (A.4) will eventually relax toward the equilibrium distribution (A.5). Indeed, the directions of cascades predicted by assuming approach toward equilibrium seem to be supported by computer simulations (Orszag, 1970).

The mean variance of the mode "α" of the stream function is given by

$$\langle y_\alpha^2(t)\rangle = \frac{1}{Z}\int\int\cdots\int y_\alpha^2 e^{-\frac{1}{2}\sum_{\beta=1}^{m} k_\beta^2\left(\sigma + \mu k_\beta^2\right) y_\beta^2} \quad dy_1 \ldots dy_m$$

$$= \frac{1}{k_\alpha^2(\sigma + \mu k_\alpha^2)} \qquad (A.7)$$

Thus,

$$\left\langle \left|\Psi\left(\underset{\sim}{k}\right)\right|^2 \right\rangle = \frac{2}{k_\alpha^2(\sigma + \mu k_\alpha^2)} \qquad (A.8)$$

and

$$E(k) = \pi k^3 \left\langle \left|\Psi\left(\underset{\sim}{k}\right)\right|^2 \right\rangle \sim \frac{k}{\sigma + \mu k^2} \qquad (A.9)$$

(A.9) shows that for the case $\sigma < 0$, the energy spectrum is dominated by the contributions form the largest wavelengths ($k = k_{\min} \approx \sqrt{-\sigma/\mu}$) which is in accord with the fact that the energy cascades toward large scales. Observe further that, for $k \Rightarrow 0$, (A.9) gives the spectrum of equipartition of kinetic energy among the modes -

$$E(k) \sim k \, . \qquad (A.10)$$

Thus, an inviscid finite system evolves towards an equipartition of energy among all Fourier modes (Orszag, 1970). However, as we have seen in Sec. 6, the situation is quite different for real flows (with infinite degrees of freedom), which evolve towards the Batchelor-Kraichnan scaling laws -

$$E(k) \sim k^{-5/3} \tag{A.11}$$

in the inverse energy cascade, and

$$E(k) \sim k^{-3} \tag{A.12}$$

in the direct enstrophy cascade. Thus, truncation of the modes acts as a barrier preventing possible cascades and can produce a significant alteration in the statistical properties of the system (Basdevant and Sadourny, 1974).

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References:

1. ARGOUL, F., ARNEODO, G., GRASSEAU, G., GAGNE, Y., HOPFINGER, E. J. and FRISCH, U.: Nature 338, 51, (1989).
2. BASDEVANT, C., LEGRAS, B., SADOURNY, R. and BELAND, M.: J. Atmos. Sci. 38, 2305, (1981).
3. BASDEVANT, C. and SADOURNY, R.: J. Fluid Mech. 69, 673, (1974).
4. BASDEVANT, C. and SADOURNY, R.: J. Mec. Theor. et. Appl., Numero Special, 243, (1983).
5. BATCHELOR, G. K.: Phys. Fluids Suppl. II, 233, (1969).
6. BENZI, R., PALADIN, G., PARISI, G. and VULPIANI, A.: J. Phys. A 17, 3521, (1984).
7. BENZI, R., PALADIN, G., PATARNELLO, S., SANTANGELO, P. and VULPIANI, A.: J. Phys. A. 19, 3771, (1986).
8. BRACHET, M. E., MENEGUZZI, M., and SULEM, P. L.: Phys. Rev. Lett. 57, 683, (1986).
9. BRACHET, M. E., MENEGUZZI, M., POLITANO, H. and SULEM, P. L.: J. Fluid Mech. 194, 333, (1988).
10. COUDER, Y.: J. Phys. Lett. 45, 353, (1984).
11. DESBOIS, M.: J. Atmos. Sci. 32, 1838, (1975).
12. FJORTOFT, R.: Tellus 5, 225, (1953).
13. FORNBERG, B.: J. Comp. Phys. 25, 1, (1977).
14. FRISCH, U. and SULEM, P. L.: Phys. Fluids 27, 1921, (1984).
15. FRISCH, U., LESIEUR, M. and SULEM, P. L.: Phys. Rev. Lett. 37, 895, (1967).
16. FRISCH, U., SULEM, P. L., and NELKIN, M.: J. Fluid Mech. 87, 719, (1978).
17. GILBERT, A. D.: J. Fluid Mech. 193, 475, (1988).
18. GREENSPAN, H. P.: Theory of Rotating Fluids, Cambridge University Press, (1968).
19. HASEGAWA, A.: Adv. Phys. 34, 1, (1985).
20. HEISENBERG, W.: Z. Phys. 124. 628, (1948).
21. HERRING, J. and McWILLIAMS, J. C.: J. Fluid Mech. 153, 229, (1985).
22. HERRING, J., ORSZAG, S. A., KRAICHNAN, R. H., and FOX, D. G.: J. Fluid Mech. 66, 417, (1974).
23. JULIAN, P. R., WASHINGTON, W. M., HEMBREE, L. and RIDLEY, C.: J. Atmos. Sci. 27, 376, (1970).

24. KELLER, J. B.: Proc. Symp. Appl. Math. 16, 84, (1964).

25. KIDA, S.: Phys. Fluids 24, 604, (1981).

26. KIDA, S.: J. Phys. Soc. Japan 54, 2840, (1985).

27. KRAICHNAN, R. H.: J. Math. Phys. 2, 124, (1961).

28. KRAICHNAN, R. H.: Phys. Fluids 10, 1417, (1967).

29. KRAICHNAN, R. H.: J. Fluid Mech. 47, 525, (1971).

30. KRAICHNAN, R. H.: In Statistical Mechanics: New Concepts, New Problems, New Applications, (ed. S. A. Rice, K. F. Freed, and J. C. Light), p. 201, University of Chicago Press, (1972).

31. KRAICHNAN, R. H.: J. Fluid Mech. 64, 737, (1974).

32. KRAICHNAN, R. H.: J. Fluid Mech. 67, 155, (1975a).

33. KRAICHNAN, R. H.: Adv. Math. 16, 305, (1975b)

34. KRAICHNAN, R. H. and MONTGOMERY, D.: Rep. Progr. Phys. 44, 35, (1980).

35. KUBO, R.: J. Phys. Soc. Japan 12, 570 & 1203, (1957).

36. LEE, T. D.: Q. Appl. Math. 10, 69, (1952).

37. LEITH, C. E.: Phys. Fluids II, 671, (1968).

38. LILLY, D. K.: Phys. Fluids Suppl. II, 240, (1969).

39. MANDELBROT, B.: In Turbulence and Navier-Stokes Equation, (ed. R. Teman). Lecture Notes In Mathematics, vol 565, p. 121, (1976).

40. McWILLIAMS, J. C.: J. Fluid Mech. 146, 21, $(1984)_a$.

41. McWILLIAMS, J. C.: In "Predictability of Fluid Motions," LaJolla Institute 1983, AIP Conference Proceedings (Ed. G. Holloway and B. J. West), 106, 205, $(1984)_b$.

42. MOREL, P. and NECCO, G.: J. Atmos. Sci. 30, 909, (1973).

43. MOREL, P. and LARCHEVEQUE, M.: J. Atmos. Sci, 31, 2189, (1974).

44. NARIMOUSA, S., MAXWORTHY, T. and SPEDDING, G. R.: J. Fluid Mech. 223, 113, (1991).

45. NELKIN, M.: Phys. Rev. A 11, 1737, (1975).

46. OGURA, Y.: J. Meteor. 15, 375, (1958).

47. OHKITANI, K.: Phys. Fluids A 3, 1598, (1991).

48. ORSZAG, S. A.: J. Fluid Mech. 41, 363, (1970).

49. PAO, Y. H.: Phys. Fluids 8, 1063, (1965).

50. POUQUET, A.: J. Fluid Mech. 88, 1, (1978).

51. POUQUET A., LESIEUR, M., ANDRE, J. C., and BASDEVANT, C.: J. Fluid Mech. 72, 305, (1975).

52. RHINES, P.: Ann. Rev. Fluid Mech. 11, 401, (1979).

53. SANTANGELO, P., BENZI, R. and LEGRAS, B.: Phys. Fluids A1, 1027, (1989).

54. SHIVAMOGGI, B. K.: Phys. Lett. A 122, 145, (1987).

55. SHIVAMOGGI, B. K.: J. Phys. A 23, 1689, $(1990)_a$.

56. SHIVAMOGGI, B. K.: Nuovo Cimento B 105, 1359, $(1990)_b$.

57. SOMMERIA, J.: J. Fluid Mech, 170, 139, (1986).

58. STANLEY, H. E. and MEAKIN, P.: Nature 335, 405, (1988).

59. SULEM, P. L. and FRISCH, U.: J. Fluid Mech. 72, 417, (1975).

60. UBEROI, M. S. and KOVASZNAY, L. S. G.: Q. Appl. Math. 10, 375, (1953).

61. UBEROI, M. S. and NARAIN, J. P.: Z. Phys. 269, 1, (1974).

62. VON KARMAN, T.: Proc. Nat. Acad. Sci., Washington 34, 530, (1984).

63. VON WEIZSACKER, C. F.: Z. Phys. 124, 614, (1948).

64. WEISS, J.: Physica D 48, 273, (1991).

65. WIIN-NIELSEN, A.: Tellus 19, 540, (1967).