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Plasmon Frequency for a Spin-Density Wave Model

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Abstract

A prototype of a system with a long range interaction, a mean field model exhibiting a phase transition showing a spin density wave, is considered, in order to explain rigorously the occurrence of a discrete plasma frequency in the fluctuation spectrum. A canonical pair is found corresponding to the fluctuations of the relative (spin up, spin down) spin density, respectively the local order parameter density. The pair decouples from the other degrees of freedom of the system.

1 Introduction

The standard Goldstone theorem [1] for short range interactions in many-body systems implies specific properties of the spectrum of the time evolution at zero momentum ($k = 0$) [2]. It is known that due to the breakdown of the Galilei invariance an energy gap in the density excitation spectrum is incompatible with short range forces, i.e. in the mathematical language of [3] the symmetry group may be not implementable if long range forces are present.

This situation beyond the Goldstone theorem, has been the subject of many studies for a long time. It is referred to as the phenomenon of oscillations with frequency spectrum taking a finite value $\omega \neq 0$ at $k = 0$. From the physical point of view the phenomenon has had much attention (see e.g. [4,5,6]). In the various discussions of this problem, one is particularly concerned with the computation of these frequencies. Different approximations, e.g. the random phase approximation, yield good experimental values. However the mechanism or the mathematical frame in which it can be understood was still lacking.

Recently we developed central limit theorems for non-commuting random variables (operators) which made it clear how to associate to quantum systems, a macroscopic system of fluctuations with a dynamics induced by the microdynamics of the system [7]. We worked out already this theory for the strong coupling BCS-model [8] and the Schwinger model [9]. We were able to describe the above phenomenon rigorously as part of the spectrum of the phonons carried by the fluctuation fields. In [10] we computed the full dynamics of the macroscopic field fluctuations in thermodynamic equilibrium for the Overhauser model [11,12]. This model describes electronic interactions in certain metals.

Below a critical temperature T_c the interactions give rise to a coherent excitation of a spin-density wave with wave vector q. The local Hamiltonian in the volume V is given by

$$\begin{aligned} H_V &= \sum_{i=1}^{2} \frac{1}{2} \int_V dx\, \nabla C_i^+(x) \cdot \nabla C_i(x) \\ &\quad - \frac{\lambda}{V} \int_V dx\, e^{iqx} C_1^+(x) C_2(x) \int_V dz\, e^{-iqz} C_2^+(z) C_1(z) \end{aligned} \tag{1.1}$$

where the C_i^+, C_i, $i = 1,2$ are the usual Fermion creation and annihilation operators satisfying the anti-commutation relations.

We consider here the translation invariant case i.e. we put the wave vector $q = 0$ or in other words we consider the limit of the wavelength of the spin wave tending to infinity. Hence the actual model is given by

$$H_V = \sum_{i=1}^{2} \frac{1}{2} \int_V dx\, \nabla C_i^+(x) \cdot \nabla C_i(x) - \frac{\lambda}{V} \int_V dx\, C_1^+(x) C_2(x) \int_V dz\, C_2^+(z) C_1(z) \tag{1.2}$$

The symmetry group of the model is the translation group and the two-dimensional gauge group $\mathcal{G}$ represented by the gauge $*$-automorphisms of the algebra of anti-commutation relations :

$$\mathcal{G} = \{\gamma_{\alpha_1,\alpha_2} | \alpha_i \in [0, 2\pi]\} \tag{1.3}$$

where

$$\begin{aligned} \gamma_{\alpha_1,\alpha_2} C_1^+(x) &= e^{i\alpha_1} C_1^+(x) \\ \gamma_{\alpha_1,\alpha_2} C_2^+(x) &= e^{i\alpha_2} C_2^+(x) . \end{aligned}$$

The infinitesimal generators of this gauge group $\mathcal{G}$ are given by

$$N_i = \int dx\, C_i^+(x) C_i(x) \ ; \ i = 1,2 . \tag{1.4}$$

Below the critical temperature T_c the symmetry $\mathcal{G}$ breaks down spontaneously to a one-dimensional subgroup $\mathcal{G}_0 = \{\gamma_{\alpha,\alpha} | \alpha \in [0, 2\pi]\}$ acting as follows :

$$\gamma_{\alpha,\alpha} C_i^+(x) = e^{i\alpha} C_i^+(x) \ , \ i = 1, 2 \, .$$

The infinitesimal generator of the gauge group $\mathcal{G}_0$ is given by $N_1 + N_2$.

In particular the operator $N_1 - N_2$ is not a constant of the motion anymore below T_c, it is the generator of a spontaneously broken symmetry. Clearly this operator is an integral over the local density $n(x) = n_1(x) - n_2(x)$,

$$N_1 - N_2 = \int dx \, n(x)$$

where $n_i(x) = C_i^+(x) C_i(x)$, $i = 1, 2$. We are particularly interested in the time evolution of this operator. As the spectrum of the microscopic time evolution will turn out to be absolutely continuous except for the invariant vector, the time evolution of $n(x)$ does not show anything interesting. However, along the lines of the general theory as exposed in [7] one considers the fluctuation $\tilde{Q}$ of the local density $n(x)$:

$$\tilde{Q} \approx \lim_V \frac{1}{\sqrt{V}} \int_V dx \, n(x) \tag{1.5}$$

together with the fluctuation $\tilde{P}$ of, essentially, the density of the order parameter operator of this model. We show that these macroscopic observables $\tilde{Q}$ and $\tilde{P}$ are operators forming a canonical pair i.e. they yield a representation of the canonical commutation relations. Furthermore the mean square deviations $\tilde{Q}^2$ of the density $n(x)$ and of the order parameter density, namely $\tilde{P}^2$ satisfy the virial theorem

$$\langle \tilde{P}^2 \rangle = \omega^2 \langle \tilde{Q}^2 \rangle \tag{1.6}$$

with a frequency ω which can be interpreted as a plasma frequency of a spin density wave, described again by the density $n(x)$. The value of this frequency is found to be equal to $\omega = 2\lambda|b|$; λ is the coupling constant and $|b|$ is the absolute value of the order parameter of the phase transition. In particular the phenomenon disappears above the critical temperature.

We are also able to solve the time evolution of the macroscopic fluctuations $\tilde{Q}$ and $\tilde{P}$. Actually suggested by the harmonic oscillator

$$\frac{\tilde{P}^2}{2} + \frac{\omega^2}{2}\tilde{Q}^2 \tag{1.7}$$

we get the virial theorem (1.6) and we prove rigorously the following oscillator solution

$$\tilde{Q}(t) = \tilde{Q}\cos\omega t + \frac{\tilde{P}}{\omega}\sin\omega t\,.$$

This proves that the frequency ω is a discrete point in the spectrum of the macroscopic system of $\{\tilde{Q}, \tilde{P}\}$-fluctuations. This system decouples completely from the other coordinates of the system, it corresponds to the spin wave mode. Finally, motivated by the above results, we compute the thermal equilibrium distributions of $\tilde{Q}^2$ and $\tilde{P}^2$ and find, as expected, that they are distributed according to the Gibbs distribution, determined by a harmonic oscillator (1.7), depending only on the temperature and on the plasma frequency. The latter computation is not based on the explicit use of the equilibrium state expectation values, but purely on correlation inequalities for equilibrium states, showing the model independence of the above results.

2 Equilibrium states

First we determine the translation invariant equilibrium states of the model (1.2). As the algebra of observables is the separable CAR-algebra $\mathcal{A}$ which is asymptotically abelian, there exists a unique decomposition of any equilibrium state into translation extremal invariant states [13, section 4.3]. Now we determine these extremal translation invariant equilibrium states which we denote by ρ.

One of the main properties of these states is that the space mean $m(A)$ of a local observable A exists and is given by :

$$m(A) = \text{weak} - \lim_{V\to\infty} \frac{1}{V}\int_V dx\, \tau_x A = \rho(A)1 \qquad (2.1)$$

where τ_x is the action or the $*$-automorphism of the CAR-algebra, representing the translation over the distance $x \in \mathbb{R}^\nu$. Clearly these means $m(A)$ are observables at infinity [14].

For the model H_V (1.2) and for ρ an extremal translation invariant state, using (2.1) one has for any local observable $A \in \mathcal{A}$:

$$\begin{aligned}
&\lim_V \rho(A^*[H_V, A]) \\
&= \lim_{V\to\infty} \{\rho(A^*[T_V, A]) - \lambda\bar{b}\rho(A^*[D_V, A]) \\
&\qquad -\lambda b\rho(A^*[D_V^*, A])\} \qquad (2.2)
\end{aligned}$$

where $D_V = \int_V dx\, C_1^+(x)C_2(x)$ and $b = \lim_V \frac{1}{V}\int_V dx\, \rho(C_1^+(x)C_2(x)) = \rho(C_1^+(x)C_2(x))$. This computation yields the effective Hamiltonian in the state ρ :

$$H_V^\rho = T_V - \lambda(bD_V^* + \bar{b}D_V)$$

such that

$$\text{weak} - \lim_V [H_V, A] = \text{weak} - \lim_V [H_V^\rho, A] \equiv \delta_\rho(A)\,.$$

Explicitly :

$$\begin{aligned}
\delta_\rho C_1^+(x) &= -\frac{\Delta}{2} C_1^+(x) - \lambda b C_2^+(x) \\
\delta_\rho C_2^+(x) &= -\frac{\Delta}{2} C_2^+(x) - \lambda \bar{b} C_1^+(x)
\end{aligned}$$

where Δ is the Laplacian.

For convenience we introduce the two-component version of the CAR-algebra $\mathcal{A}$: for

$$f = (f_1, f_2) \in L^2(\mathbb{R}^\nu) \oplus L^2(\mathbb{R}^\nu) \equiv L$$

define the annihilation operators

$$C(f) = (C_1(f_1), C_2(f_2))$$

then

$$\delta_\rho C^+(f) = C^+(h_\rho f) \tag{2.3}$$

where h_ρ is the following operator :

$$(h_\rho f)(x) = \begin{pmatrix} -\frac{\Delta}{2} & -\lambda\bar{b} \\ -\lambda b & -\frac{\Delta}{2} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

which can be defined as a self-adjoint operator on L. Let $\theta = \arg b$, for all $k \in \mathbb{R}$, denote

$$e_\pm^k(x) = \frac{1}{\sqrt{2}(2\pi)^{\nu/2}} \begin{pmatrix} 1 \\ \mp e^{i\theta} \end{pmatrix} e^{ikx}\,.$$

One has

$$h_\rho e_\pm^k = \varepsilon_\pm(k) e_\pm^k \tag{2.4}$$

where

$$\varepsilon_{\pm}(k) = \frac{1}{2}(k^2 \pm 2\lambda|b|) .$$

These are the quasi-particle energies at momentum k. Clearly the spectrum of h_ρ is absolutely continuous.

Now we are in a position to find all equilibrium or KMS-states at arbitrary but fixed inverse temperature β.

The extremal space invariant β-equilibrium states of the system (1.2) are the quasi-free states ρ of $\mathcal{A}$ determined by the two-point function

$$\rho(C^+(f)C(g)) = \left(g, \frac{1}{e^{\beta h_\rho} + 1} f\right) \tag{2.5}$$

where $f, g \in L$ and where the order parameter b is determined by the gap equation [10] :

$$0 = b\left(1 - \frac{\lambda}{(2\pi)^\nu} \int dk \frac{\sinh \beta\lambda|b|}{4\beta\lambda|b|} \frac{1}{\cosh(\beta\varepsilon_+(k)/2)\cosh(\beta\varepsilon_-(k)/2)}\right) . \tag{2.6}$$

This gap equation admits always a solution $b = 0$. It corresponds to the state of free fermions. It is not only space translation invariant but also gauge invariant for the two-dimensional gauge group $\mathcal{G}$ (1.3).

For β large enough there exists a solution b of (2.6) such that $b \neq 0$. For these solutions the gauge group $\mathcal{G}$ is broken in the sense that $\rho(C_1^+C_2) \neq 0$, i.e. there are non-trivial transitions from one spin state to the other spin state. The solution is carrying a spin density wave (see [19]).

The gap equation fixes only the absolute value of the order parameter b, but not its phase θ. In the following we consider only the generic solution $b \neq 0$ with $\theta = 0$.

The physical spectrum of the microdynamics in the state ρ is as usual given by the spectrum of the Hamiltonian H_ρ, implementing the time evolution in the GNS-representation of ρ

$$e^{iH_\rho t}C(f)e^{-iH_\rho t} = e^{it\delta_\rho}C(f)$$

where δ_ρ is given by formula (2.3). It follows from (2.4) that the spectrum of H_ρ consists of the discrete point $\{0\}$ and the absolutely continuous spectrum coinciding with the real line.

Considering the spectrum of h_ρ given in (2.4), it follows from (2.5) that the two-point function $\rho(C_i^+(x)C_j(y))$ tends to zero if $|x-y|$ tends to infinity and it tends to zero faster than any polynomial. As a consequence the truncated functions ρ_T are integrable, i.e. let $A_1, \ldots A_n$ be strictly local observables, e.g. even products of creation and annihilation operators $C^{\pm}(f)$ with support (f) bounded, then

$$\int dx_1 \ldots dx_{n-1} \left|\rho_T(\tau_{x_1}A_1, \ldots \tau_{x_{n-1}}A_{n-1}, A_n)\right| < \infty \tag{2.7}$$

for all $n \geq 2$. It is readily checked that we have the following central limit theorem.

Proposition 2.1. (Central limit theorem)

For any strictly local self adjoint observable A one has :

$$\begin{aligned} &\lim_{V\to\infty} \rho\left(e^{i\mu\frac{1}{\sqrt{V}}\int_V dx(\tau_x A-\rho(A))}\right) \\ &= \exp -\frac{\mu^2}{2}(A,A)_\sim \,;\quad \mu \in \mathbb{R} \end{aligned} \tag{2.8}$$

where

$$(A,A)_\sim = \int dx\, \rho((A-\rho(A))(\tau_x A - \rho(A)))\,.$$

Proof : Follows straightforwardly by expansion of the exponential, the definition of truncated functions and the property (2.7). ∎

This property is used to give a meaning to macroscopic fluctuations. The local fluctuation of an observable A in a volume V is of course given by

$$A_V = \frac{1}{\sqrt{V}} \int_V dx(\tau_x A - \rho(A)) . \tag{2.9}$$

We are interested in the macroscopic quantities $\lim_{V\to\infty} A_V$, called simply the *fluctuation of A in the state* ρ. The limit should be understood in the sense of the central limit theorem as given in proposition 2.1. We denote

$$\tilde{A} = \lim_V A_V . \tag{2.10}$$

Clearly by differentiation with respect to μ in (2.8) the central limit theorem also give a meaning to any power of the fluctuation e.g.

$$\tilde{A}^m = (\lim_V A_V)^m \; ; \; m = 1, 2, 3, \dots .$$

Hence we can consider the algebra of fluctuations generated by $\{\tilde{A} | A$ any strictly local observable$\}$. As is proved in [7], it is a representation of the canonical commutation relations algebra $\tilde{\mathcal{A}}$ with the commutation relations

$$[\tilde{A}, \tilde{B}] = i\sigma(A, B)\mathbb{1} \tag{2.11}$$

where $\sigma(A,B) = -i \int dx\, \rho([A, \tau_x B])$.

The representation space of the CCR-algebra $\tilde{\mathcal{A}}$ is the Hilbert space $\tilde{\mathcal{H}}$ obtained as the closure of $\tilde{\mathcal{A}}$ with respect to the scalar product

$$(\tilde{A}^m, \tilde{B}^n)_\sim = \tilde{\rho}(\tilde{A}^m \tilde{B}^n)$$

where $\tilde{\rho}$ is the state of $\tilde{\mathcal{A}}$ defined by the formula :

$$\tilde{\rho}(e^{i\tilde{A}}) \equiv \exp -\frac{1}{2}(A, A)_\sim \; , \; A \in \mathcal{A} .$$

Clearly the Hilbert space $\tilde{\mathcal{H}}$, can be called the space of (macroscopic) fluctuations.

3 A canonical pair of fluctuations

As said above we work with an equilibrium state ρ, below the critical temperature, such that $b \neq 0$ and for notational simplicity we take $\theta = \arg b = 0$.

We limit our attention to the following two strictly local observables :

$$Q(x) = \frac{1}{2\sqrt{2\lambda b}}(n_1(x) - n_2(x)) \tag{3.1}$$

$$P(x) = i\sqrt{\frac{\lambda}{2}}(C_1^+(x)C_2(x) - C_2^+(x)C_1(x)) \tag{3.2}$$

where $n_i(x) = C_i^+(x)C_i(x)$; $i = 1, 2$.

Note that $\rho(Q(x)) = \rho(P(x)) = 0$; $x \in \mathbb{R}^\nu$.

On the basis of the central limit theorem (2.8) one considers the corresponding fluctuations $\tilde{Q}$ and $\tilde{P}$.

Proposition 3.1.

The operators $\tilde{Q}$ and $\tilde{P}$ form a canonical pair and satisfy

$$[\tilde{Q}, \tilde{P}] = i\mathbb{1}\,. \tag{3.3}$$

Proof : Following formula (2.11) one has to compute

$$-i \int dx\, \rho([Q(z), P(z+x)])\,.$$

The result follows straightforwardly from the commutator

$$[C_1^+(x)C_1(x) - C_2^+(x)C_2(x), C_1^+(y)C_2(y) - C_2^+(y)C_1(y)]$$

$$= 2(C_1^+(x)C_2(x) + C_2^+(x)C_1(x))\delta(x - y)$$

and the expectation value

$$b = \lim_V \frac{1}{V} \int_V dx\, \rho(C_1^+(x) C_2(x)) .$$

■

Hence we discovered in the system a pair of canonical observables $\tilde{Q}$ and $\tilde{P}$ which are macroscopic in nature and correspond to the fluctuations of the local observables $Q(x)$ and $P(x)$ defined in (3.1) and (3.2). The algebra of canonical commutation relations generated by this $\tilde{Q}$ and $\tilde{P}$ is a subalgebra $\tilde{\mathcal{B}}$ of the total algebra of fluctuations $\tilde{\mathcal{A}}$ of the system. We will limit our attention to this subalgebra $\tilde{\mathcal{B}}$.

As any equilibrium state is time invariant, also the state ρ is time invariant : $\rho \cdot \delta_\rho = 0$. In the next proposition we prove that the mean square fluctuation of Q is proportional to the mean square fluctuation of P. In order to derive this property we do not need that ρ is an equilibrium state, its time invariance is sufficient, of course supplemented with its cluster properties.

Proposition 3.2. (Virial theorem)

One has

$$4\lambda^2 |b|^2 \tilde{\rho}(\tilde{Q}^2) = \tilde{\rho}(\tilde{P}^2)$$

where

$$\begin{aligned} \tilde{\rho}(\tilde{Q}^2) &= \lim_V \frac{1}{V} \int_V dx\, dy\, \rho(Q(x)Q(y)) \\ \tilde{\rho}(\tilde{P}^2) &= \lim_V \frac{1}{V} \int_V dx\, dy\, \rho(P(x)P(y)) \end{aligned}$$

Proof : Using the time translation invariance of the state ρ one has

$$\frac{1}{V}\int_V dx \int_V dy\{\rho(\delta_\rho(Q(x))P(y)) + \rho(Q(x)\delta_\rho(P(y))\} = 0\,. \tag{3.4}$$

Using the fact that the kinetic energy is gauge invariant for the full gauge group $\mathcal{G}$ one has that $\int dx[T, Q(x)] = 0$, where T is the kinetic energy such that we can drop this term. A straightforward computation of the commutator of $Q(x)$ with the potential part of the Hamiltonian H^ρ yields

$$\int dx\, \delta_\rho(Q(x)) = -i\int dx\, P(x)\,. \tag{3.5}$$

We compute also

$$\begin{aligned}
&\delta_\rho P(y) = i4\lambda^2|b|^2 Q(y)\\
&+\frac{1}{2\sqrt{2\lambda}}\nabla_y(-\nabla C_1^+(y)C_2(y) + C_1^+(y)\nabla C_2(y)\\
&-C_2^+(y)\nabla C_1(y) - \nabla C_2^+(y)C_1(y))\,.
\end{aligned} \tag{3.6}$$

After substitution of (3.5) and (3.6) in (3.4) one gets

$$\frac{1}{V}\int_V dx\, dy\, \rho(P(x)P(y)) = \frac{4\lambda^2|b|^2}{V}\int_V dx\, dy\, \rho(Q(x)Q(y))$$

up to a boundary term, coming from the second term in (3.6), which vanishes in the limit $V \to \infty$, by partial integration and translation invariance of the state ρ. This proves the proposition.

■

The above result has the following physical meaning. Under the ergodic hypothesis that phase space expectation values coincide with the time means, this result is expressing the quantum virial theorem on the level of the macroscopic fluctuations for a harmonic oscillator problem

$$\frac{\tilde{P}^2}{2} + \frac{\omega^2}{2}\tilde{Q}^2 \tag{3.7}$$

where $\omega = 2\lambda|b|$ is the oscillator frequency. Clearly this frequency can be called the *plasma frequency* of the spin wave density. We shall prove that this plasma frequency is indeed a point of the spectrum of the dynamics of the fluctuations algebra $\tilde{\mathcal{B}}$ generated by $\tilde{Q}$ and $\tilde{P}$. This dynamics is of course induced by the microdynamics, given by the model (1.2) :

$$A(t) = \alpha_t A(0) = e^{it\delta_\rho} A(0) \tag{3.8}$$

where A is any local observable and δ_ρ is given by formula (2.3).

As can be seen from the computation (3.6) the dynamics does not leave invariant the pair of local observables $Q(x)$ and $P(x)$, (3.1) and (3.2). However this dynamics α_t (3.8) induces a time evolution $\tilde{\alpha}_t$ of the fluctuations by the simple formulae

$$\tilde{\alpha}_t\tilde{Q} = \widetilde{\alpha_t Q} \ ; \ \tilde{\alpha}_t\tilde{P} = \widetilde{\alpha_t P} \tag{3.9}$$

i.e. the time evolved fluctuation of a local observable is defined as the fluctuation of the time evolved local observable. Immediately the question arises whether the fluctuation of a time evolved local observable still exists. It turns out that this is the case for the time evolution of $\tilde{Q}$ and $\tilde{P}$.

Mathematically we consider the Hilbert space $\tilde{\mathcal{H}}$ of fluctuations and consider the time evolution as a map of $\tilde{\mathcal{H}}$ into $\tilde{\mathcal{H}}$. We prove :

Proposition 3.3.

The infinitesimal generator δ_ρ of the microdynamics induces a map $\tilde{\delta}_\rho$ of the subset $\tilde{\mathcal{B}}$ of $\tilde{\mathcal{H}}$ into $\tilde{\mathcal{B}}$ such that

$$\text{(i)} \qquad \tilde{\delta}_\rho \tilde{Q} \equiv \widetilde{\delta_\rho Q} = -i\tilde{P}$$

$$\text{(ii)} \qquad \tilde{\delta}_\rho \tilde{P} \equiv \widetilde{\delta_\rho P} = i\omega^2 \tilde{Q}\,.$$

Hence $\tilde{Q}$ and $\tilde{P}$ are eigenvectors of $\tilde{\delta}_\rho^2$:

$$\tilde{\delta}_\rho^2 \tilde{Q} = \omega^2 \tilde{Q}\ ,\ \tilde{\delta}_\rho^2 \tilde{P} = \omega^2 \tilde{P}$$

such that

$$\tilde{\alpha}_t \tilde{Q} = e^{it\tilde{\delta}_\rho} \tilde{Q} = \tilde{Q} \cos \omega t + \frac{\tilde{P}}{\omega} \sin \omega t\,.$$

Proof : (i) follows from (3.5) and (ii) from (3.6) and the scalar product in $\tilde{\mathcal{H}}$. The rest is straightforward.

■

This property proves that the frequency ω is a discrete point in the spectrum of the macroscopic system of the Q, P-fluctuations. Although the dynamics $\tilde{\alpha}_t$ is induced by the microdynamics α_t, the spectrum of $\tilde{\alpha}_t$ is totally different in nature from that of α_t. We insist that the frequency ω is not a discrete point of the spectrum of α_t.

Note also that the frequency $\omega = 2\lambda|b|$ decreases for increasing temperature and vanishes at the critical temperature β_c determined by the equation

$$1 = \frac{1}{4} \frac{\lambda}{(2\pi)^\nu} \int dk \frac{1}{\left(\cosh \beta_c \frac{k^2}{4}\right)^2}\,.$$

The occurrence of the frequency $\omega \neq 0$ is due to the long range of the potential and is accompanied by the phenomenon of spontaneous symmetry breaking. The two-dimensional gauge group with generators N_1 and N_2 is broken down to a one dimensional gauge group with generator $N_1 + N_2$. The so-called Goldstone mode connected to the frequency ω is the fluctuation of the generator of the broken symmetry, namely $N_1 - N_2 \simeq Q$. The latter operator is not a constant of motion anymore below T_c.

The fluctuation of this operator behaves in time as a periodic fluctuation density of spin-up particles to spin-down particles and back, describing the spin wave. This spin wave seems to decouple completely from the other coordinates of the system, and seems subjected only to the harmonic force $\omega^2\tilde{Q}$. We expect that the thermal equilibrium distributions will be accordingly determined following the Gibbs distribution of the harmonic oscillator Hamiltonian (3.7).

In order to compute this distribution of $\tilde{P}^2$ or $\tilde{Q}^2$, we will have to use the properties of ρ being the equilibrium state at inverse temperature β.

Proposition 3.4.

The mean square deviation of the spin wave density is given by

$$\tilde{\rho}(\tilde{Q}^2) = \frac{1}{2\omega} \coth \beta \frac{\omega}{2}$$

where ω is the spin wave plasma frequency.

Proof : Instead of computing explicitly this quantity using the equilibrium expectation value (2.5) we use the following correlation inequalities [15,16,17] for equilibrium states : for any local observable A one has :

$$\frac{-\beta\rho(A\delta_\rho(A^*))}{\rho(AA^*)} \leq \ln\frac{\rho(A^*A)}{\rho(AA^*)} \leq \frac{\beta\rho(A^*\delta_\rho(A))}{\rho(A^*A)} \qquad (*)$$

We take for A, the operators

$$A_V = \frac{1}{\sqrt{V}}\int_V dx(Q(x) + i\omega^{-1}P(x))$$

compute the different terms in the inequality and take the limit $V \to \infty$. Using (3.3) one gets

$$\lim_{V\to\infty} \rho(A_V A_V^*) = \tilde{\rho}(\tilde{Q}^2) + \omega^{-2}\tilde{\rho}(\tilde{P}^2) + \omega^{-1}$$

and by the virial theorem, proposition 3.2

$$\lim_{V\to\infty} \rho(A_V A_V^*) = 2\tilde{\rho}(\tilde{Q}^2) + \omega^{-1}\,.$$

Analogously

$$\lim_{V\to\infty} \rho(A_V^* A_V) = 2\tilde{\rho}(\tilde{Q}^2) - \omega^{-1}\,.$$

Using (3.5) and (3.6)

$$\delta_\rho(A_V) = -i\frac{1}{\sqrt{V}}\int_V dx\, P(x) - \omega\frac{1}{\sqrt{V}}\int_V dx\, Q(x)$$

up to a boundary term, vanishing in the limit $V \to \infty$, which we drop immediately.

Hence

$$\lim_V \rho(A_V^*\delta_\rho(A_V)) = \tilde{\rho}((\tilde{Q} - i\omega^{-1}\tilde{P})(-i\tilde{P} - \omega\tilde{Q}))$$
$$= -\omega\tilde{\rho}(\tilde{Q}^2) - \omega^{-1}\tilde{\rho}(\tilde{P}^2) - i\tilde{\rho}([\tilde{Q}, \tilde{P}])\,.$$

Again using (3.3) and the virial theorem

$$\lim_V \rho(A_V^*\delta_\rho(A_V)) = 1 - 2\omega\tilde{\rho}(\tilde{Q}^2)\,.$$

Analogously

$$\lim_V \rho(A_V \delta_\rho(A_V^*)) = 1 + 2\omega\tilde{\rho}(\tilde{Q}^2)\,.$$

After substitution in the inequalities (∗) one gets

$$\ln \frac{2\omega\tilde{\rho}(\tilde{Q}^2) - 1}{2\omega\tilde{\rho}(\tilde{Q}^2) + 1} = -\beta\omega$$

or alternatively

$$\omega^2\tilde{\rho}(\tilde{Q}^2) = \tilde{\rho}(\tilde{P}^2) = \frac{\omega}{2}\coth\frac{\beta\omega}{2}\,.$$

■

Formally a similar result has been obtained before for the Jellium model [18] of fermions interacting via Coulomb potentials against a uniform background. In this model the Galilei boost transformations do not commute with the kinetic energy. It is not a model of spontaneous symmetry breaking, but a case of absence of symmetry. It is proved that under suitable sum rules and at high temperatures, a canonical pair consisting of the bulk momentum and the center of gravity splits off from the other degrees of freedom in an equilibrium state.

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