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On connections between the four-dimensional harmonic oscillator and the Coulomb-problem in the representation with the discrete basis

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Abstract. The exact quantum mechanical solution for the four-dimensional harmonic oscillator with arbitrary frequency ω' in the matrix representation using the harmonic oscillator basis with unit frequency ω is obtained. Using the KS-transformation this result is transformed into the solution of the three-dimensional Coulomb-problem (bound states) in the frame of matrix representation with Sturmian basis. The connection of this solution with a rich set of special functions is established. The group theoretical aspects of the KS-mapping are discussed.

1. Introduction

The connection between the harmonic oscillator and the Kepler- or Coulomb-problem has been investigated in many papers in the frame both of classical and quantum mechanics [1–10]. This question was considered in the Schrödinger-representation [11–15], in the representation of Wigner–Moyal [16], and by the path-integral method of Feynman [17–19]. Recently it was also attacked in the formalism of supersymmetric quantum mechanics [20].

In this paper our aim is to consider the interrelation between the quantum mechanical solutions of the four-dimensional harmonic oscillator (FDHO) and the three-dimensional Kepler–Coulomb problem (TDCP) in some additional representations, namely, in the frame of the matrix representation using a convenient discrete basis of square integrable functions for each of these problems. The initial point of our approach is based on the well-known fact [21], that the matrix of the Coulomb-problem Hamiltonian

$$H_{\text{coul}} = p^2/2m + Z_1 Z_2 e^2/r^2 \tag{1}$$

is a tri-diagonal (Jacobi) matrix in the Laguerre/Sturm basis:

$$|nlm\rangle = R_{nl}(r) Y_{lm}(\theta, \phi), \tag{2}$$

where

$$R_{nl}(r) = Q_{nl} r^l e^{-r/2} L_n^{2l+1}(\lambda r),$$

$$Q_{nl} = (-1)^n \sqrt{\frac{2\lambda\Gamma(n+1)}{(n+2l+2)}},$$

$Y_{lm}(\theta, \phi)$ are the usual spherical harmonics,

$$\text{with angular momentum } l \text{ and its projection } m \tag{3}$$

If we shall seek the eigenfunctions $\psi_l(r) = R_l(r) Y_{lm}(\theta, \phi)$ of the Hamiltonian (1) in a form of a Fourier series with respect to the radial Sturmian functions:

$$R_l(r) = \sum B_{nl} R_{nl}(r), \tag{4}$$

then the Schrödinger-equation

$$H_{\text{coul}} \psi_l(r) = E \psi_l(r)$$

transforms into the set of three-terms recurrent relation (TRR) for the coefficients B_{nl} [21]

$$\begin{aligned} &-\sqrt{n(n+2l+1)}B_{n-1l} - \sqrt{(n+1)(n+2l+2)}B_{n+1l} \\ &+ 2[(n+1+1)\cos\theta + t\sin\theta]B_{nl} = 0, \end{aligned} \tag{5}$$

where $t = \left(-\frac{2Z}{q}\tan\frac{\theta}{2}\right)$, $\quad \theta = \arccos\left(\frac{q^2-1}{q^2+1}\right)$.

It is possible to show that the eigenvectors $\{B_{nl}(q), n = 0, 1, 2, \ldots\}$ of the infinite dimensional matrix (5) might be found in the explicit analytical form. It means that the Coulomb-problem is exactly solvable in the discrete matrix representation, too. In particular its solution in the continuum was obtained in [21], i.e. the coefficients $B_{nl}(q)$ corresponding to the energy eigenvalues $E = (q^2/2) > 0$ were found. Namely these coefficients were expressed for the regular solution in the continuum in terms of the Pollaczek-polynomials $P_n^{\lambda}(\theta, w)$. As for the bound states $R_{\nu l}(r)$, with energy $E_\nu \sim \nu^{-2}$—in the attractive Coulomb-potentials—the corresponding solution of TRR (5) will be done in this paper. Taking into account the orthonormality properties of Sturmian functions:

$$\int R_{nl}(r) R_{nl}(r) r \, dr = \delta_{n'n}, \tag{6}$$

it is possible to calculate the coefficients

$$B_{nl}(\nu) = \int R_{\nu l}(r) R_{nl}(r) r \, dr \tag{7}$$

by direct integration. However in this paper the discrete spectrum of eigenvalues for the matrix (4) in the case of the attractive Coulomb-potential and the corresponding eigenvectors $\{B_{nl}(\nu)\}$ will be found in a different way using the above mentioned connection between the FDHO and the TDCP. In our opinion this approach allows to obtain a deeper insight into the interrelation of these two problems and to analyse their group theoretical structure.

The content is organized as follows: In Section 2 the solution of the FDHO problem with the Hamiltonian

$$H_{osc} = \sum \left[\frac{\pi_k^2}{2m} + \frac{m\omega^2 x_k^2}{2} \right] \tag{8}$$

will be obtained in the matrix representation with basis functions

$$\psi_{nKm_1m_2}(\rho, \alpha, \beta, \gamma) = R_{nK}(\rho/\rho_0')Y_{Km_1m_2}(\alpha, \beta, \gamma), \tag{9}$$

which are the eigenfunctions of the FDHO of different frequency ω', $\omega \neq \omega'$, i.e. [24, 27]

$$R_{nK}(r) = (-1)^n Q_{nK}(\rho/\rho_0)^{K+3/2} \exp\left[-\tfrac{1}{2}(\rho/\rho_0)^2\right] L_n^{K+1}(\rho/\rho_0) \tag{10}$$

with orthonormalization condition $\int R_{n'K}(\rho) R_{nK}(\rho)\, d\rho = \delta_{n'n}$,

where:

$\rho_0^2 = h/m\omega$ "radius" of the oscillator,

$\pi_k = -ih\, \partial/\partial x_k$ four-dim. momentum,

$\rho^2 = \sum_k x_k^2$ four-dim. radius,

$$Q_{nK} = \sqrt{\frac{2\Gamma(n+1)}{\rho_0 \Gamma(n+K+2)}},$$

$Y_{Km_1m_2}(\alpha, \beta, \gamma)$ four-dimensional spherical harmonic.

In Section 3 the connection between the obtained solution and the orthogonal polynomials of a discrete variable (namely the Meixner polynomials) will be discussed. In Section 4 the results of Sections 2 and 3 will be used in order to find the solutions corresponding to the bound states of the attractive Coulomb-system in the frame of the matrix representation with the basis (2, 3).

The point is that the FDHO (7, 8) can be transformed into the Laguerre (or Sturmian) basis (2, 3) using the nonbijective, canonical Kustaanheimo-Stiefel (KS-) transformation [10]. Simultaneously this transformation reduces the Hamiltonian (8) to the Coulomb–Hamiltonian (1). This fact allows to obtain the explicit form of eigenvectors $\{C_{nl}\}$ for bound states in the attractive Coulomb-potential in the Laguerre basis using the eigenvectors of the Hamiltonian (8) in the basis (9, 10). In the appendix the interrelations of the dynamical symmetry groups of the FDHO and the TDCP are discussed on the base of the constained Lie algebra concept [1–9].

2. The FDHO in the coordinate and matrix representation

The solution of the Schrödinger-equation for the harmonic oscillator in the space of arbitrary dimension is well known. In particular the Hamiltonian (8) has the eigenvalues:

$$E_{NK} = (2N + K + 2)\hbar\omega \tag{11}$$

and the corresponding eigenfunctions are of the form

$$\psi_{NKm'_1m'_2}(\rho, \alpha, \beta, \gamma) = R_{NK}(\rho/\rho'_0)Y_{Km'_1m'_2}(\alpha, \beta, \gamma), \tag{12}$$

where $\rho'_0 = \hbar/m\omega'$. Below we shall use the following set of hyperspherical coordinates in four-dimensional space

$$x_1 = \rho \cos\alpha \cos\beta,$$

$$x_2 = \rho \cos\alpha \sin\beta,$$

$$x_3 = \rho \sin\alpha \cos\gamma,$$

$$x_4 = \rho \sin\alpha \sin\gamma. \tag{13}$$

We will use the following choice for the hyperspherical angles:

$$0 \leqslant \alpha \leqslant 2\pi; \qquad 0 \leqslant \beta \leqslant 2\pi; \qquad 0 \leqslant \gamma \leqslant \pi/2.$$

In this case the four-dimensional spherical harmonics can be written as follows:

$$Y_{Km'm}(\alpha, \beta, \gamma) = Q_{Km'm}e^{i(m'\beta + m\gamma)}P^{(|m'|, |m|)}_{(K - |m| - |m'|)/2}(\cos 2\alpha), \tag{14}$$

where $P^{(a, b)}_n(\cos 2\alpha)$ are the Jacobi-polynomials,

$$Q_{Km'm} = \frac{\sin^{|m'|}\alpha \cos^{|m|}\alpha}{2\pi}\left\{\left[(2K+2)\left(\frac{K - |m| - |m'|}{2}\right)!\right.\right.$$
$$\left.\left.\times\left(\frac{K + |m| + |m'|}{2}\right)!\right]\Big/\left[\left(\frac{K + |m| - |m'|}{2}\right)!\left(\frac{K - |m| + |m'|}{2}\right)!\right]\right\}^{1/2}.$$

For the quantum numbers the following values are allowed:

$$K = |m'| + |m|, |m'| + |m| + 2, \ldots.$$

The harmonics (12) are orthonormalized in correspondence with the relation

$$\int Y^*_{Km_1m_2}(\alpha, \beta, \gamma)YK'_{m'_1m'_2}(\alpha, \beta, \gamma)\, d\Omega = \delta^{K'm'_1m'_2}_{Km_1m_2}, \tag{15}$$

where $d\Omega = \sin\alpha \cos\alpha\, d\alpha\, d\beta\, d\gamma$ is the four-dimensional sphere surface element.

The function (12) can be expressed in terms of the Wigner D-functions [25]

$$Y_{Km_1m_2}(\alpha, \beta, \gamma) = D^{K/2*}_{m_1+m_2, m_1-m_2}(\beta + \gamma, \alpha, \beta - \gamma). \tag{16}$$

They satisfy the differential equation

$$\left[\frac{1}{\sin\alpha\cos\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\sin\alpha\cos\alpha\,\partial}{\partial\alpha}\right) + \frac{1}{\sin^2\alpha}\frac{\partial}{\partial\beta} + \frac{1}{\cos^2\alpha}\frac{\partial}{\partial\gamma}\right]Y_{Km_1m_2}$$
$$= -K(K+2)Y_{Km_1m_2}. \tag{17}$$

In order to transform the wave functions $\psi(r)$ from the coordinate representation to the matrix representation with the discrete basis (9, 10) it is necessary to expand $\psi(r)$ in a Fourier series with respect to this basis

$$\psi(r) = \sum C_{nKm_1m_2} \psi^*_{nKm_1m_2}(r) \tag{18}$$

as a result the Schrödinger-equation with the Hamiltonian (8)

$$(H_{\mathrm{osc}} - E)\psi(r) = 0$$

will be transformed into the infinite set of coupled algebraic equations for the coefficients C_{nK}

$$\sum [\langle n'K|H_{\mathrm{osc}}|nK\rangle - E\delta_{nn'}]C_{nK} = 0, \qquad n, n' \in N_0. \tag{19}$$

The quantum numbers m_1 and m_2 are omitted here because the matrix of the Hamiltonian (8) is diagonal on K, m_1 and m_2 and does not depend on m_1 and m_2. These properties are evident from the explicit form of the matrix elements of the operator ρ^2 [23, 27]

$$\langle n'K'm_1'm_2'|\rho^2|nKm_1m_2\rangle = \delta^{Km_1'm_2'}_{Km_1m_2}[\delta_{n'n}(2n + K + 2)$$
$$+ \delta_{n'n-1}\sqrt{n(n + K + 1)} + \delta_{n'n+1}\sqrt{(n + 1)(n + K + 2)}]\hbar/m\omega. \tag{20}$$

The kinetic energy matrix [23, 27]

$$\langle n'K'|p^2|nK\rangle = \delta^{Km_1'm_2'}_{Km_1m_2}[\delta_{n'n}(2n + K + 2) - \delta_{n'n-1}\sqrt{n(n + K + 1)}$$
$$- \delta_{n'n+1}\sqrt{(n + 1)(n + K + 2)}]\hbar m\omega \tag{21}$$

can be found from equation (20) using the virial theorem or by direct calculation. Obviously, the resulting matrix of the Hamiltonian (8) in the basis (9, 10) is a tri-diagonal one and the coefficients C_{nK} satisfy the set of TRR

$$-\left(\frac{1 - A}{1 + A}\right)\sqrt{(n + 1)(n + K + 2)}C_{n+1K} + \frac{1 + A^2}{(1 + A)^2}(2n + K + 2)C_{nK}$$
$$-\left(\frac{1 - A}{1 + A}\right)\sqrt{n(n + K + 1)}C_{n-1K} = 2\varepsilon\frac{A}{(1 + A)^2}C_{nK}, \tag{22}$$

where $A := \omega'/\omega$, $\varepsilon := E/\hbar\omega'$. Using the notation $\omega'/\omega = e^\tau$ ($\tau \in R$) we can rewrite this relation as follows:

$$\tfrac{1}{2}\,\mathrm{th}(\tau/2)\sqrt{(n + 1)(n + K + 2)}C_{n+1K} + \tfrac{1}{2}\,\mathrm{th}(\tau/2)\sqrt{n(n + K + 1)}C_{n-1K}$$
$$+ \frac{2\mathrm{ch}^2\,\tau/2 - 1}{4\,\mathrm{ch}^2\,\tau/2}(2n + K + 2)C_{nK} = \frac{1}{4\,\mathrm{ch}^2\,\tau/2}\varepsilon C_{nK}. \tag{23}$$

Evidently the next step is to solve this finite difference equation (FDE) of second order, and it will be realized in Section 3, where the coefficients C_{nK} are expressed in terms of the orthogonal Meixner-polynomials on the uniform grid. In this section the coefficients C_{nK} will be found in another manner. The spectrum of eigenvalues of the Hamiltonian (8) is

$$\varepsilon_{NK} = (2N + K + 2) \quad \text{(in terms of } \hbar\omega). \tag{24}$$

The transformation (18) for the corresponding eigenfunctions coincides with the expansion of the radial wave function of the FDHO with frequency ω' on the similar functions with frequency ω (in analogy with the method used in [29]), i.e.

$$R_{nK}(\rho/\rho_0') = \sum_N C_{nK}(N) R_{NK}(\rho/\rho_0), \tag{25}$$

$$C_{nK}(N) = \langle R_{NK} | R'_{nK} \rangle = \int R_{NK} R'_{nK} \rho^3 \, d\rho. \tag{26}$$

This integral can be calculated in a direct manner [40] or by taking into account the fact that the harmonic oscillator wave functions with different frequencies can be connected with each other by a dilatation operator $D(\tau)$:

$$R_{nK}(\rho/\rho_0') = D(\tau) R_{nK}(\rho/\rho_0). \tag{27}$$

Here

$$D(\tau) = \exp(\tau Q_2), \qquad \tau = \ln(\omega'/\omega) \tag{28}$$

and

$$Q_2 = \frac{i}{4} \sum (a_k^\dagger a_k^\dagger + a_k a_k) = -\frac{1}{4} \sum \left(x_k \frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_k} x_k \right) \tag{29}$$

is the generator of Sp(2, R) (or SU(1, 1)) group (see e.g. [30]). It follows from equations (27, 28) that the coefficients C_{nK} coincide with the matrix element $d^j_{\mu\nu}(\tau)$ of the boost in the unitary irreducible representation (UIR) D^{j+} of the SU(1, 1) group belonging to the positive discrete series

$$C_{nK}(N) = \langle R_{NK}(\rho/\rho_0) | D(\tau) | R_{nK}(\rho/\rho_0') \rangle = d^j_{\mu\nu}(\tau). \tag{30}$$

The noncompact "angular momentum" j and its "projections" μ and ν are connected with the quantum numbers K, n, N by the relations:

$$j = \frac{K}{2}, \qquad \mu = n + \frac{K}{2} + 1, \qquad \nu = N + \frac{K}{2} + 1$$

(we use the same notations for the IR of SU(1, 1) group as in [30, 31], i.e. the representation D^{j+} corresponds to the eigenvalue $j(j+1)$ of the Casimir-operator J^2 and $\mu, \nu = j+1, j+2, j+3, \ldots$).

The explicit formula for the SU(1, 1) d-functions was obtained at first by V. Bargmann [32]

$$d^j_{\mu\nu}(\tau) = (-1)^{\nu-\mu} \sqrt{(j+\mu)!(j+\nu)!(\mu-j-1)!(\nu-j-1)!} \frac{(\operatorname{sh}\tau/2)^{\mu-\nu}}{(\operatorname{ch}\tau/2)^{\mu+\nu}}$$
$$\times \sum_{y=0}^{\mu-\nu} \frac{(-1)^y (\operatorname{sh}^2 \tau/2)^y}{y!(\nu-\mu+y)!(\mu+j-y)!(\mu-j-1-y)!}. \tag{31}$$

The same result can be obtained if the matrix element (30) is calculated by the method of Ref. [30]. The coefficients C_{nK} can be expressed in terms of the Gaussian hypergeometric function [28, 33, 38]

$$C_{nK}(N) = \frac{(-1)^N}{\Gamma(K+2)}\left[\frac{\Gamma(n+K+2)\Gamma(N+K+2)}{\Gamma(N+1)\Gamma(n+1)}\right]^{1/2}(\mathrm{th}\,\tau/2)^{N-n}(\mathrm{ch}\,\tau/2)^{-(K+2)}$$
$$\times F(-n, N+K+2, K+2; (\mathrm{ch}\,\tau/2)^{-2}). \tag{32}$$

If the coefficients C_{nK} are considered as a function of the continuous variable τ, then they can be expressed in terms of the Jacobi-polynomials [28, 33]

$$C_{nK}(N) = (-1)^N\left[\frac{\Gamma(N+K+2)\Gamma(n+1)}{\Gamma(N+1)\Gamma(N+K+2)}\right]^{1/2}(\mathrm{th}\,\tau/2)^{N-n}(\mathrm{ch}\,\tau/2)^{-(K+2)}$$
$$\times P_n^{(K+1,\,N-n)}\left(1-\frac{1}{\mathrm{ch}^2\,\tau/2}\right). \tag{33}$$

It should be noted that in the limit $n \gg (K+2)/2$ the TRR (22) can be approximated by the differential equation [39]

$$\left[-\frac{\partial}{\partial x^2}+2\frac{K(K+2)}{x^2}+x^2-2\varepsilon\right]\chi = 0 \tag{34}$$

for the function $\chi(x) = n^{1/4}C_{nK}$ depending on the argument $x = 2\sqrt{n}\rho_0$. This equation coincides with the radial Schrödinger-equation for the FDHO. Since its eigenvalues are equal to $\varepsilon_{NK} = (2N+K+2)\hbar\omega'$ and the eigenfunctions are of the form $x^{(K+3)/2}R_{NK}(x)$ (the analytical form of $R_{NK}(x)$ is given in (10)), we obtain the following asymptotical formula for the coefficients $C_{nK}(N)$:

$$C_{nK}(N) \sim (2n^{1/2}\rho_0)^{7/4}R_{NK}(2n^{1/2}\rho_0). \tag{35}$$

Therefore it is possible to consider the functions (32), (33) as a discrete analogue of the wave functions of the FDHO. In the limit of the continuous variable n the orthonormalization condition for the coefficients $C_{nK}(N)$

$$\sum_{n=0}^{\infty} C_{nK}(N)C_{nK}(N') = \delta_{NN'}, \tag{36}$$

which is a consequence of the unitarity of the d-functions, can be transformed into the orthogonality property of the harmonic oscillator wave functions [24]

$$\int R_{NK}(x)R_{N'K}(x)x^3\,dx = \delta_{NN'}. \tag{37}$$

In [31] it was remarked that the Bargmann d-functions can be expressed also in terms of the Meixner-polynomials [36, 37]

$$C_{nK}(N) = (-1)^N\left[\frac{\Gamma(N+K+2)}{\Gamma(n+K+2)\Gamma(n+1)\Gamma(N+1)}\right]^{1/2}(\mathrm{th}\,\tau/2)^{N+n}$$
$$\times(\mathrm{ch}\,\tau/2)^{-(K+2)}m_n^{(K+2,\,\mathrm{th}^2\tau/2)}(N) \tag{38}$$

that form the system of orthogonal polynomials on the uniform grid. It means that the coefficients $C_{nK}(N)$ can be connected with the Meixner-polynomials, too. In the next section the corresponding expressions will be found by direct solution of the FDE (22).

3. The FDHO in the matrix representation and Meixner-polynomials

The methods of solution of finite difference equations that is a discrete analogue of the hypergeometrical differential equations are described in [24] and [39]. In these monographs the following equations are considered:

$$\sigma(n)\,\Delta\nabla y_n + \tau(n)\,\Delta y_n + \lambda y_n = 0, \tag{39}$$

where

$$\Delta y_n := y_{n+1} - y_n, \qquad \nabla y_n := y_n - y_{n-1}.$$

At $\sigma(n) = n$, $\tau(n) = \gamma_n = \gamma\mu - n(1-\mu)$ the Meixner-polynomials $m_N^{(\gamma,\mu)}$ are the solutions of this equation. The eigenvalues λ have the form

$$\lambda_N = N(1-\mu), \qquad N = 0, 1, 2, \ldots. \tag{40}$$

It is clear that the finite difference equation (39) of second order can be written in form of a TRR

$$\mu(\gamma + n)y_{n+1} + n y_{n-1} + (\lambda - n - \gamma\mu - n\mu)y_n = 0. \tag{41}$$

Thus the TRR (41) has the solution

$$y_n = m_N^{(\gamma,\mu)}(n) \tag{42}$$

corresponding to the eigenvalues (40). They satisfy the orthonormality condition:

$$\sum_{n=0}^{\infty} m_N^{(\gamma,\mu)}(n) m_{N'}^{(\gamma,\mu)}(n)\,\rho(n) = \delta_{NN'} d_N^2, \tag{43}$$

where

$$d_N^2 = \frac{\Gamma(N+1)(\gamma)_N}{\mu^N(1-\mu)^\gamma}, \qquad (\gamma)_N = \frac{\Gamma(N+\gamma)}{\Gamma(\gamma)} \text{ is the Pochhammer symbol.} \tag{43a}$$

Setting $\gamma = K + 2$ and $\mu = \mathrm{th}^2(\tau/2)$ in (41) and substituting y_n by $C_{NK}(n)$ in correspondence with relation

$$y_n(N) = \sqrt{\frac{\Gamma(n+1)\Gamma(N+1)}{\mu^{u+N}(1-\mu)^\gamma}}\sqrt{\frac{(\gamma)_N}{(\gamma)_n}}\, C_{NK}(n), \tag{44}$$

we obtain the TRR (22) instead of TRR (41) with $\varepsilon = 2\lambda(1-\mu)^{-1} + \gamma$. It is clear from the expression (39) that the obtained spectrum of eigenvalues ε_N is in agreement with the standard result (11). In addition we remark that the Meixner-polynomials in (42) are symmetric with respect to the variable exchange n and N

$$m_n^{(\gamma,\mu)}(N) = (\gamma_n/\gamma_N) m_N^{(\gamma,\mu)}(n).$$

Combining the relations (42) and (44) we can write the solutions of the TRR (22) as follows:

$$C_{nK}(N) = (-1)^N \left[\frac{\Gamma(N+K+2)}{\Gamma(n+K+2)\Gamma(N+1)\Gamma(n+1)} \right]^{1/2} (\operatorname{th} \tau/2)^{N+n}$$
$$\times (\operatorname{ch} \tau/2)^{-(K+2)} m_n^{(K+2, \operatorname{th}^2 \tau/2)}(N). \tag{45}$$

They are orthonormalized in accordance with condition (36). This result is in agreement with the earlier given relations (32) and (38) of course. Equation (42) permits to obtain the solution for the Coulomb-problem in the Laguerre basis for the bound states, using the KS-transformation.

4. KS-transformation and the solution of the Coulomb-problem in Sturmian basis

The KS-mapping of the states of the FDHO onto the Sturmian basis of the TDCP includes the following points [4]:

1. In the space of the FDHO states (9), only the states with even K

$$K = 2l, \qquad m_1 = m_2 \tag{46}$$

must be selected.

2. New angular variables [22] . . .

$$\theta := 2\alpha, \qquad \phi := \tfrac{1}{2}(\beta - \gamma), \qquad \psi := \tfrac{1}{2}(\beta + \gamma) \tag{47}$$

must be introduced.

3. Finally the following substitution for the radial variables

$$\rho := r^{1/2} \tag{48}$$

must be fulfilled.

As a result of the substitution (47) the four-dimensional hyperspherical harmonics $Y_{Kmm}(\alpha, \beta, \gamma)$-independent of the angle ψ, because of condition (46), will be transformed into the usual three-dimensional spherical harmonics

$$Y_{K=2lmm'}(\alpha, \beta, \gamma) = Y_{lm}(\theta, \phi). \tag{49}$$

It follows from the fact that the substitution (47) transforms equation (17) (for those functions not depending on ψ) into the Laplace-equation for the three-dimensional spherical harmonics with angular momentum l.

$$\left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial}{\partial\phi^2} \right] Y_{Kmm} = \frac{-K(K+2)}{4} Y_{Kmm} \tag{49a}$$

also the well-known relations [25, 28]

$$P_n^{(a,a)}(\cos\beta) = P_n(\cos 2\beta) \tag{49b}$$

(connection between the Jacobi- and Legendre-polynomials)

$$D^{l\,*}_{m,m'=0}(\phi, \theta, \psi) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\theta, \phi) \tag{49c}$$

(the generalized hyperspherical functions in R^4 are equal to $Y_{lm}(\theta, \phi)$ in R^3 if $m' = 0$)

must be taken into account. The comparison of equations (3) and (9) shows that the substitution (48) transforms exactly the radial wave functions for the FDHO into the functions of Laguerre basis (3). It is rather natural, since the radial wave functions $R_{nK}(\rho)$ satisfy the Schrödinger-equation

$$\left[\frac{1}{\rho^3}\frac{\partial}{\partial\rho}\left(\rho^3\frac{\partial}{\partial\rho}\right) - \frac{K(K+2)}{\rho^2} + \rho^2 + E_{nK}\right] R_{nK}(\rho) = 0 \tag{50}$$

after substitution (48) equation (50) might be written in the form

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2} + \frac{E_{nl}}{4r} + \frac{1}{4}\right] R_{nl}(r) = 0. \tag{51}$$

This coincides with the equation for the Sturmian functions $R_{nl}(r)$ of the TDCP. Because $E_{nl} = 2(n + l + 1)$ the solutions of (51) have the same analytical form as the hydrogen-like atomic wave functions with electric charge $Z_{\text{eff}} = 2(n + l + 1)$, linearly increasing with the radial quantum number n, so that the energy of each level with fixed orbital momentum l remains constant and equal to $-(2l)^{-2}$. The relation between the harmonic oscillator radial quantum number n and the main quantum number v determining the energy $E_v \sim v^{-2}$ of the hydrogen-like atom is of the form

$$v = (n + l + 1). \tag{52}$$

Applying the KS-transformation (46)–(48) to each side of the expansion (18) for the wave function of the FDHO

$$\psi^{\omega}_{NK=2lmm}(\rho, \alpha, \beta, \gamma) = \sum C_{nK}(N)\psi^{\omega'}_{nK=2lmm}(\rho, \alpha, \beta, \gamma), \tag{53}$$

we obtain the following expansion of the Sturmian functions with parameter $\gamma' \sim \omega'$ in terms of Sturmian functions with parameter $\lambda \sim \omega$:

$$R^{\Lambda}_{n'K}(\lambda' r) = \sum C_{nn'}(\tau) R^{\Lambda}_{nK}(\lambda r), \qquad \tau = \ln(\lambda'/\lambda), \tag{54}$$

where the coefficients $C_{nn'}(\tau)$ are of the form

$$C_{nn'}(\tau) = \int R^{\Lambda}_{n'K} R^{\Lambda}_{nK}\, r\, dr,$$

$$C_{nn'}(\tau) = (-1)^N 2^{2l+1}(\lambda\lambda')^{l+1}\left[\frac{\Gamma(N+2l+2)}{\Gamma(n+2l+2)\Gamma(n+1)\Gamma(N+1)}\right]^{1/2}$$

$$\times (\text{th}\,\tau/2)^{N+n}(\text{ch}\,\tau/2)^{-(2l+2)} m_n^{(2l+2,\,\text{th}^2\tau/2)}(N). \tag{54a}$$

Using this relation it is possible to find also the expansion

$$R^{c}_{n'l}(r) = \sum B^{l}_{nn'} R^{\Lambda}_{nl}(r) \tag{55}$$

of the bound state wave functions of the Coulomb-problem

$$R^{c}_{nl}(r) = (-1)^{-n+2l+1} 2(n+l+1)^{-3/2} R^{\Lambda}_{nl}(r) \tag{56}$$

in terms of the Sturmian functions (3) with parameter λ. But it is necessary to take into account two conditions:

1. In expansion (55) should be used formula (54) with the parameter $\tau = \ln [\lambda'/\lambda(n+l+1)]$ depending on the quantum number n.
2. It is important to remark that Sturmian and Coulomb wave functions satisfy two different orthonormalization conditions [41]

$$\int R^{c}_{nl}(\lambda r) R^{c}_{nl}(\lambda r) r^2\, dr = \delta_{nn'}, \tag{57a}$$

$$\int R^{\Lambda}_{nl}(\lambda r) R^{\Lambda}_{n'l}(\lambda r) r\, dr = \delta_{nn'}. \tag{57b}$$

As a consequence of these properties we have that

$$R^{c}_{nl}(\lambda r) = T_{nl} R^{\Lambda}_{nl}(\lambda r), \tag{58}$$

where $T_{nl} = 2/(n+l+1)^{3/2}$. Finally we obtain for the coefficients $B^{l}_{nn'}$ in expansion (55)

$$B^{l}_{nN}(\tau) = 2^{2l+1} \frac{(\lambda\lambda')}{(N+l+1)^2} \left[\frac{\Gamma(n+2l+2)\Gamma(N+2l+2)}{\Gamma(N+1)\Gamma(n+1)}\right]^{1/2} (-1)^N \frac{1}{\Gamma(2l+2)}$$
$$\times (\text{th}\, \tau/2)^{N-n} (\text{ch}\, \tau/2)^{-(2l+2)} F(-n, N+2l+2, 2l+2; (\text{ch}\, \tau/2)^{-2}) \tag{59a}$$

and in terms of the coefficients $C_{nK}(N)$ (32) if we remember the KS-transformation $K = 2l$

$$B^{l}_{nN}(\tau) = T_{nl} C_{nl}(N), \tag{59b}$$

$$B^{l}_{nN}(\tau) = 2^{2l+1} \frac{(\lambda\lambda')}{(N+l+1)^2} (-1)^N \left[\frac{\Gamma(N+2l+2)}{\Gamma(n+2l+2)\Gamma(n+1)\Gamma(N+1)}\right]^{1/2}$$
$$\times (\text{th}\, \tau/2)^{N+n} (\text{ch}\, \tau/2)^{-(2l+2)} m_n^{(2l+2,\, \text{th}^2 \tau/2)}(N). \tag{59c}$$

It is possible to compare this result with the analytical continuation of the regular solution of the TRR (5) obtained in [21]

$$S_{nl}(\theta, t) = 2^l \frac{\Gamma(n+1)|\Gamma(l+1-it)|}{\Gamma(n+2l+2)} e^{(\pi/2+\varepsilon\pi)t} e^{\theta t}$$
$$\times (\sin\theta)^{l+1} P^{l+1}_n(\cos\theta, 2z/\lambda, -2z/\lambda) \tag{60}$$

that might be reduced to the expression (59c) if the connection between Meixner- and Pollaczek-polynomials [24]

$$P_n^\lambda(s, \phi) = \frac{e^{in\phi}}{\Gamma(n+1)} m_n^{(2\lambda, \mu)}(-\lambda - is), \qquad \mu = e^{2i\phi}$$

and the differences of their normalization properties are considered. The correctness of result (59) can be proved by comparison with the integral [40]

$$\int e^{-bx} x^\alpha L_n^\alpha(\lambda x) L_m^\alpha(\mu x)\, dx = \frac{\Gamma(m+n+\alpha+1)}{\Gamma(m+1)\Gamma(n+1)}$$
$$\times \frac{(b-\lambda)(b-\mu)}{b^{m+n+\alpha+1}} F\left(-m, -n, -m-n-\alpha; \frac{b(b-\lambda-\mu)}{(b-\lambda)(b-\mu)}\right).$$

5. Conclusion

In the previous sections the explicit expression for the wave functions of the bound states in the attractive Coulomb-potential was obtained of the matrix representation with Sturmian basis. The problem was attacked from different standpoints giving the connection of this solution with a rich set of special functions: Jacobi-, Meixner-, Pollaczek-polynomials.

The KS-mapping of the FDHO states allows to consider only bound states of the Coulomb-problem. In order to obtain the Coulomb-continuum states the KS-mapping has to be applied to the repulsive oscillator in a four-dimensional space [4]. The results of this paper can be considered as a starting point for future similar analysis of the KS-mapping in continuum, that is poorly investigated until now.

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