Axiomatic description of irreversible and reversible evolution of a physical system

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Axiomatic description of irreversible and reversible evolution of a physical system

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Abstract. In the framework of Piron–Aerts theory the notion of a general evolution of a physical system is studied. Axiomatic characterizations of a general irreversible deterministic evolution and of a reversible evolution are given. Related mathematical structures are studied and appropriate mathematical categories within which the theory can be formulated either as a property-lattice description or as a state-space description are exhibited. Also, within this framework, a natural construction has been found which enables the axiomatic characterization of the set of trajectories of the given entity.

0. Introduction

In the Piron–Aerts theory a physical system, i.e. a part of reality on which one can act and describe the results of these actions without any appeal to the 'rest of the universe' is described (and defined) by a set of its properties. Each property is defined by some experimental project with a certain well-defined result, which is called 'yes'. When for the given physical system one can claim with certainty, that in the case that the experiment defining some property were performed, the result 'yes' would be obtained, or equivalently, that all the other possible results of this experiment denoted by 'no' would be impossible, then this property is called actual. It is precisely an "element of reality" as it was defined in the famous EPR paper [1].

The set of all actual properties which the system has at a given moment of time is a state of the system. Starting from the notion of question one can develop a mathematical model for such a description which after imposing some axioms, can be specified, such that the Hilbert space formalism of quantum mechanics or the formalism of classical mechanics can be recovered. However, some time ago it has been proved by Aerts [2], that two of the axioms, responsible for the Hilbert space structure, are not satisfied in a physically relevant situation, namely in the case of the system which consists of two separated quantum entities. It turns out that – roughly speaking – there is not a Hilbert space description for

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such system. It seems therefore natural to look for the description of a dynamical evolution of a system at the most basic level, where no such particular axioms have to be taken into account at the beginning. So, we shall start with a general theory, which includes as particular cases classical and quantum mechanics, and which is able to describe the system consisting of two separated entities. Let us mention here that, owing to such a theory, we have one and the same interpretational framework for classical and for quantum mechanics: entirely realistic, non-probabilistic and non-statistical.

We shall not confine ourselves at the beginning to any particular kind of the evolution: deterministic, irreversible or reversible. What they are, has to be defined – possibly in terms of the primitive notions of the theory. We shall begin with the following simple observation: as time passes the set of actual properties (i.e. state) changes. Certain properties cease to be actual. Others, which have not been actual, become actual. It is now the aim of the theory to describe these changes in such a way, that one can make predictions concerning the properties which the system may have in future.

1. The description of a physical system

We shall give here a brief survey of the Piron–Aerts formalism. For more detailed exposition, justification and proofs, the reader is referred to [2], [3].

As we have said, a physical system is described by a set of its properties. Since every property is defined by an experimental project, it is natural to begin with the set Ω of experimental projects. Thus, every $\alpha \in \Omega$ is a project of an experiment which may be performed on the physical system under consideration. Moreover, among possible results of this experiment, there is a uniquely specified one which is considered to be positive, and which is denoted by "yes". All the other results are denoted by "no". For the sake of shortness the elements of Ω will be also called *questions*. We shall assume that the set Ω of questions is closed with respect to the following operations denoted by $^\sim$ and $\sqcap$ respectively.

If α is a question, then by $\alpha^\sim$ we denote the same experimental project but for which the results "yes" and "no" have been interchanged. $\alpha^\sim$ is called an *inverse question*. Let $\{\alpha_i \mid i \in J\}$, J – any index set, be a family of questions. By the *product* of a family of questions $\{\alpha_i\}$ we mean the question denoted by $\sqcap_i \alpha_i$ and defined in the following way: choose in any way, random or not, one from the experimental projects $\{\alpha_i\}$, perform the corresponding experiment, and attach to $\sqcap_i \alpha_i$ the result obtained.

It may happen, that for a question $\alpha \in \Omega$ in the given situation for the physical system under consideration, it is certain, that if this experiment were performed, then the result "yes" would be obtained. Or, equivalently, in other words, the result "no" is certainly impossible. In such case we say that the question α is *true*.

We shall assume that there is always a trivial question I in Ω, which consists in doing anything (or nothing) with the system and attaching to it always the result "yes". Clearly, I is the question which is always true and $I^{\sim}$ is the question which is never true.

We say, that a question α is *stronger* than β if whenever α is true, also β is true. We shall denote this by $\alpha < \beta$. This is quasi-ordering relation on Ω and as such enables to define the equivalence relation "$\sim$" on Ω in an obvious way:

$$\alpha \sim \beta \Leftrightarrow \alpha < \beta \quad \text{and} \quad \beta < \alpha$$

From a physical point of view, equivalent experimental projects can be practically identified. The equivalence class $a \equiv [\alpha]$ containing a question α is called the *property* defined by a question (experimental project) $\alpha \in \Omega$.

Let $\mathscr{L} \equiv \Omega/\sim$ be the set of all equivalence classes in Ω, i.e. the set of properties of a physical system. A property $a \in \mathscr{L}$ is said to be *actual* if any (and hence every) question $\alpha \in a$ is true. A property $a \in \mathscr{L}$ is said to be *stronger* than a property $b \in \mathscr{L}$ if for any $\alpha \in a$, $\beta \in b$, $\alpha < \beta$. We denote this by $a < b$. With the relation "$<$" $\mathscr{L}$ is a partially ordered set and, moreover, one can prove that it is a complete lattice, where the greatest lower bound $\bigwedge_i a_i$ for any family is a property defined by the question $\sqcap_i \alpha_i$, where $\alpha_i \in a_i$. That is

$$\bigwedge_i a_i = [\sqcap_i \alpha_i]$$

The maximal and the minimal elements in this lattice are respectively

$$\mathbf{1} \equiv [I], \qquad \mathbf{0} = [I^{\sim}],$$

The lattice $\mathscr{L}$ is called the *property lattice* of a physical system.

A *state* ε of a system is the set of all actual properties of it, i.e. $\varepsilon \equiv \{a \in \mathscr{L} \mid a\text{-actual}\}$. As one can show, states can be uniquely represented by certain elements from the property lattice $\mathscr{L}$, namely every state S is uniquely defined by a property $p \in \mathscr{L}$, such that $p = \bigwedge_{a \in S} a$. This property p is called a *state-property*. For the sake of simplicity in what follows we shall always identify states with state properties. So, if a system is in a state represented by a state – property $p \in \mathscr{L}$ (or shortly: in a state p), the fact that a property $a \in \mathscr{L}$ is actual is expressed by the relation: $p < a$.

Let $\Sigma \subset \mathscr{L}$ be the set of state-properties. Then every property $a \in \mathscr{L}$ can be written as:

$$a = \bigvee_{p \in \Sigma, p < a} p$$

The mapping $\mu : \mathscr{L} \to P(\Sigma)$ defined by

$$\mu(a) \equiv \{p \in \Sigma \mid p < a\}$$

is called a *Cartan mapping*. With this mapping every property lattice can be represented by some family of subsets of the state space of a system. Its is easy to check that μ is injective, $\mu(\mathbf{0}) = \varnothing$, $\mu(\mathbf{1}) = \Sigma$, $\mu(\bigwedge_i a_i) = \bigcap_i \mu(a_i)$.

Definition 1. Two states p, $q \in \Sigma$ are said to be *orthogonal* and we shall denote this by $p \perp q$ if there exists a question $\gamma \in \Omega$, such that γ is true whenever the system is in a state p and $\gamma^{\sim}$ is true whenever the system is in a state q; shortly:

$$p \perp q \Leftrightarrow \exists \gamma \in \Omega : p < \gamma \quad \text{and} \quad q < \gamma^{\sim}$$

This relation induce the relation $\perp$ on the property lattice L in the following way:

$$a \perp c \Leftrightarrow [(p < a, q < b, p, q \in \Sigma) \Rightarrow p \perp q)]$$

The relation $\perp$ on $\mathscr{L}$ enjoys the following features

$$a \perp b \Rightarrow b \perp a$$

$$a < b, c < d, b \perp d \Leftrightarrow a \perp c$$

$$a \perp b \Leftrightarrow a \wedge b = \mathbf{0} \tag{1}$$

$$a \perp \mathbf{0} \; \forall a \in \mathscr{L}$$

One observes, that if $a \perp b$ then $\mu(a) \cap \mu(b) = \varnothing$.

Let $p \in \mathscr{L}$ be a property and let us suppose that p is an atom of the lattice $\mathscr{L}$. Then p must be a state property. Indeed, since $p \neq \mathbf{0}$, there is a state – property $q \in \Sigma$ in which p is actual, i.e. $q < p$ holds and since p is an atom, $q = p$. The converse is imposed by the following axiom.

Axiom 1. State-properties are just atoms of the property lattice.

Consequently, the image of a state – property under the Cartan mapping is always a singleton, i.e. for $p \in \Sigma$, $\mu p = \{p\} \in P(\Sigma)$. In what follows we shall denote the singletons $\{p\}$ simply by p.

Let for $M \subset \Sigma$, $M^{\perp} \equiv \{p \in \Sigma \mid p \perp q \; \forall q \in M\}$. $\mathscr{L}(\Sigma, \perp) \equiv \{M \subset \Sigma \mid M^{\perp\perp} = M\}$ is a complete orthocomplemented lattice with the partial ordering of set inclusion, set intersection as lattice meet and with orthocomplementation defined by

$$\mathscr{L}(\Sigma, \perp) \ni M \mapsto M^{\perp} \in \mathscr{L}(\Sigma, \perp).$$

If we, furthermore, assume that $p^{\perp\perp} = p$ then this lattice is also atomistic.

In general, a property lattice $\mathscr{L}$ can not be identified with the corresponding $\mathscr{L}(\Sigma, \perp)$. It is the case when for each property there is the opposite property and, on the other hand, every property is the opposite of some property. This is the contents of the following axiom:

Axiom 2. (i) For every state $p \in \Sigma$ there exists a question $\beta \in \Omega$ which is true iff the system is in a state q, such that $q \perp p$.

(ii) If $p' \equiv [\beta]$, where β is postulated by (i), then the mapping $\mathscr{L} \ni a \mapsto \bigwedge_{\mu(a)} p' \equiv a'$ is surjective.

If Axiom 2 is satisfied, then the property lattice is an orthocomplemented lattice, obviously $\mu a' = (\mu a)^{\perp}$ and in a consequence the image of $\mathscr{L}$ under the Cartan mapping is the lattice $\mathscr{L}(\Sigma, \perp)$. This means that a property lattice is represented by the set of all biorthogonal subsets of the set of states of a system.

Corollary 1. *If axiom* 1 *is satisfied then the mathematical structure of the set* $\mathscr{L}$ *of properties of a physical system is that of a complete, atomistic lattice with an orthogonality relation satisfying the conditions* (1); *state-properties are atoms of this lattice. If, in addition, axiom* 2 *is satisfied, then* $\mathscr{L}$ *is an orthocomplemented lattice and for any* $a, b \in \mathscr{L}$ $a \perp b \Leftrightarrow a < b'$.

Two other axioms must be imposed on a property lattice of a physical system in order to obtain the Hilbert–space model of quantum mechanics. These are:

Axiom 3 (Weak modularity). If $a, b \in \mathscr{L}$, and $a < b$, then $b = (a' \wedge b) \vee a$

Axiom 4 (Covering law). If $p \in \mathscr{L}$ is an atom, $a \in \mathscr{L}$ and $a \wedge p = \mathbf{0}$ then $a \vee p$ covers p.

We recall here that these two axioms cannot be satisfied by a property lattice, which describes a system consisting of two separated quantum entities. This fact justifies the terminology which has been adopted. By a physical entity we shall mean a physical system for which axioms 3 and 4 are satisfied.

To end this brief survey of the Piron–Aerts axiomatics, let us mention the following remarkable feature which it enjoys. Suppose that an abstract complete, orthocomplemented atomistic lattice is given. Let us interpret atoms of this lattice as possible states of a physical system and let for each such state an experimental project can be admitted, such that it gives with certainty the result "yes" when the system is in the considered state and the result "no" when the system is in a state orthogonal to it. One proves, cf. [3], that the property lattice constructed from this set of experimental projects coincides with the initial lattice. This shows the consistency of the formalism.

2. The description of an evolution of a physical system

Let us consider now the description of a physical system when time is taken into account. It is natural to suppose that at any moment the system under consideration is described by a property lattice $\mathscr{L}_t \equiv \Omega_t/\sim$. That is, $\mathscr{L}_t$ is a set of properties which a system may have at the moment t. Every property is now defined by some experimental project (question) $\alpha_t \in \Omega_t$, which–being labelled by t–is an experiment which may be (or not) performed at the moment t. The decision to perform or not given experiment α_t is to be taken by a physicist not later than at the moment t. The state of the system at the moment t is the collection $\varepsilon_t \subset \mathscr{L}_t$ of all actual properties from the property lattice $\mathscr{L}_t$. Let Σ_t denote the set of all possible states of the system at the moment t, identified with respective state-properties in the lattice $\mathscr{L}_t$. Thus, at every moment t the system is in a certain $p_t \in \Sigma_t$. Due to the system itself, and due to the influence of the surrounding, the set of the actual properties changes, that is the system passes from $p_t \in \Sigma_t$ at the moment t to $p_s \in \Sigma_s$ at the moment $s > t$. To describe the evolution of a system means to make at the moment t predictions concerning the properties which the system may have at the moment $s > t$. Obviously, it is possible only if the external conditions which influenced the system between the moment t and s are specified. This includes also the case of so called "isolated" physical system.

Definition 2. We say that a dynamical evolution of en entity is given, if for every $t \in \mathbf{R}$ a family of mappings $\varphi_{t,s}: \mathscr{L}_s \to \mathscr{L}_t$, $s \geq t$, is defined such that for any $a_s \in \mathscr{L}_s$, whenever $\varphi_{t,s}a_s$ is actual at the moment t, a_s will be actual at the moment s, i.e.

$$\varphi_{t,s}^{-1}\varepsilon_t \equiv \{a_s \in \mathscr{L} \mid \varphi_{t,s}a_s \in \varepsilon_t\} \subset \varepsilon_s$$

where ε_t, ε_s denote sets of actual properties at the moments t and s respectively.

The following theorem enables the interpretation of the mapping $\varphi_{t,s}$ in terms of experimental projects.

Theorem 1. *To define a family of mappings $\varphi_{t,s}$ it is equivalent to say that for any $s \geq t$ and any $a_s \in \mathscr{L}_s$ there exists a question in Ω_t which we shall denote by $\varphi_{t,s}\alpha_s$, $\alpha_s \in a_s \in \mathscr{L}_s$ and which is defined by the project of the following experiment: let the system evolve in the given conditions between the moments t and s and then perform the question $\alpha_s \in a_s$; the result of this experiment is the result of α_s obtained at the moment s.*

Proof. Let a family of mappings $\varphi_{t,s}: \mathscr{L}_s \to \mathscr{L}_t$ be given. Let us consider any $a_s \in \mathscr{L}_s$. Let $\beta_t \in \varphi_{t,s}a_s$. Then, if β_t is true, $\varphi_{t,s}a_s$ is actual, a_s will be actual at time s, and consequently $\varphi_{t,s}\alpha_s$ (as it is defined in the theorem) is true. On the other hand, if $\varphi_{t,s}\alpha_s$ is true, then by definition of this question, when one decide at the moment t to perform the corresponding experiment, one is certain that he will obtain answer "yes" for the question $\alpha_s \in a_s$ at time s. But this means that $\varphi_{t,s}a_s$ is actual at the moment t, i.e. $\beta_t \in \varphi_{t,s}a_s$ is true. Thus $\beta_t \sim \varphi_{t,s}a_s$. Let us suppose now that for any $a_s \in \mathscr{L}_s$, there is $\beta_t \in \Omega_t$ such that $\beta_t \sim \varphi_{t,s}\alpha_s$. We can therefore define $\varphi_{t,s}: \mathscr{L}_s \to \mathscr{L}_t$ by: $a_s \mapsto [\varphi_{t,s}\alpha_s]$ where $[\varphi_{t,s}\alpha_s]$ is an equivalence class of questions from Ω_t defined by $\varphi_{t,s}\alpha_s$, and $\alpha_s \in a_s$. This mapping is well-defined, since if $\alpha_s \sim \gamma_s$ then $\varphi_{t,s}\alpha_s \sim \varphi_{t,s}\gamma_s$. Indeed, whenever $\varphi_{t,s}\alpha_s$ is true, the result "yes" of an eventual measurement of α_s at the moment s is already certain at the moment t and hence, by the assumption also the result "yes" of an eventual measurement of γ_s is certain. Thus, whenever $\varphi_{t,s}\alpha_s$ is true at s, γ_s will be true at s, what means that whenever $\varphi_{t,s}\alpha_s$ is true, $\varphi_{t,s}\gamma_s$ is true, i.e. $\varphi_{t,s}\alpha_s < \varphi_{t,s}\gamma_s$. In the same way we prove the converse. Manifestly, if $\varphi_{t,s}a_s$ is actual, then a_s will be actual at the moment s.

Theorem 2. The mappings $\varphi_{t,s}$, $t \leq s$, enjoy the following features:

(i) $\varphi_{t,s}\left(\bigwedge_i a_s^i\right) = \bigwedge_i \varphi_{t,s}a_s^i$

(ii) $\varphi_{t,s}\mathbf{1}_s = \mathbf{1}_t$

(iii) $a_s \perp b_s \Rightarrow \varphi_{t,s}a_s \perp \varphi_{t,s}b_s$

(iv) $\varphi_{t,t'}\varphi_{t',s} = \varphi_{t,s}$, where $t \leq t' \leq s$.

Proof. (i) Whenever $\varphi_{t,s}(\bigwedge_i a_s^i)$ is actual, a_s^i for every i will be actual at the moment s, what means that whenever $\varphi_{t,s}(\bigwedge_i a_s^i)$ is actual, $\varphi_{t,s}\, a_s^i \forall i$ is actual, i.e. $\bigwedge \varphi_{t.s}\, a_s^i$ is actual. When $\bigwedge_i \varphi_{t,s} a_s^i$ is actual, then $\forall i \varphi_{t,s} a_s^i$ is actual, i.e. $\forall i\, a_s^i$ will be actual and consequently $\bigwedge_i a_s^i$ will be actual at s. Thus, whenever $\bigwedge_i \varphi_{t,s} a_s^i$ is actual, $\bigwedge_i a_s^i$ will be actual at s, what means that when $\bigwedge_i \varphi_{t,s} a_s^i$ is actual, $\varphi_{t,s}(\bigwedge_i a_s^i)$ is actual.

(ii) Obviously, $\varphi_{t,s}\mathbf{1}_s$ whenever tested gives the answer "yes", so $\varphi_{t,s}\mathbf{1}_s = \mathbf{1}_t$.

(iii) By definition $a_s \perp b_s$ if for any p_s, $q_s \in \Sigma$ such that $p_s < a_s$, $q_s < b_s$, there exists $\gamma_s \in \Omega_s$ such that $p_s < \gamma_s$ and $q_s < \gamma_s^{\sim}$. Let p_t, $q_t \in \Sigma_t$ and $p_t < \varphi_{t,s} a_s$ and $q_t < \varphi_{t,s} b_s$. Whenever the system is in the state p_t at the moment t, it will be at the moment s in such a state p_s, that a_s will be actual, i.e. in such a state that $p_s < a_s$. The same is for q_t: when it is the state of the system at the moment t, the state q_s at the moment s will be such that $q_s < b_s$. But for such pair of states there exists by the assumption a question γ_s such that $p_s < \gamma_s$ and $q_s < \gamma_s^{\sim}$. Hence we see that for any pair of states p_t, q_t such that $p_t < \varphi_{t,s} a_s$, $q_t < \varphi_{t,s} b_s$ there is a question $\gamma_s \in \Omega_s$, such that whenever the system is in the state p_t at the moment t, γ_s will be true at time s, and whenever the system is in the state q_t at t, $\gamma_s^{\sim}$ will be true. But this implies that in the first case always $\varphi_{t,s}\gamma_s$ is true at t, and in the second case $\varphi_{t,s}\gamma_s^{\sim}$ is true. Since obviously $\varphi_{t,s}\gamma_s^{\sim} = (\varphi_{t,s}\gamma_s)^{\sim}$ it follows that $p_t \perp q_t$, where it is the question $\varphi_{t,s}\gamma_s$ making them orthogonal.

(iv) If $a_s \in \mathscr{L}_s$, then $\varphi_{t',s} a_s$ is defined by a question $\varphi_{t',s}\alpha_s$, i.e. the project of an experiment in which system evolves during $s - t'$ seconds from the moment t'. According to the same definition, the question $\varphi_{t,t'}(\varphi_{t's}\alpha_s)$ is a project of an experiment in which the system evolves from time t till t', and then $\varphi_{t's}\alpha_s$ is performed. Obviously, this is just the question defining $\varphi_{t,s} a_s$.

The set of properties $\varphi_{t,s}^{-1}\varepsilon_t$ is the subset of all properties which the system will have actual at the moment s. Properties in $\varphi_{t,s}^{-1}\varepsilon_t$ are precisely those properties which can be predicted from actual properties (state of the system) at the moment t. In general, not every property which will be actual for the system, can be predicted in advance. This is the case of deterministic evolution.

Definition 3. The evolution of a system given by a family of mappings $\varphi_{t,s}$ is called deterministic if for every t, $s \in R$, $t \leq s$, and for any state ε_t

$$\varphi_{t,s}^{-1}\varepsilon_t = \varepsilon_s$$

where ε_s is the state of the system at the moment s.

Theorem 3. *If the evolution of a system is deterministic, then mappings $\varphi_{t,s}$ are injective.*

Proof. Since we already know from the Theorem 2 that $a_s < b_s \Rightarrow \varphi_{t,s} a_s < \varphi_{t,s} b_s$, we have only to show that the converse holds. Let $\varphi_{t,s} a_s < \varphi_{t,s} b_s$ and suppose that a_s is actual at the moment s. According to the definition 3, $\varphi_{t,s} a_s$ was actual at t, consequently $\varphi_{t,s} b_s$ was actual and b_s is actual.

Lemma 1. *Let the deterministic evolution of a physical system be given. If, for some $a_t \in \mathscr{L}_t$ and some $s_0 > t$, $a_t \notin \varphi_{t,s_0}\mathscr{L}_{s_0}$, where $\varphi_{t,s_0}\mathscr{L}_{s_0} \subset \mathscr{L}_t$ is the image of $\mathscr{L}_{s_0}$, then for every $s > s_0$, $a_t \notin \varphi_{t,s}\mathscr{L}_s$.*

Proof. Let us suppose that $a_t \notin \varphi_{t,s_0}\mathscr{L}_{s_0}$ and that there exists $s_1 > s_0$ such that $a_t = \varphi_{t,s_1} a_{s_1}$ for some $a_{s_1} \in \mathscr{L}s_1$. According to the preceding theorem we have $\varphi_{t,s_1} = \varphi_{t,s_0}\varphi_{s_0,s_1}$, i.e. $a_t = \varphi_{t,s_0}(\varphi_{s_0,s_1} a_{s_1})$ what contradicts the assumption.

Theorem 4. *If $s < s'$, then $\varphi_{t,s'}\mathscr{L}_{s'} \subset \varphi_{t,s}\mathscr{L}_s$.*

Proof. From the preceding lemma we have $s < s' \Rightarrow \{a_t \mid a_t \notin \varphi_{t,s}\mathscr{L}_s\} \subset \{a_t \mid a_t \notin \varphi_{t,s'}\mathscr{L}_{s'}\}$ from what the conclusion follows.

In view of the above theorem, it is reasonable to adopt the following assumption:

Assumption 1. The description of a deterministic evolution of a physical system is irredundant, i.e. all the mappings $\varphi_{t,s}$, $t, s \in R$, $t < s$ are surjective.

Corollary 2. *Under all the above assumptions, the deterministic evolution of a physical system is given by a family of mappings $\varphi_{t,s} : \mathscr{L}_s \to \mathscr{L}_t$, $t, s \in \mathbf{R}$, $t \leq s$ which has the properties (i) $\div$ (iv) of Theorem 2. Moreover,*

(i) *$\varphi_{t,s} a_s$ is actual at the moment t if a_s will be actual at the moment s.*
(ii) *$\varphi_{t,s}$ are bijective*
(iii) *If $p_t \in \Sigma_t$ is a state (atom) in $\mathscr{L}_t$ then there exists a state (atom) in $\Sigma_s \subset \mathscr{L}_s$ such that $\varphi_{t,s}^{-1}\{a_t \in \mathscr{L}_t \mid p_t < a_t\} = \{b_s \in \mathscr{L}_s \mid p_s < b_s\}$.*

3. The mathematical framework

In order to get a mathematical structure which will enable us to handle with the model of evolution which has been formulated in the proceeding section, it is useful to study appropriate mathematical structures from a categorial point of view. As we shall see in the section 4, this shall give us immediately a dual description of an evolution in terms of morphisms of state spaces and the description of a dynamical system in terms of its trajectory space.

Let $K_\wedge$ denote a category which objects are complete, atomistic lattices with orthogonality relation $\perp$ satisfying the following conditions. If $a, b \in \mathscr{L} \in ObK_\wedge$ and Σ is the set of atoms in $\mathscr{L}$ then:

$$\begin{aligned} & a \perp b \Rightarrow b \perp a \\ & a \perp a \Rightarrow a = \mathbf{0} \\ & [(p < a,\, q < b,\, p,\, q \in \Sigma) \Rightarrow p \perp q] \Leftrightarrow a \perp b \end{aligned} \tag{2}$$

Morphisms in this category are defined to be mappings $\varphi : \mathscr{L}_2 \to \mathscr{L}_1$, $\mathscr{L}_1$, $\mathscr{L}_2 \in ObK_\wedge$, such that:

$$\begin{aligned} &\text{(i)}\quad \varphi\Big(\bigwedge_i a_2^i\Big) = \bigwedge_i \varphi a_2^i, \qquad \varphi\mathbf{1}_2 = \mathbf{1}_1 \\ &\text{(ii)}\quad a_2 \perp b_2 \Rightarrow \varphi a_2 \perp \varphi b_2 \\ &\text{(iii)}\quad \text{for any atom } p_1 \in \mathscr{L}_1 \text{ there exists an atom} \\ &\qquad p_2 \in \mathscr{L}_2 \text{ such that } \varphi^{-1} S_{p_1} = S_{p_2} \end{aligned} \tag{3}$$

where $S_p \equiv \{a \in \mathscr{L} \mid p < a\}$.

We shall be also considering another category, denoted by K_Σ and defined as follows. Objects of K_Σ are sets with a distinguished family of subsets and an irreflexive, symmetric relation defined on it. That is, an object of K_Σ is a set Σ with the relation $\perp$ which is symmetric and irreflexive and with a family of subsets $\mathscr{L}(\Sigma)$ such that:

$$\text{(i) } \varnothing, \Sigma, \{p\} \in \mathscr{L}(\Sigma)$$

$$\text{(ii) } (M_i \in \mathscr{L}(\Sigma) \forall i) \Rightarrow \bigcap_i M_i \in \mathscr{L}(\Sigma) \tag{4}$$

where $p \in \Sigma$ and $\{M_i\}$ is any indexed family.

Morphisms in K_Σ are mappings $T: \Sigma_1 \to \Sigma_2$, $(\Sigma_1, \Sigma_1 \in ObK_\Sigma)$ such that:

$$\text{(i) } M \in \mathscr{L}(\Sigma_1) \Rightarrow T^{-1}M \in \mathscr{L}(\Sigma_1)$$

$$\text{(ii) } Tp_1 \perp Tq_1 \Rightarrow p_1 \perp q_1 \tag{5}$$

Let us define the following mapping $F: K_\wedge \to K_\Sigma$. For $\mathscr{L} \in ObK_\wedge$, $F\mathscr{L} \equiv \Sigma$, where Σ is the set of atoms of $\mathscr{L}$; if $\varphi: \mathscr{L}_2 \to \mathscr{L}_1$ is a morphism of $K_\wedge$, $\mathscr{L}_1$, $\mathscr{L}_2 \in ObK_\wedge$, then according to (3) (iii) we can define:

$$F\varphi: \Sigma_1 \to \Sigma_2 \quad \text{as} \quad (F\varphi)p_1 \equiv p_2 \tag{6}$$

Theorem 5. *F is a contravariant functor from $K_\wedge$ to K_Σ. Moreover, this functor is*

(1) *full, i.e. for any $\mathscr{L}_1$, $\mathscr{L}_2 \in ObK_\Sigma$ and any morphism $T: F\mathscr{L}_1 \to F\mathscr{L}_2$ there exists morphism $\varphi: \mathscr{L}_2 \to \mathscr{L}_1$ in K_Σ such that $F\varphi = T$.*

(2) *representative, i.e. for any $\Sigma \in ObK_\wedge$ there exists $\mathscr{L} \in ObK_\wedge$ such that $F\mathscr{L}$ and Σ are isomorphic.*

(3) *faithfull, i.e. $F: Mor(\mathscr{L}_2, \mathscr{L}_1) \to Mor(F\mathscr{L}_1, F\mathscr{L}_2)$ is $1-1$, where $Mor(\ ,\ )$ denotes the set of morphisms between the two given objects of respective category.*

Remark. Here and after by isomorphism of objects of respective category we mean a morphism which is bijective and which inverse is also a morphism in this category.

Before giving the proof of the theorem we shall prove following lemma.

Lemma 2. *Let $\mathscr{L}_1$, $\mathscr{L}_2 \in ObK_\wedge$, $p_1 \in \mathscr{L}_1$ be an atom, $b_2 \in \mathscr{L}_2$ and let $\varphi: \mathscr{L}_2 \to \mathscr{L}_1$ be a morphism. Then*

$$p_1 < \varphi b_2 \Leftrightarrow p_2 \equiv (F\varphi)p_1 < b_2.$$

Proof. Let $p_1 < \varphi b_2$. Then $b_2 \in \varphi^{-1}S_{p_1}$ and

$$(F\varphi)p_1 = \bigwedge_{a_2: p_1 < \varphi a_2} a_2 < a_2$$

and in particular $(F\varphi)p_1 < b_2$. Conversely, let $(F\varphi)p_1 < b_2$. Then

$$\varphi((F\varphi)p_1) = \bigwedge_{a_2:p_1<\varphi a_2} \varphi a_2 < \varphi b_2$$

and in a consequence $p_1 < \varphi b_2$.

Proof of Theorem 5. Let $\Sigma \subset \mathscr{L} \in ObK_\wedge$ be the set of atoms in $\mathscr{L}$. Then we can define

$$\mathscr{L}(\Sigma) \equiv \{M \subset \Sigma \mid M = \mu a,\ a \in \mathscr{L}\},$$

where $\mu a \equiv \{p \in \Sigma \mid p < a\}$. The family of subsets $\mathscr{L}(\Sigma)$ satisfies the conditions (6). Also, the orthogonality relation $\perp$ on Σ inherited from $\mathscr{L}$ is symmetric and irreflexive. Thus, $F\mathscr{L}$ is an object of K_Σ. If $M \in \mathscr{L}(\Sigma_2)$, i.e. $M = \mu a_2$, $a_2 \in \mathscr{L}_2$, then from the Lemma 2, we obtain $(F\varphi)^{-1}M = (F\varphi)^{-1}\mu a_2 = \mu\varphi a_2$, i.e. $(F\varphi)^{-1}M \in \mathscr{L}(\Sigma_1)$. Finally, if $(F\varphi)p_1 \perp (F\varphi)q_1$, then from the Lemma 2 $p_1 < \varphi((F\varphi)p_1)$, $q_1 < \varphi((F\varphi)q_1)$, and by (3)(ii) and (2) it follows that $p_1 \perp q_1$. We have proved, that $F\varphi$ is a morphism of K_Σ. By a straightforward manipulation one can check that F is indeed a contravariant functor, i.e. if $\nu: \mathscr{L}_3 \to \mathscr{L}_{2,\varphi}: \mathscr{L}_2 \to \mathscr{L}_3$ are morphisms, then $F\nu F\varphi = F(\varphi\nu)$.

Proof of assertion 1. Let $T: F\mathscr{L}_1 \to F\mathscr{L}_2$ be a morphism of K_Σ. It is easy to check, that the mapping:

$$\mathscr{L}_2 \ni a_2 \mapsto \bigvee_{T^{-1}\mu a_2} p_1 \equiv \varphi a_2 \in \mathscr{L}_1 \tag{7}$$

is a morphism in $K_\wedge$. Indeed, (7) is equivalent to

$$\mu(\varphi a_2) = T^{-1}\mu a_2 \tag{8}$$

since by (5) (i) $T^{-1}\mu a_1 \in \mathscr{L}(\Sigma_2)$. From (8) one can immediately see that φ preserves $\wedge$, $\mathbf{1}$, $\perp$. Since (8) reads:

$$p_1 < \varphi a_2 \Leftrightarrow Tp_1 < a_2$$

for $p_1 \in \Sigma_1$, $a_2 \in \mathscr{L}_2$, we see that also (3) (iii) is satisfied, i.e. for any atom $p_1 \in \Sigma_1$ there exists an atom $p_2 \in \Sigma_2$ such that $\varphi^{-1}S_{p_1} = S_{p_2}$. In fact $p_2 = Tp_1$ in this case. Moreover, for φ just defined we have

$$(F\varphi)p_1 = \bigwedge_{a_2:p_1<\varphi a_2} a_2 = \bigwedge_{a_2:Tp_1<a_2} a_2 = Tp_1$$

Proof of assertion 2. If $\Sigma \in ObK_\Sigma$, then $\mathscr{L}(\Sigma)$ is an object of $K_\wedge$. Indeed, it is a complete lattice, with the partial ordering given by set-theoretical inclusion and $\bigwedge_i M_i = \bigcap_i M_i$, $M_i \in \mathscr{L}(\Sigma)$. Since by definition singletons belongs to it, it is atomistic. The orthogonality is defined by

$$M \perp N \Leftrightarrow [(p \in M,\ q \in N) \Rightarrow p \perp q]$$

It is also clear, that $F\mathscr{L}(\Sigma) = \Sigma$.

Proof of assertion 3. Let us assume that $F\varphi = F\nu$, where φ, $\nu \in Mor\,(\mathscr{L}_2, \mathscr{L}_1)$. From the Lemma 2, we have $(F\varphi)^{-1}\mu a_2 = \mu(\varphi a_2)$, $(F\varphi)^{-1}\mu a_2 =$

$\mu(\nu a_2)$, for any $a_2 \in \mathscr{L}_2$, and by the assumption $\mu(\varphi a_2) = \mu(\nu a_2)$ what by the injectivity of the mapping μ implies that $\varphi = \nu$.

As a matter of fact as it can be seen from the proof of the assertion 2 of the theorem, we have more than this. For any object $\Sigma \in ObK_\Sigma$ there exists $\mathscr{L} \in ObK_\wedge$ such that $F\mathscr{L} = \Sigma$.

Corollary 3. *The contravariant functor $F: K_\wedge \to K_\Sigma$ defines correspondence between these categories such that:*

(1) *For any object $\Sigma \in ObK_\Sigma$ there exists a unique (up to an isomorphism) object $\mathscr{L} \in ObK_\wedge$ such that $F\mathscr{L} = \Sigma$.*
(2) The mapping $F: Mor(\mathscr{L}_2, \mathscr{L}_1) \to Mor(F\mathscr{L}_1, F\mathscr{L}_2)$ is bijective for any $\mathscr{L}_1$, $\mathscr{L}_2 \in ObK_\wedge$.

The inverse functor $G \equiv F^{-1}$ has the form: $G\Sigma = \mathscr{L}(\Sigma)$ for $\Sigma \in ObK_\Sigma$ and if $T \in Mor(\Sigma_1, \Sigma_2)$ then

$$(GT)M_2 = \bigvee_{T^{-1}M_2} p_1 \tag{9}$$

where $M_2 \in \mathscr{L}(\Sigma_2)$ and $\bigvee p_1$ denotes the lattice join in $\mathscr{L}(\Sigma_1)$ over all singletons $\{p_1\}$ such that $Tp_1 \in M_2$.

From the point of view of physics, the situation of particular interest is, when orthogonality on a lattice is defined by an orthocomplementation. That is on $\mathscr{L} \in ObK_\wedge$ there is a mapping $': \mathscr{L} \to \mathscr{L}$ such that:

$$a'' = a,\ a < b \Rightarrow b' < a',\ a' \wedge a = \mathbf{0}$$

$$a \perp b \Leftrightarrow a < b' \tag{10}$$

Let $K_\wedge^0$ denote a category, such that $ObK_\wedge^o$ are orthocomplemented, complete, atomistic lattices, and if $\mathscr{L}_1$, $\mathscr{L}_2 \in ObK_\wedge^o$, then morphisms are mappings $\varphi: \mathscr{L}_1 \to \mathscr{L}_2$ satisfying:

$$\begin{aligned} &\text{(i)}\ \ \varphi\Big(\bigwedge_i a_1^i\Big) = \bigwedge_i \varphi a_1^i,\ \varphi \mathbf{1}_1 = \mathbf{1}_2 \\ &\text{(ii)}\ \ \varphi(a_1') = (\varphi a_1)' \\ &\text{(iii)}\ \ \text{for any atom } p_2 \in \mathscr{L}_2 \text{ there exists an atom} \\ &\qquad p_1 \in \mathscr{L}_2 \text{ such that } \varphi^{-1} S_{p_2} = S_{p_1} \end{aligned} \tag{11}$$

Obviously, $K_\wedge^o$ is a subcategory of $K_\wedge$, i.e. every object $\mathscr{L} \in ObK_\wedge^o$ with $\perp$ defined by (10) and if $\varphi: \mathscr{L}_1 \to \mathscr{L}_2$ is a morphism in $K_\wedge^o$, then it is also a morphism in $\mathscr{L}_\wedge$. Indeed, if $\varphi a_1' = (\varphi a_1)'$, then $a_1 \perp b_1 \Rightarrow \varphi a_1 \perp \varphi b_1$. The converse is not true in general, i.e. if $\mathscr{L}_1$, $\mathscr{L}_2 \in ObK_\wedge^o$ and $\varphi: \mathscr{L}_1 \to \mathscr{L}_2$ is a morphism in the category $K_\wedge$, then it need not be a morphism in the category $K_\wedge^o$.

Let us denote by $K_\Sigma^{\perp\perp}$ the category, which objects are sets with a symmetric and irreflexive relation $\perp$ such that $p^{\perp\perp} = p$ for any $p \in \Sigma$, where $p^\perp \equiv \{q \in \Sigma \mid q \perp p\}$. Let Σ_1, $\Sigma_2 \in ObK_\Sigma^{\perp\perp}$. Morphisms are defined to be the mappings

$T:\Sigma_1 \to \Sigma_1$ such that for any $M_2 \subset \Sigma_2$ satisfying $M_2^{\perp\perp} = M_2$:

$$T^{-1}M_2^{\perp} = (T^{-1}M_2)^{\perp}, \tag{12}$$

where $M_2^{\perp} \equiv \{q_2 \in \Sigma_2 \mid q_2 \perp p_2 \, \forall p_2 \in M_2\}$.

Lemma 3. *If the mapping T satisfies* (12) *then*

$$Tp_1 \perp Tq_1 \Rightarrow p_1 \perp q_1. \tag{13}$$

Proof. $Tp_1 \perp Tq_1$ is equivalent to $p_1 \in (T^{-1}(Tq_1))^{\perp}$ by (12). Since $q_1 \in T^{-1}(Tq_1)$, $(T^{-1}(Tq_1))^{\perp} \subset q_1^{\perp}$, i.e. $p_1 \in q_1^{\perp}$.

Theorem 6. (i) $K_{\Sigma}^{\perp\perp}$ *is a subcategory of* K_{Σ}.

(ii) *the functor F maps the subcategory* $K_{\wedge}^{0}$ *into the subcategory* $K_{\Sigma}^{\perp\perp}$.

(iii) *for any object* $\Sigma \in ObK_{\Sigma}^{\perp\perp}$ *there exists a unique (up to an isomorphism) object* $\mathscr{L} \in ObK_{\wedge}^{o}$ *such that* $F\mathscr{L} = \Sigma$.

(iv) the mapping $F: Mor(L_2, \mathscr{L}_1) \to Mor(F\mathscr{L}_1, F\mathscr{L}_2)$, where $Mor(\,,\,)$ denotes morphisms in $K_{\wedge}^{o}$ and $K_{\Sigma}^{\perp\perp}$ respectively, is bijective for any $\mathscr{L}_2, \mathscr{L}_1 \in ObK_{\wedge}^{o}$.

Proof. (i) If $\Sigma \in ObK_{\Sigma}^{\perp\perp}$, then defining

$$\mathscr{L}(\Sigma, \perp) \equiv \{M \subset \Sigma \mid M^{\perp\perp} = M\} \tag{14}$$

we immediately see that with this family of subsets Σ satisfies the conditions (4) and therefore $\Sigma \in ObK_{\Sigma}$. If $T:\Sigma_1 \to \Sigma_1$ is a morphism in $K_{\Sigma}^{\perp\perp}$, then for $M_2 = M_2^{\perp\perp}$ we obtain from (12)

$$T^{-1}M_2 = T^{-1}(M_2^{\perp\perp}) = (T^{-1}M_2)^{\perp\perp} \tag{12a}$$

and the condition (5) (i) is satisfied. By the Lemma 3 also (5) (ii) is satisfied, such that T is a morphism in K_{Σ}.

(ii) Let $\mathscr{L} \in ObK_{\wedge}^{o}$. From the definition of F, we have $F\mathscr{L} = \Sigma$, Σ being the set of atoms of $\mathscr{L}$ with the family of subsets $\mathscr{L}(\Sigma) = \{M \subset \Sigma \mid M = \mu a,\, a \in \mathscr{L}\}$. But $\mathscr{L}$ is orthocomplemented, so that for any $M \subset \Sigma$ we have $M = \mu a \Leftrightarrow M^{\perp\perp} = M$. Consequently, $\mathscr{L}(\Sigma) = \mathscr{L}(\Sigma, \perp)$ and $F\mathscr{L} \in ObK_{\Sigma}^{\perp\perp}$. Let $\varphi: \mathscr{L}_2 \to \mathscr{L}_1$ be a morphism of $K_{\wedge}^{0}$. Since $F\varphi$ is a morphism in $K_{\wedge}$, then (5) (i) is satisfied, what means in this case that for any $a_2 \in \mathscr{L}_2$, $(F\varphi)^{-1}\mu a_2 \equiv \mu\varphi a_2$. Let $M_2 \subset \Sigma_2$ be such that $M_2^{\perp\perp} = M_2$. Since $\mathscr{L}(\Sigma_2, \perp) = \mathscr{L}(\Sigma_2)$, $(F\varphi)^{-1}M_2^{\perp} = (F\varphi)^{-1}(\mu a_2)^{\perp} = (F\varphi)^{-1}\mu(\varphi a') = \mu(\varphi a)' = (\mu\varphi a)^{\perp} = ((F\varphi)^{-1}\mu a_2)^{\perp} = ((F\varphi)^{-1}M_2)^{\perp}$, where we have denoted $\mu a_2 \equiv M_2$.

(iii) If $\Sigma \in ObK_{\Sigma}^{\perp\perp}$, then obviously $\mathscr{L}(\Sigma, \perp)$ is a complete, atomistic, orthocomplemented lattice and $F\mathscr{L}(\Sigma, \perp) = \Sigma$.

(iv) Proceeding in the same way as in the proof of the assertion 3 of the theorem 5 we can show that $F: Mor(\mathscr{L}_2, \mathscr{L}_1) \to Mor(F\mathscr{L}_1, F\mathscr{L}_2)$ is $1-1$. If $T: F\mathscr{L}_1 \to F\mathscr{L}_2$ is a morphism in $K_{\Sigma}^{\perp\perp}$, then defining φ in the same way as in the proof of the assertion 1 of the Theorem 5 we can easily see that $F\varphi = T$.

Let us pass now to the discussion of products and coproducts in the introduced categories.

Theorem 7. (i) *In the category $K_\wedge$ there exists a product, i.e. by definition, for any indexed family $\mathscr{L}_i$ of objects of $K_\wedge$ there is an object denoted by $\prod_i \mathscr{L}_i$ and a family of morphisms $\pi_k : \prod_i \mathscr{L}_i \to \mathscr{L}_k$, such that for any object $\mathscr{L}$ and any family of morphisms $\phi_i : \mathscr{L} \to \mathscr{L}_i$ there exists a unique morphism $\phi : \mathscr{L} \to \prod_i \mathscr{L}_i$ which makes the following diagram commutative:*

$$\begin{tikzcd} \mathscr{L} \arrow[r, "\Phi"] \arrow[dr, "\phi_k"'] & \prod_i \mathscr{L}_i \arrow[d, "\pi_k"] \\ & \mathscr{L}_k \end{tikzcd} \tag{15}$$

(ii) *If $\mathscr{L}_i \in ObK^o_\wedge$ is a family of objects of the subcategory $K^o_\wedge$, then $\prod_i \mathscr{L}_i \in ObK^o_\wedge$. Moreover, if $\mathscr{L}_i$ are weakly modular and/or satisfy the covering law, so does $\prod_i \mathscr{L}_i$.*

Proof. (i) Define the set:

$$\prod_i \mathscr{L}_i \equiv \{(a_i) \mid a_i \in \mathscr{L}\ \forall i\},$$

where (a_i) denotes an indexed family of elements, equipped with the following relations:

$$(a_i) < (b_i) \Leftrightarrow a_i < b_i\ \forall i$$
$$(a_i) \perp (b_i) \Leftrightarrow a_i \perp b_i\ \forall i$$

$\prod_i \mathscr{L}_i$ is a complete, atomistic lattice with orthogonality relation, such that

$$\bigwedge_k (a_i^k) = \Big(\bigwedge_k a_i^k\Big), \qquad \bigvee_k (a_i^k) = \Big(\bigvee_k a_i^k\Big)$$

and with

$$\Sigma = \{(\mathbf{0}_{i, i \neq k}, p_k) \mid p_k \in \Sigma_k\}$$

as its set of atoms. Mappings

$$\pi_k : \prod_i \mathscr{L}_i \to \mathscr{L}_k, \qquad \pi_k(a_i) \equiv a_k$$

are morphisms. Now, if $\phi_i : \mathscr{L} \to \mathscr{L}_i$ is any family of morphisms, then the mapping

$$\phi : \mathscr{L} \to \prod_i \mathscr{L}_i, \qquad \phi a \equiv (\phi_i a) \tag{16}$$

is a morphism in $K_\wedge$ which makes (15) commutative.
(ii) – see [4, p. 34].

Theorem 8. *In the category $K_\wedge$ there exists a coproduct, i.e. by definition, for any family of objects $\mathscr{L}_i \in ObK_\wedge$, there is an object denoted by $\coprod_i \mathscr{L}_i$ and a family of morphisms $i_k : \mathscr{L}_k \to \coprod_i \mathscr{L}_i$ such that for any object $\mathscr{L}$ and any family of morphisms $\varphi_i : \mathscr{L}_i \to \mathscr{L}$ there exists a unique morphism $\psi : \coprod_i \mathscr{L}_i \to \mathscr{L}$ which makes* *for any k the following diagram commutative:*

$$\begin{tikzcd} \mathscr{L} & & \coprod_i \mathscr{L}_i \arrow[ll, "\psi"'] \\ & \mathscr{L}_k \arrow[ul, "\varphi_k"] \arrow[ur, "i_k"'] & \end{tikzcd} \tag{17}$$

Proof. Let us define:

$$\coprod_i \mathscr{L}_i \equiv \{(a_i) \mid a_i \in \mathscr{L}, a_i \neq \mathbf{0}_i\} \cup \{\mathbf{0}\}$$

$$(a_i) < (b_i) \Leftrightarrow a_i < b_i \forall i \tag{17a}$$

$$(a_i) \perp (b_i) \Leftrightarrow \exists i_0 : a_{i_0} \perp b_{i_0}$$

$\coprod_i \mathscr{L}_i$ is a complete, atomistic lattice with orthogonality relation, such that

$$\bigwedge_k (a_i^k) = \begin{cases} \mathbf{0}, & \text{if } \exists i_0 : \bigwedge_k a_{i_0}^k = \mathbf{0}_{i_0}; \\ (\bigwedge_k a_i^k), & \text{otherwise.} \end{cases}$$

$$\bigvee_k (a_i^k) = \left(\bigvee_k a_i^k\right)$$

and $\Sigma = \{(p_i) \mid p_i \in \Sigma_i\}$ is the set of atoms in $\coprod_i \Sigma_i$. The mappings

$$i_k : \mathscr{L}_k \to \coprod_i \mathscr{L}_i, \qquad i_k a_k \equiv (\mathbf{1}_{i, i \neq k}, a_k)$$

are morphisms. Now, if $\mathscr{L}$ is an object and $\varphi_i : \mathscr{L}_i \to \mathscr{L}$ is a family of morphisms then the mapping $\psi : \coprod_i \mathscr{L}_i \to \mathscr{L}$ defined by

$$\psi(a_i) \equiv \bigwedge_i \varphi_i a_i \tag{18}$$

is a morphism which makes the diagram (16) commutative.

Contrary to the case of product, the coproduct defined above is not a coproduct for the subcategory $K_\wedge^o$.

Lemma 4. *Let $\mathscr{L}_1$, $\mathscr{L}_2 \in ObK_\wedge$. If $\mathscr{L}_1 \amalg \mathscr{L}_2$ is an orthocomplemental lattice, then either on $\mathscr{L}_1$ or on $\mathscr{L}_2$ the orthogonality relation is empty, i.e. no two elements are orthogonal, except* $\mathbf{0}$ *and* $\mathbf{1}$.

Proof. Let us suppose that $\mathscr{L}_1 \amalg \mathscr{L}_2$ is orthocomplemented. Then $(a_1, a_2) \perp (b_1, b_2) \Leftrightarrow (a_1, a_2) < (b_1, b_2)'$. Let us suppose, that there are $a_1, b_1 \neq \mathbf{0}_1$, $a_2, b_2 \neq \mathbf{0}_2$ and such that $a_1 \perp b_1$, $a_2 \perp b_2$. It follows, that $(\mathbf{1}_1, a_2) \perp (b_1, b_2)$ and $(a_1, \mathbf{1}_2) \perp (b_1, b_2)$. Hence

$$(\mathbf{1}_1, a_2) < (b_1, b_2)', \quad (a_1, \mathbf{1}_2) < (b_1, b_2)'$$

and

$$(\mathbf{1}_1, a_2) \vee (a_1, \mathbf{1}_2) = (\mathbf{1}_1, \mathbf{1}_2) < (b_1, b_2)'.$$

Therefore, either $b_1 = \mathbf{0}_1$ or $b_2 = \mathbf{0}_2$.

This lemma is a reformulation of an argument given originally by Aerts, cf [5]. The following two theorems are generalizations of the theorems also given by Aerts, [5].

Theorem 9. *Let $\mathscr{L}_i \in ObK_\wedge$. If $\coprod_i \mathscr{L}_i$ satisfies the covering law, then all $\mathscr{L}_i$ with the except of at least one are the lattices containing only two elements* **0** *and* **1**.

Proof. Let $\coprod_i \mathscr{L}_i$ satisfy the covering law and let us suppose that in the lattice $\mathscr{L}_k$, there exist two different atoms $p_k \neq q_k$. Let $p_i, q_i \in \Sigma_i$, $i \neq k$, be atoms in the lattices $\mathscr{L}_i$. Let us denote: $(a_i, a_k) \equiv (a_{i,i\neq k}, a_k)$. Since $p_k \neq q_k$, $(p_i, p_k) \neq (q_i, q_k)$, i.e. (p_i, p_k) and (q_i, q_k) are two different atoms of $\coprod_i \mathscr{L}_i$. From the covering law we have:

$$(p_i, p_k) \prec (p_i, p_k) \vee (q_i, q_k). \tag{$*$}$$

But

$$(p_i, p_k) < (p_i \vee q_i, p_k) < (p_i \vee q_i, p_k \vee q_k) < (p_i, p_k) \vee (q_i, q_k)$$

so, from (*)

$$(p_i, p_k) = (p_i \vee q_i, p_k) \quad \text{or} \quad (p_i \vee q_i, p_k) = (p_i \vee q_i, p_k \vee q_k)$$

i.e. $p_i = p_i \vee q_i \; \forall i \neq k$ or $p_k = p_k \vee q_k$. The second possibility is excluded by the assumption $p_k \neq q_k$. Consequently, $p_i = q_i \; \forall i \neq k$, what implies that each lattice $\mathscr{L}_i$, $i \neq k$, is of the form $\{\mathbf{0}, \mathbf{1}\}$.

Theorem 10. The coproduct $\coprod_i \mathscr{L}_i$ is a weakly modular lattice iff each $\mathscr{L}_i$ is.

Proof. Suppose, that each $\mathscr{L}_i$ is weakly modular, i.e. if $a_i < b_i$ then there is c_i such that $c_i \perp a_i$ and $c_i \vee a_i = b_i$, $a_i, b_i, c_i \in \mathscr{L}_i$. Now, if $(a_i), (b_i) \in \coprod_i \mathscr{L}_i$, and $(a_i) < (b_i)$, then $\forall_i \; a_i < b_i$ and it follows that $(c_i) \vee (a_i) = (b_i)$ where $(c_i) \perp (a_i)$. On the other hand, if $\coprod_i \mathscr{L}_i$ is weakly modular, then $a_k, b_k \in \mathscr{L}_k$, $a_k < b_k$, implies $(\mathbf{1}_{i,i\neq k}, a_k) > (\mathbf{1}_{i,i\neq k}, b_k)$ and $(\mathbf{1}_{i,i\neq k}, b_k) = (c_i) \vee (\mathbf{1}_{i,i\neq k}, a_k)$, where $(c_i) \perp (\mathbf{1}_i, a_k)$. It follows, that $c_k \perp a_k$ and $b_k = c_k \vee a_k$.

The following theorems give an account of product and coproduct in the category K_Σ.

Theorem 11. *In the category K_Σ there exists a product, i.e. for any family of objects $\Sigma_i \in ObK_\Sigma$ there is an object $\prod_i \Sigma_i$ and a family of morphisms $\pi_k : \prod_i \Sigma_i \to \Sigma_k$, such that for any $\Sigma \in ObK_\Sigma$ and any family of morphisms $T_i : \Sigma \to \Sigma_i$ there exists a morphism $T : \Sigma \to \prod_i \Sigma_i$ which makes the following diagram commutative for every k:*

$$\begin{tikzcd} \Sigma \arrow[rr, "T"] \arrow[dr, "T_k"'] & & \prod_i \Sigma_i \arrow[dl, "\pi_k"] \\ & \Sigma_k & \end{tikzcd} \tag{19}$$

Proof. Let us define:

$$\prod_i \Sigma_i \equiv \{(p_i) \mid p_i \in \Sigma_i \forall i\}$$

$$(p_i) \perp (q_i) \Leftrightarrow \exists i_0 : p_{i_0} \perp q_{i_0} \tag{20}$$

We distinguish the family of subsets $\mathscr{L}(\prod_i \Sigma_i)$ of $\prod_i \Sigma_i$ in the following way. Let $\pi_i : \prod_i \Sigma_i \to \Sigma_i$ be the following mappings: $\pi_i(p_i) \equiv p_i$. We define

$$\mathscr{L}\left(\prod_i \Sigma_i\right) \equiv \left\{M \subset \prod_i \Sigma_i \mid M = \bigcap_i \pi_i^{-1} M_i,\ M_i \in \mathscr{L}(\Sigma_i)\right\}. \tag{21}$$

It is now easy to check that mappings π_i are morphisms, that $\prod_i \Sigma_i$ with the family of subsets distinguished above satisfies the conditions (4), and that the morphism which makes the diagram (18) commutative is:

$$Tp \equiv (T_i p). \tag{22}$$

For the sake of completeness we shall also state the following theorem:

Theorem 12. *In the category K_Σ there exists a coproduct, i.e. for any family of objects $\Sigma_i \in Ob K_\Sigma$ there exists on object denoted by $\coprod_i \Sigma_i$ and a family of morphisms $i_k : \Sigma_k \to \coprod_i \Sigma_i$ such that for any family of morphisms $T_i : \Sigma_i \to \Sigma$ there exists a morphism $T : \coprod_i \Sigma_i \to \Sigma$ which makes the following diagram commutative:*

$$\begin{tikzcd} \Sigma & & \coprod_i \Sigma_i \arrow[ll, "T"'] \\ & \Sigma_k \arrow[ul, "T_k"] \arrow[ur, "i_k"'] & \end{tikzcd} \tag{23}$$

Obviously, by applying the functor F (6) to the diagrams (15) and (17) one can pass to the diagrams (23) and (19) respectively. From the Lemma 4, we know, that the product of orthocomplemented lattices cannot be an orthocomplemented lattice. The same can be expressed in terms of coproduct of respective atom spaces.

Lemma 5. *Let $\pi_k : \prod_i \Sigma_i \to \Sigma_k$ be the same as defined in the proof of the Theorem* 11 *and let $M_k \subset \Sigma_k$. Then,*

$$\pi_k^{-1} M_k^\perp = (\pi_k^{-1} M_k)^\perp.$$

Proof. Let $p_k \in \Sigma_k$ and let $(r_i) \in \prod_i \Sigma_i$, be such that $(r_i) \in \pi_k^{-1} p_k^\perp$. Then $(r_i) \perp (q_i)$ for every $(q_i) \in \prod_i \Sigma_i$ and $\pi_k(q_i) = p_k$ i.e. $(r_i) \in (\pi_k^{-1} p_k)^\perp$. On the other hand, if $(r_i) \in (\pi_k^{-1} p)^\perp$ then $(r_i) \perp (q_i)$ for every (q_i) such that $\pi_k(q_i) = p_k$ whereas all the other components of (q_i) are arbitrary. If r_k was not orthogonal to p_k, then it would follow that in particular $(r_{i,i\neq k}, r_k) \perp (r_{i,i\neq k}, p_k)$ what is impossible. Therefore for any $p_k \in \Sigma_k$, $\pi_k^{-1} p_k^\perp = (\pi_k^{-1} p_k)^\perp$. Since for $M_i \subset \Sigma_i$ we have $M_i^\perp = \bigcap_{M_i} p_i^\perp$, the conclusion follows.

Theorem 13. *Let $\Sigma_i \in ObK_\Sigma^{\perp\perp}$, and let $\mathscr{L}(\prod_i \Sigma_i)$ be the family of subsets of $\prod_i \Sigma_i$ defined as in* (20) *and let $\mathscr{L}(\prod_i \Sigma_i, \perp)$ be a family of subsets defined by* (14). *Then $\mathscr{L}(\prod_i \Sigma_i) \subset \mathscr{L}(\prod_i \Sigma_i, \perp)$ and this is a proper subset.*

Proof. Let $M \in \mathscr{L}(\prod_i \Sigma_i)$. By definition $M = \bigcap_i \pi_i^{-1} M_i$, where $M_i \in \mathscr{L}(\Sigma_i) \equiv \mathscr{L}(\Sigma_i, \perp)$, and $M_i^{\perp\perp} = M_i$. From the previous lemma it follows that $M^{\perp\perp} = (\bigcap_i \pi_i^{-1} M_i)^{\perp\perp} = \bigcap_i \pi_i^{-1} M_i$, i.e. $M \in \mathscr{L}(\prod_i \Sigma_i, \perp)$. $\mathscr{L}(\prod \Sigma_i, \perp)$ is an orthocomplemented lattice with $M \to M^\perp$ as an orthcomplementation, what implies that in particular $(\bigcap_i \pi_i^{-1} M_i)^\perp \in \mathscr{L}(\prod_i \Sigma_i, \perp)$. However, from the definition of orthogonality in $\prod_i \Sigma_i$ (20) it immediately follows, that

$$\left(\bigcap \pi_i^{-1} M_i\right)^\perp = \bigcup_i \pi_i^{-1} M_i^\perp \tag{24}$$

what shows that $(\bigcap_i^{-1} M_i)^\perp$ is not an element of $\mathscr{L}(\prod_i \Sigma_i)$.

Corollary 4. *The product defined in the Theorem* 12 *is not a product for the subcategory $K_\Sigma^{\perp\perp}$, i.e. $\Sigma_i \in ObK_\Sigma^{\perp\perp}\ \forall i$ does not imply $\prod_i \Sigma_i \in ObK_\Sigma^{\perp\perp}$.*

However, it is still possible to define a product for the subcategory $K_\Sigma^{\perp\perp}$.

Theorem 14. *Let $\Sigma_i \in ObK_\Sigma^{\perp\perp} \forall i$, and let $\prod_i' \Sigma_i$ denote the set with the orthogonality relation* (20) *as defined in* (20) *but with the distinguished family of subsets*

$$\mathscr{L}\left(\prod_i \Sigma_i, \perp\right) \equiv \left\{M \subset \prod_i \Sigma_i \mid M^{\perp\perp} = M\right\}$$

$\prod_i' \Sigma_i$ is a product in the category $K_\Sigma^{\perp\perp}$.

Proof. By definition $\prod_i' \Sigma_i \in ObK_\Sigma^{\perp\perp}$. We have to check the same condition as in the Theorem 11, with the only difference that all the morphisms involved instead of (5) should now satisfy the condition (12). Let π_k be the same as in the proof of Theorem 12 and let for a family of morphisms $T_k : \Sigma \to \Sigma_k$, the mapping $T : \Sigma \to \prod_i' \Sigma_i$ be defined as in (22). From the Lemma 4 it follows that π_k are morphisms of $K_\Sigma^{\perp\perp}$. We shall show that T is also a morphism. From definitions and (12) we have: $T^{-1}(q_i)^\perp = [\bigcap_i T_i^{-1} q_i]^\perp$ for any $(p_i) \in \prod_i' \Sigma_i$. Therefore for any $M \in \mathscr{L}(\prod_i \Sigma_i, \perp)$ we have: $T^{-1} M^\perp = \bigcap_M [\bigcap_i T^{-1} q_i]^\perp = [\bigcup_M \bigcap_i T_i^{-1} q_i]^\perp = [T^{-1} M]^\perp$, i.e. T satisfies the condition (12). Obviously, T makes the diagram (19) commutative.

The lattice $\mathscr{L}(\prod_i \Sigma_i, \perp)$ is complete, atomistic and orthocomplemented. The following theorem, being the generalization of theorem stated originally by Aerts, [2], gives more insight into its structure.

Theorem 15. *If the lattice $\mathscr{L}(\prod_i \Sigma_i \perp)$ is weakly modular or satisfy the covering law, then the orthogonality relation in at least one of Σ_i is trivial, i.e. every two different elements of Σ_i are orthogonal.*

For the proof the following two lemmas are needed.

Lemma 6. *Let $\Sigma \in ObK_{\Sigma}^{\perp\perp}$. If the lattice $\mathscr{L}(\Sigma, \perp)$ is weakly modular or satisfy the covering law then for any two elements $p, q \in \Sigma$*

$$(p \neq q, \{p, q\}^{\perp\perp} = \{p, q\}) \Rightarrow p \perp q$$

Proof. The condition $\{p, q\}^{\perp\perp} = \{p, q\}$ reads: $p \vee q = p \cup q$. If follows

$$\mathbf{0} < p^{\perp} \cap (p \vee q) = p^{\perp} \cap q < q.$$

Hence, either $p^{\perp} \cap q = \mathbf{0}$ or $p \perp q$. Let us suppose that the lattice is weakly modular. Then

$$p \vee q = [p^{\perp} \cap (p \vee q)] \vee p = (p^{\perp} \cap q) \vee p.$$

The case $p^{\perp} \cap q = \mathbf{0}$ is excluded, since we would have $p \vee q = p$, i.e. $p = q$ what is impossible by the assumption. Let us suppose that the lattice satisfies the covering law. Then again the case $p^{\perp} \cap q = \mathbf{0}$ is excluded. Indeed, if $p^{\perp} \cap (p \cup q) = \mathbf{0}$ then $p \vee (p^{\perp} \cap q^{\perp}) = \mathbf{1}$ and since $p \cap (p^{\perp} \cap q^{\perp}) = \mathbf{0}$ it follows from the covering law, that

$$p^{\perp} \cap q^{\perp} < p \vee (p^{\perp} \cap q^{\perp}) = \mathbf{1}.$$

However, since $p^{\perp} \cap q^{\perp} < p^{\perp} < \mathbf{1}$ it follows that $p^{\perp} \cap q^{\perp} = p^{\perp}$, i.e. $p = q$, what by the assumption is impossible.

Lemma 7. *If $(p_i), (q_i) \in \prod_i' \Sigma_i$ and $p_i \neq q_i\ \forall_i$, then*

$$\{(p_i), (q_i)\}^{\perp\perp} = \{(p_i), (q_i)\}$$

Proof.

$$\{(p_i), (q_i)\} = \bigcap_i \pi_i^{-1} p_i \cup \bigcap_j \pi_j^{-1} q_j = \bigcap_{i,j} (\pi_i^{-1} p_i \cup \pi_j^{-1} q_j) = \bigcap_{i,j} (\pi_i^{-1} p_i \cup \pi_j^{-1} q_j),$$

where we have used the formula (2′) from [6, p. 119]. But $\pi_i^{-1} p_i \cup \pi_j^{-1} q_j \in \mathscr{L}(\prod_i \Sigma_i\, \perp)$, intersections belongs to $\mathscr{L}(\prod_i \Sigma_i, \perp)$ and consequently $\{(p_i), (q_i)\} \in \mathscr{L}_i(\prod_i \Sigma_i, \perp)$.

Proof of Theorem 15. Let us suppose that in all Σ_i, $i \neq k$, there are p_i, q_i such that $p_i \neq q_i$, and p_i is not orthogonal to q_i. Let $p_k, q_k \in \Sigma_k$ and let $p_k \neq q_k$. From the Lemma 7 it follows that $\{(p_i), (q_i)\}^{\perp\perp} = \{(p_i), (q_i)\}$, what by the Lemma 6 implies $(p_i) \perp (q_i)$, hence by the assumption it follows that $p_k \perp q_k$.

Let us end this section with a short summary of the results. We have studied a category of complete, atomistic lattices with the relation $\perp$. Within this category there exist product and coproduct for any family of lattices.

If the lattices are in addition orthocomplemented, weakly modular and satisfy the covering law, so does their product. This is not the case for the coproduct. In the considered category the coproduct of nontrivial orthocomplemented lattices is neither orthocomplemented nor does satisfy the covering law.

These facts can be stated dually for the category of atom spaces, i.e. sets with distinguished family of subsets and endowed with symmetric and irreflexive relation $\perp$.

Obviously, via the respective functor, the coproduct of lattices can be transformed into the product of their atom spaces. An orthocomplemented lattice corresponds to the set with the set with the family of all its biorthogonal subsets. Although the product of such atom spaces corresponding to the coproduct of respective lattices, does not have the structure of biorthogonal subsets of a set, one can still study the lattice of biorthogonal subsets of this set. The lattice of these subsets being naturally orthocomplemented is neither weakly modular nor does satisfy the covering law, except the case when at least in one of the factors in the coproduct the orthogonality relation is trivial. Clearly, this lattice is not a coproduct of lattices. It is in fact bigger then the coproduct. Indeed, one can consider the morphism

$$\Lambda: \coprod_i \mathscr{L}(\Sigma_i, \perp) \to \mathscr{L}\Big(\prod_i \Sigma_i, \perp\Big) \tag{25}$$
$$(M_i) \mapsto \Lambda(M_i) \equiv (M_i).$$

Of course, Λ is injective, but due to the Theorem 13 it is not surjective. The coproduct of lattices has been introduced on a physical ground for the two lattices by Aerts in [5], where it was called a 'tensor product'. In [2] he has also introduced in the connection with the description of two separated physical entities the lattice, which has the structure $\mathscr{L}(\Sigma_1 \times \Sigma_2, \perp)$ and called it a 'separated product' of the lattices $\mathscr{L}(\Sigma_1, \perp)$ and $\mathscr{L}(\Sigma_2, \perp)$. This is a particular case of our $\mathscr{L}(\prod_i \Sigma_i, \perp)$.

4. The mathematical model of an evolution of a physical system

Using the mathematical framework exhibited in the preceding section, we shall make now the description of a deterministic evolution as put forward in the Section 2 more complete and precize. In what follows we shall be considering property lattices $\mathscr{L}_t$ which describe the physical system under consideration mainly as objects of the category $K_\wedge$ and the respective state spaces Σ_t mainly as objects of the category K_Σ, where for every Σ_t, $\mathscr{L}(\Sigma_t) \equiv \{\mu a_t \mid a_t \in \mathscr{L}_t\}$, where μ is a Cartan mapping. From the Theorem 2 we see, that mappings $\varphi_{t,s}$, $t \le s$, which define the evolution are morphisms in the category $K_\wedge$. Therefore, from the Corollaries 2 and 3, we obtain immediately:

Corollary 5. *To give a deterministic evolution of a physical system in terms of the family of mappings $\varphi_{t,s}: \mathscr{L}_s \to \mathscr{L}_t$, $t \le s$ is equivalent to give it in terms of the (unique) family of bijective mappings $T_{s,t}: \Sigma_t \to \Sigma_s$, $t \le s$, which satisfy*:

(i) $T_{s,t}^{-1} \mu a_s = \mu(\varphi_{t,s} a_s)$ *for any* $a_s \in \mathscr{L}_s$, *or equivalently*:
$p_t < \varphi_{t,s} a_s \Leftrightarrow T_{s,t} p_t < a_a$

where $p_t \in \Sigma_t$;

(ii) $T_{s,t}p_t \perp T_{s,t}q_t \Rightarrow p_t \perp q_t$

(iii) $T_{s,t'}T_{t',t} = T_{s,t}$, $t \leq t' \leq s$

In the case when the property lattices $\mathscr{L}_t$ of the system are of the form $\mathscr{L}_t = \mathscr{L}(\Sigma_t, \perp)$, which is e.q. the case of a quantum entity, according to the Theorem 6 (cf. (12a)), the condition (i) takes the form:

(i)′ for any $M_s \in \mathscr{L}(\Sigma_s, \perp)$, $T_{s,t}^{-1}M_s = (T_{s,t}^{-1}M_s)^{\perp\perp}$

The interpretation of the mappings $T_{s,t}$ follows directly from (i): the system is at the moment t in the state p_t iff it will be in the sate $T_{s,t}p_t$ at the moment $s \geq t$. It is the functor F(cf. (6)) or the functor G (cf (9)) which enables one to pass from the property lattice description of an evolution to the state space description and vice versa.

Lemma 8. *Let* p_t, $q_t \in \Sigma_t$. *Then* $p_t \perp q_t \Leftrightarrow$ *there exists* $s_o \geq t$ *such that* $T_{s_0,t}p_t \perp T_{s_0,t}q_t$.

Proof. $\Rightarrow$ is trivial; $\Leftarrow$ is (ii) of the Corollary 4.

This lemma clearly indicates, that the orthogonality relation is strictly related to the dynamical properties of a system. In fact any conceivable evolution defines certain orthogonality relation. In particular, one can consider an orthogonality relation on a property lattice of the system defined by the evolution satisfying:

$$p_t \perp q_t \Leftrightarrow T_{s,t}p_t \perp T_{s,t}q_t.$$

As we shall see, (Section 5) this is the case of a reversible evolution. Such an evolution had been postulated to be described by symmetries of the property lattice, [4]. When the orthogonality relation for a given system once has been defined by presupposing such type of evolution, that is all the states-spaces Σ_t has a fixed orthogonality relation, then of course any other evolution given by $T_{s,t}$ need not satisfy longer the condition (ii) of the Corollary 4.

Let us consider certain moment of time t and the morphisms $T_{s,t}: \Sigma_t \to \Sigma_s$, $t \leq s$. Since $T_{s,t}$ are morphisms in the category K_Σ, according to the Theorem 11 the morphism $\mathscr{T}_t: \Sigma_t \to \prod_{s \geq t} \Sigma_s$ given by

$$\mathscr{T}_t p_t \equiv (T_{s,t}p_t)$$

is a unique morphism which makes the following diagram commutative:

$$\begin{tikzcd} \Sigma_t \arrow[rr, "\mathscr{T}_t"] \arrow[dr, "T_{r,t}"'] & & \prod_{s \geq t} \Sigma_s \\ & \Sigma_r \arrow[ur, "\pi_r"'] & \end{tikzcd}$$

The interpretation of $\prod_{s \geq t} \Sigma_s$ is natural: it is the set of all *a priori* possible trajectories at the moment t for the system under consideration. Let us denote:

$$\tilde{\Sigma}_t \equiv \mathscr{T}_t \Sigma_t = \{(T_{s,t}p_t) \mid p_t \in \Sigma_t\} \subset \prod_{s \geq t} \Sigma_s.$$

Since the morphism $\mathcal{T}_t$ is manifestly injective, $\tilde{\Sigma}_t$ is itself an object of K_Σ (subobject of $\prod_{s \geq t} \Sigma_s$) such that

$$\mathcal{L}(\tilde{\Sigma}_t) = \left\{ M \cap \tilde{\Sigma}_t \mid M \in \mathcal{L}\left(\prod_{s \geq t} \Sigma_s \right) \right\}. \tag{26}$$

Obviously, the morphism $\mathcal{T}_t$ defines an isomorphism between Σ_t and $\tilde{\Sigma}_t$. Thus, every deterministic dynamics determines an object–subset of the set of all *a priori* possible trajectories of the system–which, when considered within the respective category, is isomorphic with the state space of the system. It turns out, that the converse is also true.

Theorem 16. *Let $\tilde{\Sigma}_t$ be a subobject of $\prod_{s \geq t} \Sigma_t$ isomorphic with Σ_t. There exists a unique family of morphisms $T_{s,t} : \Sigma_t \to \Sigma_s$, $s \geq t$ such that for any $p_t \in \Sigma_t$ the injection $\mathcal{T}_t : \Sigma_t \to \prod_{s \geq t} \Sigma_s$ may be written as*

$$\mathcal{T}_t p_t = (T_{s,t} p_t).$$

Proof. Let us define $T_{s,t} : \Sigma_t \to \Sigma_s$

$$T_{s,t} p_t \equiv \pi_s \mathcal{T}_t p_t.$$

Since this is a composition of morphisms it is a morphism. According to the definition of product, there exists a unique morphism $\mathcal{T}'_t : \Sigma_t \to \prod_{s \geq t} \Sigma_s$ such that for every $s \geq t$, $\pi_s \mathcal{T}'_t = T_{s,t}$, cf. (22). It follows that for any $p_t \in \Sigma_t : \pi_s(T_{s,t}) = \pi_s \mathcal{T}_t p_t$ and the conclusion follows.

The following definition is therefore natural.

Definition 4. A subobject $\tilde{\Sigma}_t$ or $\prod_{s \geq t} \Sigma_s$ is called a *trajectory space* of a physical system at the moment t iff it is isomorphic with the state space Σ_t of the system.

Since a trajectory space $\tilde{\Sigma}_t$ itself is not an element of $\mathcal{L}(\prod_{s \geq t} \Sigma_s)$, $\mathcal{L}(\tilde{\Sigma}_t)$ is not a sublattice of it. The following theorem gives some insight into the nature of the lattice $\mathcal{L}(\Sigma_t)$.

Theorem 17. The lattice $\mathcal{L}(\tilde{\Sigma}_t)$ is isomorphic with a certain sublattice $\tilde{\mathcal{L}}_t$ of $\coprod_{s \geq t} \mathcal{L}_s$.

Proof. According to the (ii) of the Corollary 2, mappings $\varphi_{t,s} : \mathcal{L}_s \to \mathcal{L}_t$ are bijective. It follows that the inverse mappings $\varphi_{s,t}^{-1}$ preserve **1** and lattice join and meet for any family of elements. Moreover, from the Corollary 5 (i) we immediately see that $\varphi_{t,s|\Sigma_t}^{-1} = T_{s,t}$. Let us define:

$$\tilde{\mathcal{L}}_t \equiv \{ (\varphi_{t,s}^{-1} a_t) \mid a_t \in \mathcal{L}_t \} \subset \coprod_{s \geq t} \mathcal{L}_s. \tag{27}$$

Obviously, $\tilde{\mathcal{L}}_t$ is a sublattice of $\coprod_{s \geq t} \mathcal{L}_s$. Taking into account (26) we see that $\mu(\varphi_{t,s}^{-1} \cap \Sigma_t) \in \mathcal{L}(\tilde{\Sigma}_t)$ for any $a_t \in \Sigma_t$. On the other hand, if $M \in \mathcal{L}(\prod_{s \geq t} \Sigma_s)$ then according to (21) $M = \mu(a_s)$, where $(a_s) \in \coprod_{s \geq t} \mathscr{L}_s$, and if $M \cap \tilde{\Sigma}_t \neq \emptyset$ then for every s, $\mu a_s = T_{s,t}\mu a_t \equiv \{T_{s,t}p_t \mid p_t \in \mu a_t\}$. Consequently, for every s, $a_s = \varphi_{t,s}^{-1} a_t$. Therefore,

$$\mathscr{L}(\tilde{\Sigma}_t) = \{\mu(\varphi_{t,s}^{-1} a_t) \cap \tilde{\Sigma}_t \mid a_t \in \mathscr{L}_t\} \tag{28}$$

Clearly, the mapping:

$$\tilde{\mathscr{L}}_t \ni (\varphi_{t,s}^{-1}) \mapsto \mu(\varphi_{t,s}^{-1} a_t) \cap \tilde{\Sigma}_t \in \mathscr{L}(\tilde{\Sigma}_t) \tag{29}$$

is an isomorphism.

From the preceding section (cf. Lemma 4) we know that there is no orthocomplementation on the coproduct of orthocomplemented lattices which would be compatible with the orthogonality (17a) defined on it, i.e. such that (10) would be satisfied. This is however possible for the sublattice $\tilde{\mathscr{L}}_t$.

Theorem 18. *Let the lattices $\mathscr{L}_s$, $s \geq t$, be orthocomplemented. The sublattice $\tilde{\mathscr{L}}_t$* (27) *is orthocomplemented such that the relation* (10) *is satisfied if and only if the morphisms $\varphi_{t,s}^{-1}$ preserve orthocomplementation.*

Proof. $\Rightarrow$ Let $(\varphi_{t,s}^{-1} a_t) \mapsto (\varphi_{t,s}^{-1} a_t)'$ be an orthocomplementation on $\tilde{\mathscr{L}}_t$. Using the Cartan mapping and definitions we have:

$$\mu(\varphi_{t,s}^{-1} a_t)' = [\mu(\varphi_{t,s}^{-1} a_t)]' = \{(T_{s,t}p_t) \in \tilde{\Sigma}_t \mid T_{s,t}p_t < \varphi_{t,s}^{-1} a_t' \forall s\} = \mu(\varphi_{t,s}^{-1} a_t').$$

On the other hand we have also:

$$\mu(\varphi_{t,s}^{-1} a_t)' = \mu((\varphi_{t,s}^{-1} a_t)')$$

where $((\varphi_{t,s}^{-1} a_t)') \in \coprod_{s \geq t} \mathscr{L}_s$. The last equality follows from the definition of orthogonality in $\coprod_{s \geq t} \mathscr{L}_s$ and from the fact that for any s, $[\mu a_t]^{\perp} = T_{s,t}^{-1} \mu[\varphi_{t,s}^{-1} a_t]'$. Thus, $\mu(\varphi_{t,s}^{-1} a_t') = \mu((\varphi_{t,s}^{-1} a_t)')$, i.e. for every s and any $a_t \in \mathscr{L}_t$, $\varphi_{t,s}^{-1} a_t' = [\varphi_{t,s}^{-1} a_t]'$.

$\Leftarrow$ If the mappings $\varphi_{t,s}^{-1}$ preserve orthocomplementation, then obviously $\tilde{\mathscr{L}}_t \ni (\varphi_{t,s}^{-1} a_t) \mapsto (\varphi_{t,s}^{-1} a_t)' \equiv (\varphi_{t,s}^{-1} a_t') \in \tilde{\mathscr{L}}_t$ is an orthocomplementation. Moreover, $(\varphi_{s,t}^{-1} a_t) \perp (\varphi_{t,s}^{-1} b_t)$ iff there exists s_0 such that $\varphi_{t,s_0}^{-1} a_t \perp \varphi_{t,s_0}^{-1} b_t$, what by the assumptions is equivalent to $a_t < b_t'$ and consequently to $\varphi_{t,s}^{-1} a_t < \varphi_{t,s}^{-1} b_t'$ for every s, i.e. $(\varphi_{t,s}^{-1} a_t) < (\varphi_{t,s}^{-1} b_t)'$.

5. The reversible evolution

The notion of reversibility which we shall adopt here is suggested by thermodynamics. In thermodynamics an evolution of a system from the state A to the state B is said to be reversible if the system can evolve also from the state B to the state A following exactly all the intermediary states but in the reversed order. This notion of reversibility rests on two presuppositions. The first is that it presupposes the very possibility of meaningful speaking about "the same" state of the physical system at different moments of time. We shall accomplish this by adopting the following assumption.

Assumption 2. For each $t \in \mathbf{R}$, $\Sigma_t \equiv \Sigma$.

It should be therefore kept in mind that from now the product of state spaces is a product of the copies of the same object.

The second presupposition is connected with the notion of the "reversed evolution". It is assumed that states of the system when evolving "backward" from B to A are in some sense "the same" as respective states of this system evolving from A to B. Such identification is carried out by a certain symmetry R of the state space Σ. That is, if the system evolves from the state $p \in \Sigma$ to the state $q \in \Sigma$ along certain trajectory, then the evolution from the state Rq to the state Rp along the image of this trajectory under the symmetry R, is conceived as the 'reversed evolution'. The 'velocity reversal' transformation in classical mechanics, i.e. the transformation which changes the sign of all generalized momenta, leaving generalized coordinates invariant is an example of such symmetry, cf [7].

Assumption 3. There is an automorphism $R:\Sigma \to \Sigma$ satisfying the condition $RR = id$ which enables to define the 'reversed evolution'.

The notion of such a symmetry is prior to the notion of reversibility. It expresses our idea of what the 'reversed evolution' could be, but the question whether such a reversed evolution can occur or not is a question of dynamics of the system under consideration.

It is also clear, that in order to discuss the reversibility one has to assume that the evolution under consideration is not bounded in time, that is together with the set $\prod_{s \geq t} \Sigma_s$ of all *a priori* possible trajectories which has the origin at the moment t, we have to take into account the set $\prod_R$ which is interpreted as the set of all *a priori* possible trajectories for the evolution which extends in time from $-\infty$ to $+\infty$.

Let

$$\prod_R \Sigma_s \ni (p_s) \mapsto c_t(p_s) \equiv (p_s)_{s \geq t} \in \prod_{s \geq t} \Sigma_s.$$

Clearly, c_t is a morphism of the category K_Σ.

Definition 5. An element $(p_s) \in \prod_R \Sigma_s$ is called a *dynamically admissible trajectory* iff for every t, $c_t(p_s) \in \tilde{\Sigma}_t$, where $\tilde{\Sigma}_t$ is a trajectory space for the given physical system at the moment t (cf. Definition 3).

Let $\Sigma^\infty \subset \prod_R \Sigma_s$ be the set of all dynamically admissible trajectories and let us put for any $s < t$, $T_{s,t} \equiv T_{t,s}^{-1}$. It follows that

$$\Sigma^\infty = \bigcap_t c_t^{-1} \tilde{\Sigma}_t = \left\{ (T_{s,t_0} p) \in \prod_R \Sigma_t \mid p \in \Sigma \right\}$$

for any t_0. The mapping:

$$\Phi_t : \Sigma \to \Sigma^\infty, \ \Phi_t p \equiv (T_{s,t} p) \tag{30}$$

is not a morphism (in the sense of the category K_Σ) in general. The reason is that the condition (5)(i) need not be satisfied in a general case. It is however, when the evolution preserves orthogonality of states.

Theorem 19. *The mappings Φ_t (30) are isomorphisms of Σ onto Σ^∞ iff for any s*

$$p \perp q \Rightarrow T_{s,t}p \perp T_{s,t}q. \tag{31}$$

Proof. From (30) it follows that Φ_t is bijective and manifestly it satisfies the condition (5)(i). Therefore the necessary and sufficient condition for it to be a morphism (in fact an isomorphism, since obviously $p \perp q \Rightarrow \Phi_t p \perp \Phi_t t$) is

$$\Phi_t p \perp \Phi_t q \Rightarrow p \perp q. \tag{32}$$

It is clear from the definitions that if (31) is satisfied for any s, then (32) holds. Let us assume that (32) holds and let $p \perp q$. For $s \leq t$ (31) holds by the definition (30). From the assumptions it follows that $\Phi_t p \perp \Phi_t q$. But for any s, $\Phi_t p = (T_{r,t}p) = (T_{r,s}T_{s,t}p) = \Phi_s T_{s,t}p$ and similarily $\Phi_t q = \Phi_s T_{s,t}q$. Thus for any s, $\Phi_s T_{s,t}p \perp \Phi_s T_{s,t}q$ and by (32) $T_{s,t}p \perp T_{s,t}q$.

There is a natural interpretation of Φ_t: to every state $p \in \Sigma$ of the system it attaches a trajectory along which the system would evolve if it was in the state p at the moment t.

Let $\tilde{R}$ be the following mapping:

$$\prod_R \Sigma_s \ni (p_s) \mapsto \tilde{R}(p_s) \equiv (p_{-s}) \in \prod_R \Sigma_s \tag{33}$$

Therefore $\tilde{R}$ reverses the order of a given sequence of states. From the Assumption 3 it follows that for a given trajectory (p_s) the 'reversed' trajectory is (q_s), where $q_s = Rp_s$. Thus, a reversed trajectory is obtained by applying first the symmetry R to every state on the trajectory and then by applying the mapping $\tilde{R}$. For any trajectory (p_s) let us denote:

$$\mathcal{R}(p_s) \equiv \tilde{R}(Rp_s)$$

Definition 6. The deterministic evolution of a physical system is called reversible if whenever $(p_s) \in \prod_R \Sigma_s$ is a dynamically admissible trajectory, then also $\mathcal{R}(p_s)$ is a dynamically admissible trajectory.

We can formulate now the main theorem of this section.

Theorem 20. *If the evolution described by a family of morphisms $T_{s,t}: \Sigma \to \Sigma$, $s \geq t$, describes is reversible, then for any p, $q \in \Sigma$ and every $s \geq t$:*

(i) $RT_{s,t} = T_{-s,-t}R$

(ii) $p \perp q \Rightarrow T_{s,t}p \perp T_{s,t}q$ (34)

Proof. (i) For any state $p \in \Sigma$, $(T_{s,t}p)_{s \in R}$ is a dynamically admissible trajectory such that $\pi_t(T_{s,t}p_t) = p$. According to the definition 6, the trajectory

(q_s) where $q_{-s} = Rp_s = RT_{s,t}p$ is also a dynamically admissible trajectory, i.e. $(q_s) = (T_{s,t}q_t)$, what implies: $q_{-s} = T_{-s,-t}q_{-t}$. Hence: $RT_{s,t}p = T_{-s,-t}q_{-t} = T_{-s,-t}RT_{t,t}p = T_{-s,-t}Rp$.

(ii) From (i) it follows that $T_{s,t} = RT_{-s,-t}R$. Since by definition for $u \leq v$ we have $T_{u,v} = T^{-1}_{v,u}$ and from the Corollary 5 (ii) we know that for $u \leq v$ mappings $T_{u,v}$ preserve orthogonality, the conclusion follows.

Corollary 6. *When the evolution is reversible, then any mapping Φ_t (30) defines an isomorphism of Σ onto Σ^∞.*

If a property lattice of a physical system under consideration is the lattice of all biorthogonal subsets of its state space, i.e. is of the form $\mathscr{L}(\Sigma, \perp)$, then – in view of the above corollary $-\Sigma^\infty \in ObK_\Sigma^{\perp\perp}$ and consequently it may be considered as the set of atoms of an orthocomplemented lattice. In a particular case of the property lattice of a system being a Hilbert space lattice, i.e. lattice of closed subspaces of a complex Hilbert space $\mathscr{H}$, Σ^∞ may be considered also as the set of rays of some Hilbert space $\tilde{\mathscr{H}}$. Although the family of automorphisms $T_{s,t}$ on Σ (being the set of rays of $\mathscr{H}$) which describes a reversible evolution is not a group, one can still describe this evolution by a one parameter group of automorphisms of Σ^∞ (being in this particular case the set of rays of $\tilde{\mathscr{H}}$). We shall construct such a description in a general case of deterministic evolution of any physical system satisfying only the Axiom 1.

Theorem 21. *For a reversible evolution there exists a one parameter group $\{V_\tau\}_{\tau\in\mathbf{R}}$ of automorphisms of $\prod_{\mathbf{R}} \Sigma_s$ such that for any τ, V_τ maps Σ^∞ onto itself and for any t, $r \in \mathbf{R}$ the following diagram is commutative:*

$$\begin{array}{ccc} \prod_{\mathbf{R}} \Sigma_s & \xrightarrow{V_{r-t}} & \prod_{\mathbf{R}} \Sigma_s \\ \downarrow{\scriptstyle \pi_t} & & \downarrow{\scriptstyle \pi_t} \\ \Sigma & \xrightarrow{T_{r,t}} & \Sigma \end{array} \tag{35}$$

where $\pi_t(p_s) \equiv p_t$. Moreover, $\mathscr{R}$ is an automorphism of $\prod_{\mathbf{R}} \Sigma_s$ which maps Σ^∞ onto itself and such that $\mathscr{R}\mathscr{R} = id$ and:

$$\mathscr{R}V_{\tau|\Sigma^\infty} = V_{-\tau|\Sigma^\infty}\mathscr{R}. \tag{36}$$

Proof. For any $\tau \in \mathbf{R}$ let us define:

$$V_\tau(p_s) = (T_{s+\tau,s}p_s)$$

Clearly it is an automorphism of $\prod_{\mathbf{R}} \Sigma_s$, it makes (35) commutative, maps Σ^∞ onto itself and all the V_τ form a one parameter group. Also, the asserted features of $\mathscr{R}$ are immediate consequences of definitions and reversibility. Let us check (36). If $(p_s) \in \Sigma^\infty$ then we have: $V_{-\tau}\mathscr{R}(p_s) = (T_{s-\tau,s}RT_{-s,t}p_t) = (T_{s-\tau,s}T_{s,-t}Rp_t) = (T_{s-\tau,-t}Rp_t) = (RT_{s+\tau,t}p_t)$. On the other hand: $\mathscr{R}V_\tau(p_s) = \mathscr{R}(T_{s+\tau,t}p_t) = (RT_{-s+\tau,t}p_t)$.

Since V_τ map Σ^∞ onto itself their restrictions to Σ^∞ form a one parameter group of orthogonality preserving automorphisms of Σ^∞. The interpretation of these automorphisms follows directly from (35): V_τ shift the state of the system along the trajectory.

In the particular case mentioned just before the Theorem 21, when Σ^∞ is a set of rays of some Hilbert space $\tilde{\mathscr{H}}$, by applying the theorem of Wigner one can find a group of unitary transformations of $\tilde{\mathscr{H}}$ together with its generator which induce automorphisms V_τ. Therefore a non-homogeneous, reversible evolution can be described by means of a one parameter group of unitary operators on the Hilbert space $\tilde{H}$ or – owing to the Stone theorem – by its generator. Such construction had been described in details by Piron in [4, pp. 110–112]. The Hilbert space $\tilde{\mathscr{H}}$ called by Piron 'a large Hilbert space' is of course not the Hilbert space of the system. As it was made clear in [4], and as it follows from our present discussion, it is another Hilbert space related to the trajectories of the system.

When looking at the definition (30) of Φ_t and their interpretation, one can easily understand that in general there is no cannonical isomorphism between the trajectory space Σ^∞ and the state space Σ of the system. The point is that in a general case of a reversible evolution each of the isomorphisms Φ_t, $t \in \mathbf{R}$, attaches to a given state $p \in \Sigma$ a different trajectory $(T_{s,t}p) \in \Sigma^\infty$, for the law of evolution given by the family of morphisms $\{T_{s,t}\}_{s \in \mathbf{R}}$ depends on the moment t, i.e. is time dependent. This is however no longer the case when the evolution is homogeneous.

Definition 7. The evolution of a physical system defined by the family of morphisms $\{T_{s,t}\}_{s,t \in \mathbf{R}}$ is called *homogeneous* if whenever $s - t = s' - t'$ then $T_{s,t'}p = T_{s',t'}p$ for any $p \in \Sigma$.

Corollary 7. *A reversible, homogeneous evolution is described by a one parameter group $\{T_r\}_{r \in \mathbf{R}}$ of orthogonality preserving automorphisms of Σ.*

Explicitly, this group is defined by:

$$T_r \equiv T_{s-t} \equiv T_{s,t}.$$

Moreover, in this case

$$\Sigma^\infty = \{(T_r p)_{r \in \mathbf{R}} \mid p \in \Sigma\}$$

and for any t, $\Phi_t p = \Phi p \equiv (T_r p)_{r \in \mathbf{R}}$. Therefore, whatever is the moment of time, the trajectory along which the system being in a given initial state will evolve, is the same. Obviously, Φ defines a cannonical isomorphism between Σ and Σ^∞, and instead of (35) we can write the following commutative diagram showing the connection (which in this case is in fact trivial) between shifts on the trajectory space and dynamics defined on the state space:

$$\begin{array}{ccc} \Sigma^{\infty} & \xrightarrow{V_r} & \Sigma^{\infty} \\ \Phi \uparrow & & \uparrow \Phi \\ \Sigma & \xrightarrow{T_r} & \Sigma \end{array} \tag{37}$$

For a homogeneous evolution the first relation (34) reads:

$$RT_r = T_{-r}R \tag{38}$$

for any $r \in \mathbf{R}$.

In the particular case of quantum entity, i.e. when Σ is a set of rays of the Hilbert space, from the Corollary 7, by applying again Wigner and Stone theorems one can obtain the description of an evolution by means of a one parameter group of unitary operators on the Hilbert space of the entity and its generator and thus recover an abstract Schrödinger equation.

As we have already mentioned the symmetry R postulated by the assumption 3 is an abstract counterpart of 'velocity reversal' in classical mechanics. In [7] such mapping satisfying the condition (38) had been postulated in order to define reversibility of a dynamical system. In quantum mechanics the same role is played by Wigner's time reversal operator. Let us stress however, that in the present framework the condition (38) is not postulated but appears as a consquence of reversibility of the system.

6. Concluding remarks

In view of above discussion it is clear that from the axiomatic point of view, it is the notion of a deterministic, irreversible evolution which is more fundamental. As we he have shown, such general, deterministic evolution is characterized (within the state space description) by two conditions: 1) property is mapped onto the property; 2) two states orthogonal at the given moment of time was orthogonal before (Corollary 5). A homogeneous evolution of a quantum entity satisfying these two conditions has been first defined in [8]. We have shown there, that an example of such evolution is provided by a certain non-linear equation proposed by Gisin [9]. In a general form this equation reads:

$$\dot{x} = -iHx + k(\langle B \rangle_x - B)x$$

where H is a hamiltonian of the system, B is a self-adjoint operator and $\langle B \rangle_x = (x, Bx)/(x, x)$. Setting $B = H$ one obtains a model for the deterministic evolution during which the energy of a system decreases, (see [9], [10] for applications). In [11] Gisin proved that starting with the condition 1) mentioned above, by a suitable generalization of the Wigner theorem, one can recover the above equation.

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