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Fractals in Classical Mechanics

Martin C. Gutzwiller
IBM T.J.Watson Research Center, Yorktown Heights, NY 10598

The relations between classical and quantum mechanics are not understood sufficiently well as yet, when the classical behavior of the dynamical system is ergodic. The main unsolved task is to give a simple scheme for enumerating and effectively describing all the classical trajcetories, in particular the periodic orbits which enter into the trace formula. When the classical system shows hard chaos, i.e. all trajectories are exponentially unstable, a code can usually be found which is in 1-to-1, but only Hölder-continuous correspondence with the physical variables, such as the momentum and position coordinates. The latter define an invariant volume in phase space according to Liouville's theorem, but so does the code for the trajectories. The relation between these two measures is multifractal, i.e. the sets of points in phase space with a given Hölder-exponent α are a fractal set of dimension $f(\alpha)$. The presence of such fractals in systems without dissipation was found only very recently by the author and Benoit Mandelbrot; the spread of the distribution of fractal dimensions is much larger than in the examples known sofar.

Complex Systems in the Perspective of an Atomic/Molecular Physicist

Complex Systems have achieved general popularity in a very short time; does this seemingly global idea have any validity such as the variety of the topics in this conference would suggest? If so, the representatives of the various branches should be able to switch their field of activity with relative ease and impunity. People coming from entirely different backgrounds would assemble in this room, listen to the talks on the program, and then proceed to start working in a completely new direction.

Such an interdisciplinary free-for-all seems very unlikely to me. Contributions to science do not start at some lofty intellectual level where issues of global impact are debated; on the contrary, their origin is buried deep inside some particular individual who cannot help but concentrate on some special problem in a rather narrow field. Why the attention of one person gets focused in this way, and how the inner resources are mobilized, and then sustained for the long haul, is quite mysterious. Any interesting results have to be appreciated in the special context in which they were first found; and definitely not as one more symptom of a general philosophical trend.

I will try to tell you what special problem got me started some twenty years ago, and how I came to think recently about fractals in the context of classical mechanics. But first, I want to stake out my scientific territory by drawing some intellectual boundaries between various disciplines which are sometimes thrown together rather carelessly.

On my far left are a number of great mathematicians whose work I have come to admire: in particular Henri Poincaré, Jacques Hadamard, and George David Birkhoff; they are not far removed from what we have been hearing. Most remarkably, I am able to read and appreciate what they say, whereas that is no longer true for my contemporary colleagues in mathematics. Somewhat closer, still on my left, are the particle physicists who are almost equally estranged; while they worry about string theories for which there is not a thread of evidence, they still have to realize that their non-linear field-theories are probably full of chaotic features.

Finally, my neighbors to the left are the nuclear physicists; unfortunately, after more than 50 years of effort, they still don't know exactly which set of

mathematical equations applies to their field; but that very uncertainty has produced some ingenious new ideas such as Wigner's random Hamiltonians.

Many people do not realize that both nuclear and subnuclear physics have almost nothing to contribute to the understanding of the world in which we live. A rather limited data-bank from nuclear physics, maybe less than 1 Megabyte altogether, is all the information needed to understand the physics of atoms, molecules, condensed matter, out to biology and medicine, including even geophysics for a few billion years.

I want to start at the very bottom of this awesome and incredibly rich collection of phenomena: the study of the simplest atoms and small molecules, just a couple of nuclei and some electrons around them, with no more then electrostatic forces; a few degrees of freedom, and so small in size that quantum mechanics is required. A prime example which has been investigated with great success in recent years, is the humble hydrogen atom, one proton and one electron, under various unusual conditions such as a strong magnetic field or an intense low-frequency electric field.

When the boundary of my territory on the right is crossed, all kinds of new possibilities arise. The interaction of many particles leads to statistical mechanics, problems of noise and energy dissipation; not to mention reactions between molecules of moderate size, the conversion of sunlight in photosynthesis, the replication of genetic information, the propagation of pulses along nerves, and so on. Before we make any claims as to the universality of our insights on Complex Systems, we do well to look at this vast arena in front of us.

Connecting Classical and Quantum Mechanics

Niels Bohr succeeded in 1913 in explaining the optical spectrum of the hydrogen atom on the basis of ordinary Kepler orbits, plus some rather arbitrary rules involving Planck's quantum $\hbar$. These quantization conditions were superseded in 1925 by Heisenberg's matrix mechanics and Schrödinger's partial differential equation. But before that happened, Einstein in 1917 had turned the problem around, and asked the question: What type of a mechanical system is susceptible to Bohr's idea? Like all his papers, this one is a model of simplicity and clarity; it was written at the height of his career, when he was president of the German

Physical Society; and yet, I have found only one reference to it for the next 40 years! Obviously, its content was not appreciated by anybody.

Einstein pointed out that Bohr's rules were of no use, unless the system is integrable, i.e. it can be broken into as many independent subsystems as there are degrees of freedom. Not surprisingly, he gave this idea a more geometrical twist: the typical trajectory in phase space, instead of roaming freely over the $(2f-1)$-dimensional surface of constant energy E, remains glued to an $f-$ dimensional torus, where f is the number of degrees of freedom.

While some physicists have meanwhile become aware of the way Einstein views mechanics, and have learned of his elegant way to deal with the integrable systems, they have yet to read the last sentence of his paper: "If there exist fewer than f constants of motion, as for instance in the problem of three bodies according to Poincaré, then the quantization condition of Sommerfeld and Epstein fails even in the present, slightly generalized form." Unfortunately, I have to admit that I had been working for many years on this general problem, before I paid attention to Einstein's work to the extent of quoting him.

In the form of Hamilton and Jacobi, classical mechanics requires solving a first-order partial differential equation; the solution is easily reduced to a system of ordinary differential equations, i.e. Newton's equations of motion. Quantum mechanics raises the order of the partial differential equation from one to two; the new terms are proportional to Planck's constant $\hbar^2$. What happens when $\hbar$ turns out to be a small number in the natural units of the problem at hand, e.g. in an atom or a molecule near the ionization threshold?

In purely mathematical terms, the situation must be comparable to the flow of a fluid when its viscosity can be considered as small, or its Reynolds number as large. I have yet to see any serious effort to use the experience of hydrodynamics to get new insights into quantum mechanics, and vice versa. But then there is the spectacular case of Heisenberg himself whose Ph.D.-thesis, two years before discovering matrix mechanics, consisted mainly in applying to a problem in hydrodynamics what is now called the WKB-approximation in quantum mechanics. Did Heisenberg miss his chance in Complex Systems?

Nevertheless, it is absolutely intolerable that we are unable to establish a working link between classical and quantum mechanics, in those cases where the classical dynamical system is ergodic, to use the old-fashioned word, or equivalently, chaotic in the more dramatic language of today. I have no doubt that in

the overwhelming majority of problems, the typical trajectory is no longer confined to a submanifold of phase space of dimension at most equal to the number of degrees of freedom. No text-book in quantum mechanics raises the issue, however, and much less tells you how to deal with it.

Since 1970, we have learned a lot in this area: while some mathematically inclined people have found rigorously proved theorems, most theoreticians have been happy with heuristic arguments; most of the hard work, however, has gone into large computing enterprises and some very sophisticated experiments, both of which became technologically feasible only in the last decade.

Personally, I have been committed to an approach where a sufficiently detailed understanding of the classical chaotic behavior can be used to obtain at least approximate energy levels in the corresponding quantum system. The tool is a formula which the Norwegian mathematician Atle Selberg had proven in a special case (Balazs and Voros (1986) have written a fine survey for physicists), and which I was able to extend (Gutzwiller 1971). Ever since Selberg's paper in 1956 , number theoreticans and geometers had hoped that his trace formula would provide new insights; I don't think their expectations have been realized.

The trace formula, however, does make a very powerful case for establishing the connection I am looking for. For Selberg's case it claims the equality, and for the more general setting it claims the asymptotic equality in the limit of small $\hbar$, of the two sides of an equation: on the left we have an analytic function of the energy E whose only poles are the quantum-mechanical energy-levels, whereas on the right is another function of E which is the sum over all the periodic orbits in the classical mechanical system; periodic orbits are trajectories which close smoothly in finite time.

It became clear to me that our difficulty lies with the sum over all periodic orbits. Most physicists don't realize that a chaotic system teems with periodic orbits; their number increases exponentially with their length, whereas in an integrable system it increases as a polynomial of degree f, the number of freedoms. In the words of Poincaré (1892), "what makes these periodic solutions so valuable, is that they offer, in a manner of speaking, the only opening through which we might try to penetrate into a forteress which has the reputation of being impregnable."

These few remarks should give you an idea of my motivation for dealing with chaotic systems. Although I really want to understand quantum mechanics, I

have to settle accounts with classical mechanics first. Poincaré did not know, that the forteress holds a lot of the physics that was discovered only after his untimely death in 1913. There is also some unexpected mathematics: contrary to what you may think, classical mechanics has not been a fertile ground for fractals, until last year, when I ran into them only to find out that my eminent colleague Benoit Mandelbrot had just seen the same fractals in a different context, namely random flights between circular reflecting walls.

The Geometry of Classical Mechanics

A lot of the work on chaotic systems is done under the general heading of Dynamical Systems. These come in two closely related varieties: either ordinary non-linear differential equations, or discrete maps; the Lorenz model of meteorology, and the Hénon-map are the best known examples. (Many other mathematical models of interest involve infinitely many variables such as in hydrodynamics, or when noise is taken into account; we will not discuss them.) Most dynamical systems are not relevant to the issue which Einstein first raised, because they imply dissipation of energy.

There is no loss of energy in nature at the most fundamental level, such as in atoms or molecules, unless the interaction with the electromagnetic field is taken into account. Dissipation of energy comes about only, because energy gets transferred from some part of the system, the source, with a finite number of freedoms, to another part, the sink, with an infinite number of freedoms, like the radiation field in a cavity. Usually, this energy can not be recovered; the loss of energy can best be viewed as due to some source of friction.

All studies of chaotic systems with friction which I have seen, treat the dissipation of energy in a phenomenological manner, such as the viscosity in hydrodynamics. Schroedinger's equation in quantum mechanics does not lend itself to include a term which would describe the presence of friction, whereas classical mechanics has been treated in this manner since Newton's Principia. The connection between classical and quantum mechanics can be established, therefore, only in systems without friction.

Technically speaking, we need a Hamiltonian, i.e. a function $H(\mathbf{p}, \mathbf{q}, t)$ of the momentum $\mathbf{p}$, the position $\mathbf{q}$, and the time t; we shall call $\mathbf{p}$ and $\mathbf{q}$ the physical

coordinates. In its more refined realizations, the position coordinates $\mathbf{q}$ specify a point in a Riemannian space of f dimensions and with the metric $ds^2 = g_{jk}\, dq_j\, dq_k$, and the momentum $\mathbf{p}$ is a cotangent vector at such a point. In most examples, the Hamiltonian is the sum of the kinetic energy $T = g^{jk}\, p_j\, p_k/2m$ and the potential energy $V(\mathbf{q})$. Although a relativistic particle in an external electromagnetic field has a more complicated Hamiltonian, we will only consider cases where $H = T + V$.

Provided the physical coordinates, $\mathbf{p}$ and $\mathbf{q}$, have been chosen so as to form f canonical pairs, the Hamiltonian is used in two ways: classically it yields the equations of motion,

$$dp_j/dt = -\partial H/\partial q_j\ ,\ dq_j/dt = \partial H/\partial p_j\ , \tag{1}$$

which define a flow in the $2f$-dimensional phase space of the coordinates $(\mathbf{p}, \mathbf{q})$, also called the cotangent bundle. A solution of this system of ordinary differential equations is called a trajectory of the dynamical system.

Quantum-mechanically, the Hamiltonian leads to Schroedinger's equation,

$$i\hbar\partial\psi/\partial t = H(\mathbf{p}_{op}, \mathbf{q}, t)\psi\ , \tag{2}$$

where the momentum p_j has been replaced by the operator $p_{j\,op} = (\hbar/i)\partial/\partial q_j$ which acts on the wave function $\psi(\mathbf{q}, t)$; in most cases, there is very little difficulty in defining the Hamilton-operator in agreement with this recipe, although it is important to state the boundary condition to be satisfied by the wave function ψ.

The flow (1) has the characteristic property of conserving the 2-form

$$\Omega(\delta, \Delta) = \sum_j (\delta p_j\, \Delta q_j - \delta q_j\, \Delta p_j)\ , \tag{3}$$

i.e., if we start with a central trajectory $(\mathbf{p}(t), \mathbf{q}(t))$, and consider two neighboring trajectories, $(\mathbf{p}(t) + \delta\mathbf{p}(t), \mathbf{q}(t) + \delta\mathbf{q}(t))$ and $(\mathbf{p}(t) + \Delta\mathbf{p}(t), \mathbf{q}(t) + \Delta\mathbf{q}(t))$, then the expression (3) does not change with time, provided we take only the lowest non-trivial powers of δ and Δ into account. The central trajectory and its two neighbors define a small two-dimensional parallelogram, to which the symplectic area (3) can be assigned; this area stays the same as time evolves.

The conservation of the 2-form was originally discovered by Poincaré, who called it an integral-invariant; it is a generalization of Liouville's theorem which claims the invariance with time of the $2f-$ dimensional volume of phase space. The 2-form is invariant even when the Hamiltonian depends explicitly on the time, and energy is pumped into the dynamical system, or drained from it. An important example is the hydrogen-atom in an intense microwave-field where ionization takes place, although the frequency is so low that it would take some hundred photons piled on top of one another to get to the ionization threshold. The only viable explanation has to invoke the chaotic nature of the electronic motion, even in quantum mechanics.

If the Hamiltonian does not depend on the time t explicitly, however, then the numerical value of H stays the same along any particular trajectory, $H(\mathbf{p}, \mathbf{q}) = E$; the trajectory remains confined to a $(2f-1)$-dimensional surface of constant energy E. Then a construction, again due to Poincaré, helps greatly in visualizing the flow in phase space. A $(2f-2)$-dimensional surface Σ is chosen in the constant-energy surface, which intersects the flow transversely, and is called the surface of section. A point P_0 on this surface Σ, together with the value of the energy E, defines uniquely the initial conditions for a particular trajectory, i.e. a solution of (1).

This trajectory will eventually cut Σ again, in some other point P_1, provided Σ has been chosen so as to cover the whole flow. The transition from P_0 to P_1 defines a map T of Σ into itself, $T(P_0) = P_1$. One can show that $f-1$ canonical coordinate-pairs can be defined on Σ so that the reduced 2-form $\omega(\delta, \Delta)$, given by the same formula (3) and applied to the new coordinate-pairs, is invariant under the transformation T of Σ into itself.

The simplest classical systems with chaos have either only one degree of freedom with a time-dependent Hamiltonian, or two degrees of freedom with a time-independent Hamiltonian so that energy is conserved, $H(\mathbf{p}, \mathbf{q}) = E$. In the first case, the 2-form (3) is no more than the element of volume in phase space, while in the second case, the reduced 2-form in the the surface of section Σ is the element of volume in Σ; both times, Liouville's theorem coincides with Poincaré's integral-invariant. Very little work has been done as yet on systems where the full force of Poincaré's integral-invariants is required. Nevertheless, the resulting symplectic geometry in phase space is a fascinating intermediate between a simple measure space with no more than a element of volume, and a

full-fledged Riemannian space with a distance; notice that the metric in the position-coordinates extends quite naturally to the cotangent-bundle.

Liouville's measure plays a central role not only in statistical mechanics, but also in all kinds of elementary models of physical processes, such as the theory of collisions in chemistry, atomic, nuclear and subnuclear physics. One sometimes gets the impression from reading accounts of these fields, that there is nothing besides this special measure to determine the probability of a particular event in terms of its physical coordinates. Does it help us when we try to make the hazardous transition from classical to quantum mechanics?

Coding the Trajectories in the Anisotropic Kepler Problem

From what was said earlier, our main task is to find all the periodic orbits, i.e. trajectories which form a closed loop in phase space. In terms of Poincare's surface of section Σ, we need to know all the fixed points for the powers T^n of the transformation T, $n = 1, 2, \ldots$. Many people have attacked this problem in a systematic manner, by using the basic properties of the more interesting Hamiltonians. Very often, the classical system is integrable at low energies E, and becomes progressively less so as E gets larger; an example is the hydrogen atom in a magnetic field.

The dissolution of Einstein's invariant tori takes place preferentially through bifurcation, sometimes also trifurcation, and so on. The recent work by the group of Welge (Holle et al. 1988) in Bielefeld gives a practical, i.e. experimental, account of this approach. Other investigations have been made along this line with the help of computers; most of the early work was done in astronomy, going back all the way to Sir George H. Darwin (the son of Charles) around 1900; cf. Szebehely (1967).

With due respect for these interesting results, the method appears only partially useful at this time for the application in the trace formula. The reason seems to be that it penetrates into the ergodic jungle of phase space from the wrong side, i.e. starting from the integrable parts. I hope that this procedure will eventually be made to work, because it may be the only one available for some systems, such as the hydrogen atom in a magnetic field which is integrable at both ends, weak fields and strong fields. I like to call this situation 'soft chaos';

it is an intimate mixture of both stable and unstable orbits, the former with a minuscule set of invariant tori around them.

Hard chaos, by contrast, has only unstable orbits, and leads to a much simpler organization of the trajectories. Phase space gets foliated into two families of smooth submanifolds with codimension f, and therefore the surface of section Σ into two families of submanifolds with codimension $f-1$. Two members from different families intersect transversely, and each trajectory is the intersection of two such members. Figure 1 shows the surface of section for the anisotropic Kepler problem (AKP) which is a particularly clear cut case that I was lucky enough to find (Gutzwiller 1973 and 1988b).

Although hard chaos is exceptional among dynamical systems, it is much more frequent than integrable behavior. Perhaps the most telling argument in support of this claim comes from geometry: consider the two-dimensional Riemannian surfaces of constant Gaussian curvature K, normalized to $|K| = 1$. For $K = 1$ there is the sphere, and nothing else; the free motion of a particle on a sphere is integrable. For $K = -1$, there is an incredible variety of surfaces, some compact and some not, but all of them with a set of continuous parameters generating non-isometric examples with the same topology; the free motion of a particle (along geodesics) on each of them is a case of hard chaos, as Hadamard (1898) was the first to establish.

In systems with hard chaos, the trajectories can be coded; Birkhoff (1917) was probably the first to exploit this idea. Without giving all the details, the result is well worth explaining for the AKP which is the simplest example. Figure 1 shows the surface of section Σ; each line (submanifold) slanting roughly from the upper left to the lower right gets a binary number $(a_1, a_2, \ldots)$ describing the future; each line slanting from upper right to lower left gets a binary number written in the reverse order $(\ldots, a_{-2}, a_{-1}, a_0)$ describing the past, where in both cases $a_j = \pm 1$. The intersection of the two lines, i.e. a particular point in Σ, gets the binary sequence $a = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$. According to this assignment of binary sequences, the transformation T of Σ into itself becomes a simple shift, i.e. $T(a) = a'$ where $a'_j = a_{j+1}$.

Another way of looking at this little miracle is to express the two initial binary sequences in terms of two real numbers,

$$\xi = \sum_{j=0}^{\infty} a_{-j} 2^{-j-1} \ , \ \eta = \sum_{j=1}^{\infty} a_j 2^{-j} \ . \tag{4}$$

These two numbers define a coordinate system in Σ, where $-1 \leq \xi, \eta \leq +1$; the lines in figure 1 correspond either to ξ = constant, or to η = constant; the values chosen are 0, $\pm 1/2$, $\pm 1/4$, $\pm 3/4$, $\pm 1/8$, $\pm 3/8$, and so forth. There is a one-to-one and conntinuous corespondence between the coordinates (ξ, η) and the points in Σ as defined in the original canonical variables **p** and **q**.

Periodic orbits are described by periodic binary sequences, and these can be enumerated without difficulty because they are repetitions of the same basic pattern. E.g. the shortest pattern is either (+) or (−) , the next are (++), (+ −), (− +), (--), and so on. The corresponding coordinates (ξ, η) can be obtained from figure 1, which then yields the initial conditions for these orbits. Finally, the required information about the orbit, such as the action integral, the period, the stability exponent, and the number of conjugate poins, is obtained by numerical integration.

Even after all these steps have been succesfully completed, we are not in a position as yet to calculate the sum over all periodic orbits as demanded by the trace formula. In order to accomplish this last feat, the information about each classical orbit has to be expressed directly in the code, i.e. in terms of the binary sequences for the AKP; the information has to be translated from the framework of the physical coordinates **p** and **q** into the code. If the code is good, all of its words are actually represented in the dynamical system; the sum over periodic orbits becomes a sum over code-words.

The summation can be carried out with a trick from statistical mechanics, known as the method of transfer matrices (cf. Gutzwiller 1988b). The approximate energy levels for the AKP could be effectively calculated in this manner (Gutzwiller 1980 and 1982); the agreement with the exact eigenvalues of the Hamilton operator as computed by Wintgen et al. (1987) is very good, and certainly better than expected.

Coding the Geodesics in Spaces of Constant Negative Curvature

The description of trajectories in the AKP by an appropriate code has been presented in some detail, to make it clear how it enters into our basic problem, the connection between classical and quantum mechanics, at least in dynamical systems with hard chaos. Since the code effectively replaces the original physical coordinates **p** and **q** , one would like to understand how these new coordinates are related to the old ones; that might eventually help in getting a better handle on the working of the trace formula.

The first thing to notice about the (ξ, η) coordinates, is the fact that they also conserve the volume, or more appropriately, the area since the surface of section Σ has two dimensions. The shift in the binary sequence which corrsponds to the Poincaré map T transforms (ξ, η) according as

$$\xi \to \xi' = (\xi + \text{sign}(\eta))/2 \ , \ \eta \to \eta' = 2\eta - \text{sign}(\eta) \ . \tag{5}$$

This map of the square $-1 \leq \xi, \eta \leq +1$ into itself is known as the baker's transformation, because it gives an abstract imitation of the way in which a baker kneads the dough. It is trivial to check that the Jacobian $\partial(\xi' \eta')/\partial(\xi \eta) = 1$, i.e. the area is conserved in this transformation.

The (ξ, η) coordinates, therefore, provide an invariant measure in phase space; but is it different from Liouville's? A close look at figure 1 shows some surprising irregularities; if the relation between the two measures were simple and smooth, adjacent little parallelograms would have the same area, since the constant values for ξ and η were chosen in equal steps.

A detailed discussion of the way the binary code is constructed shows that we have only Hölder continuity between the original coordinates (p, q) in Σ and (ξ, η). If the coordinates of two neighboring points in Σ differ by δ with respect to (ξ, η) and by ε with respect to (p, q) , then all we can say is that $\varepsilon \simeq \delta^{\alpha}$ where $\alpha \neq 1$. In order to have a first derivative, one would need $\alpha = 1$.

The detailed nature of the relation between the two measures can only be established through extended computations which require the integration of many trajectories. Therefore, we will switch to another dynamical system with hard chaos, the geodesics on a surface of constant negative curvature, which has some advantages over the AKP: the trace formula is exact rather than only asymptotic in the limit of small Planck's quantum; integrating the trajectories reduces to

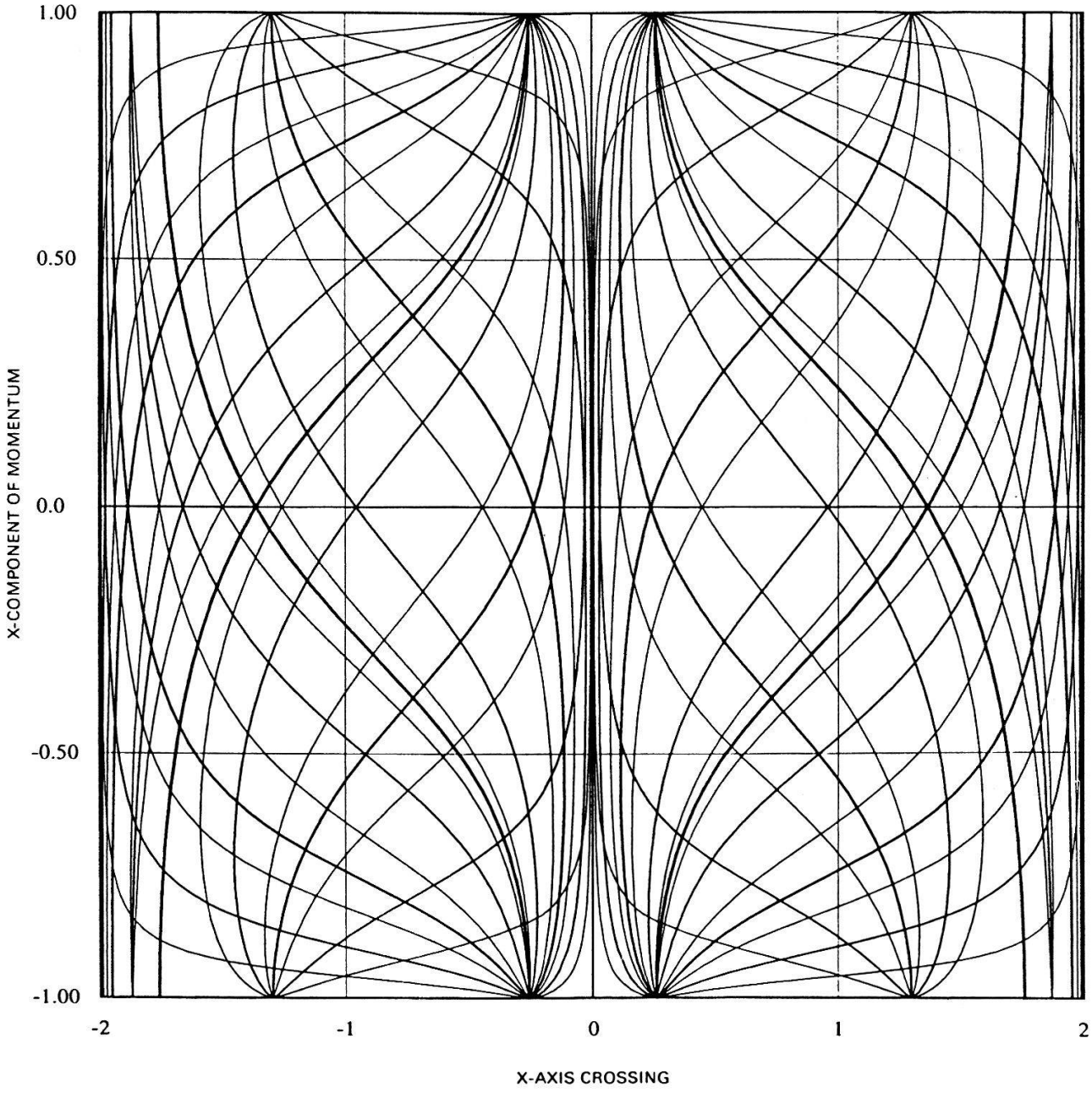

Figure 1: Stable and unstable manifolds in the surface of section for the Anisotropic Kepler Problem with the mass-ratio 3.

multiplying 2 by 2 matrices, with essentially one such multiplication for each intersection with the surface of section.

The coding of the geodesics, on the other hand, can become quite tricky. Some of the fundamental work has already been done by the mathematicians, in

particular Series (1986), as well as Adler and Flatto (1984); the physicists have caught on meanwhile, e.g. Aurich, Sieber and Steiner (1988) as well as Bohigas and coworkers (private communication). The first explicit comparison of the two measures in phase space, however, is due to Gutzwiller and Mandelbrot (1988a). I will discuss here only the quite unexpected appearance of fractals which seemingly nobody had foreseen.

The simplest example of coding arises on a surface of constant negative curvature whose topology is the same as a torus (an ordinary box with periodic boundary conditions in the language of theoretical physics), except that an exit-entry point is provided infinitely far from the main body of the torus. This surface can also be regarded as a quadrangle in the plane of hyperbolic geometry; opposite sides are identified; the four corners are at infinity.

The natural code is ternary because when the trajectory enters on one side, it then has a choice of exiting on one of the three other sides. The 1-to-1 correspondence of this code with the two physical coordinates specifying the geodesic is easy to show, because the quadrangle and all its copies by translation tile the hyperbolic plane, i.e. they cover it without gaps nor overlaps. Again the code breaks naturally into the future and the past, and both are required to specify the whole trajectory.

The physical coordinates in the surface of section can also be chosen so as to separate the future completely from the past; in other words, it is possible to define the physical attributes of a trajectory so that its story in the past is fixed by one number, while its behavior in the future is fixed independently by another number. The relation between the physical coordinates and the code breaks into the product of two maps in one dimension, one for the past and one for the future. By contrast, there seems to be no simple way of making this separation in the physical coordinates for the AKP.

A remarkable special case is the singular triangle; when the freely moving particle hits one of its sides, the trajectory can be continued as in the quadrangle, because the triangle also tiles the hyperbolic plane. As first proposed by Series (1985), the binary code is natural in this case, beceuse the particle entering on one side has only the choice of the two other sides to exit. If the binary expansion is written as $b = (b_1, b_2, \ldots)$, with $b_j = 0$ or 1, then, as in the AKP, one can use the real number $\beta = \sum_j b_j 2^{-j}$.

The physical coordinate ζ, on the other hand, is best represented by a continued fraction according to a classic paper by Artin (1924),

$$\zeta = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ldots}} , \tag{6}$$

where the integer $n_k \geq 1$. The relation with the binary expansion b could not be simpler: b starts with $n_1 - 1$ 0's, followed by n_2 1's, followed by n_3 0's, and so on. While this representation of the physical coordinate ζ, and the relation with the binary code b has been known for a long time, nobody had considered β and ζ as defining two alternative measures in the phase space of this geodesic flow.

The origin of this peculiar function $\beta(\zeta)$ of the unit-interval onto itself, or its inverse $\zeta(\beta)$, has been explained so carefully, because it was actually found in this round- about way, starting with the problem of coding the trajectories in a Hamiltonian dynamical system. The plot of β versus ζ is shown in figure 2; the function is monotonically increasing; but all its derivatives vanish at the points with a rational value of ζ. The analogous functions for the quadrangle with the ternary code have the same unusual behavior, whereas in the AKP the local distortions between the variation in the code as a function of physical coordinates are not so extreme.

Measures in Phase Space other than Liouville's

The properties of such functions are most naturally analyzed in terms of the local Hoelder exponent, $\alpha = \log[\beta(\zeta + \Delta\zeta) - \beta(\zeta)]/\log[\Delta\zeta]$. If the values of α are put into bins of width $\delta\alpha$, the number of occurrences of α in a particular bin can be written as $N(\alpha)\delta\alpha$ in terms of a density $N(\alpha)$. The points on the unit-interval corresponding to a particular bin form a fractal set, whose dimension $f(\alpha)$ can be calculated as $f(\alpha) = \log N(\alpha)/\log[1/\Delta\zeta]$. The $f(\alpha)$ curve is plotted in figure 3; it is limited for large α by the quadruple accuracy of the computations.

No multi-fractal measure of this extreme kind had ever been published before this spring. My purpose here, however, is not to present some new mathematical monstrosity; but to get a better understanding of the classical trajectories in a

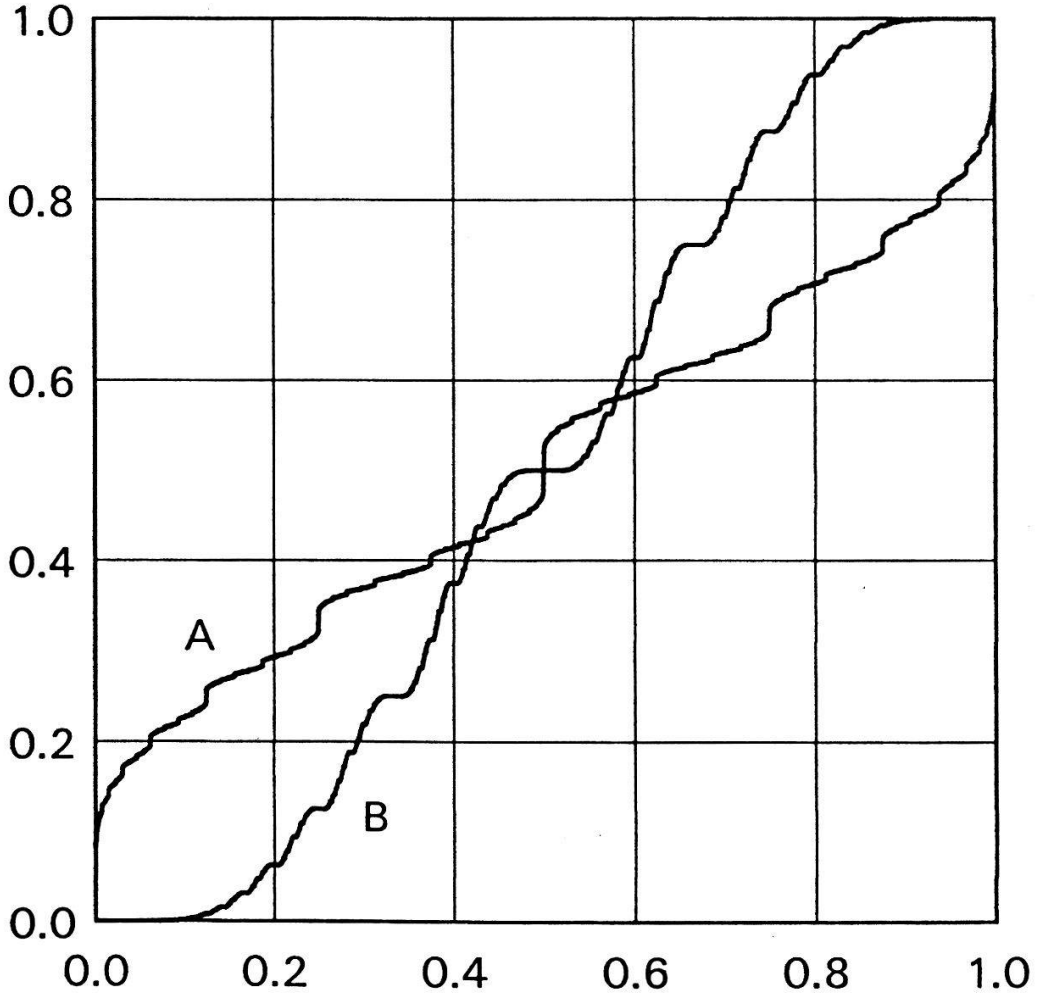

Figure 2: The functions A: $\zeta(\beta)$ and B: $\beta(\zeta)$; both are monotonically increasing, but the latter has all its derivatives vanishing at the points with rational coordinates ζ.

system with hard chaos. The almost flat portions in $\beta(\zeta)$ show that the physical coordinate ζ can be changed locally over a wide interval without any noticeable change in the corresponding code; in other words, the qualitative description of the trajectory stays almost the same in spite of a drastic change in the starting conditions (initial values of momentum and position).

Physically, the particle is seen to get trapped. In the case of a box with an exponential horn attached (the hyperbolic torus with an entry-exit at infinity), the trapping takes place in the horn. The anisotropic Kepler problem differs

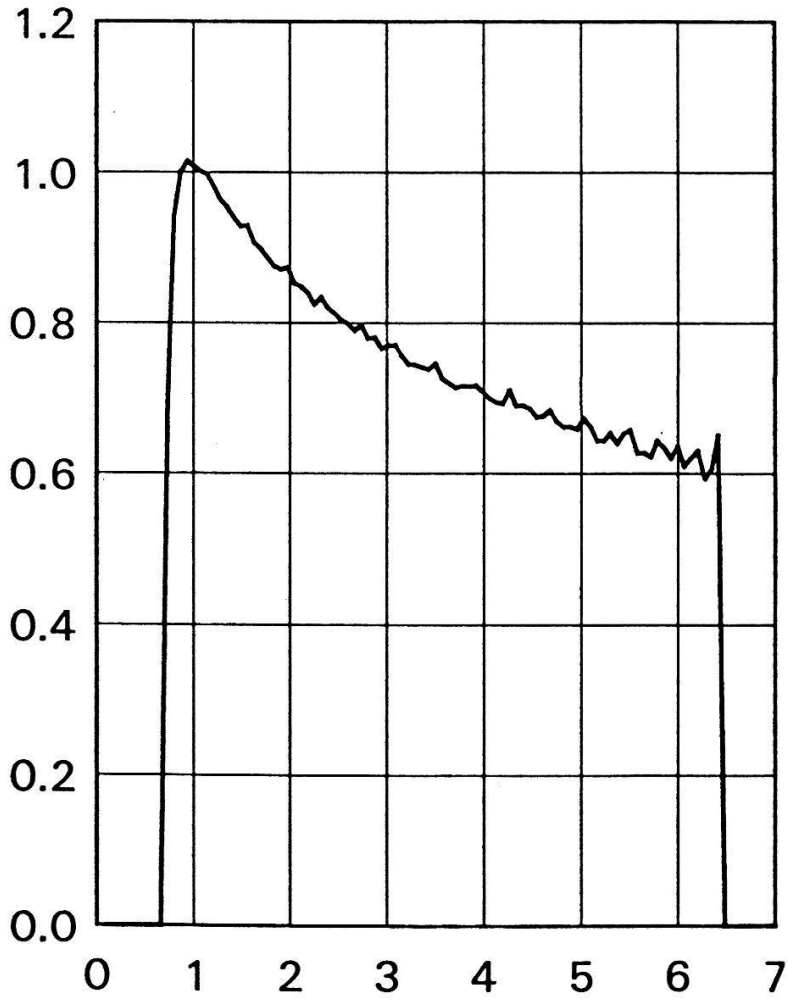

Figure 3: The distribution $f(\alpha)$ of Hölder exponents α for the curve $\beta(\zeta)$; the upper limit of $\alpha = 6.5$ is due to the quadruple accuracy of the calculations which sets every number smaller than 10^{-33} equal to 0.

from the ordinary Kepler problem only because the mass-tensor is anisotropic, as for the donor impurity in silicon or germanium; when the anisotropy becomes small, one expects to return to the trajectories resembling the Kepler ellipses; the binary code for them is the alternating sequence (... + − + − + − + − ...); this special code-word becomes then a trap, i.e. trajectories whose code has long alternating subsequences, occupy an ever larger portion of the surface of section Σ, as the anisotropy goes to 0. Figure 3 shows this effect for the mass-ratio 3: the relatively large rhombus on the middle horizontal line between the physical coordinates .25 and .5 belongs to these Kepler-type trajectories.

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