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Autor(en): **Perroud, M.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **62 (1989)**

Heft 4

PDF erstellt am: **26.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-116038

Helvetica Physica Acta, Vol. 62 (1989) 410–419

0018–0238/89/040410–10$1.50 + 0.20/0

Thermodynamics of dissipative systems II

By M. Perroud

Département de Mathématiques Appliquées Ecole polytechnique de Montréal,
C.P. 6079, Succ "A", Montréal, H3C 3A7, Canada

(5. VIII. 1988, revised 30. X. 1988)

Abstract. A new model of phenomenological thermodynamics for continua is presented. This is a sequel to the paper intitled "Thermodynamics of dissipative systems", devoted to discrete systems [1] and covers the case of fluids.

1. Introduction

The Hamiltonian structure of continuum systems has been discussed in several papers, particularly by V. Arnold and J. E. Marsden et al. [2, 3]. However, they restrict themselves to ideal incompressible or compressible, but adiabatic fluids rather than real fluids. The entropy production is always neglected, therefore the systems are never dissipative. The symplectic structure of the models is probably the main reason for this. The extension that we proposed in [1] for a discrete system applies as well and rather easily for a continuous medium and takes into account the dissipative effects. We follow again the approach used by E. C. G. Stueckelberg [4].

2. Continuum mechanics

As usual in non relativistic physics, we postulate that the space-time E_4 is fibred over the time axis $\mathfrak{R}$ and that each fibre is diffeomorphic to a flat Riemannian manifold E_3 equipped with a volume form dV; therefore, we have $E_4 = \mathfrak{R} \times E_3$.

A reference configuration D of a body (a fluid) is a Riemannian manifold. A configuration is a diffeomorphism $\Psi : D \rightarrow E_3$ and a motion of D is a time dependent family of configurations $\Psi_t : D \rightarrow E_3$ written as $x = \Psi(t, X) = \Psi_t(X)$. We call $X \in D$ a Lagrangian point and $x \in E_3$ an Eulerian point with coordinates (x^1, x^2, x^3). The Lagrangian velocity is defined by

$$V(t, X) = \frac{\partial \Psi(t, X)}{\partial t}$$

and its corresponding Eulerian-velocity is given by

$$v(t, x) = v_t(x) = V(t, \Psi_t^{-1}(x)) = V_t \circ \Psi_t^{-1}(x).$$

Hence, each point $x \in E_3$ belongs to a motion

$$x = \Psi(t, X) \quad \text{or} \quad X = \Psi_t^{-1}(x)$$

and

$$\phi(\lambda, t, x) = \Psi(t + \lambda, \Psi_t^{-1}(x))$$

is the one parameter diffeomorphism of E_3 generated by the vector field $\dot{\phi}_t = v_t = V_t \circ \Psi_t^{-1}$, with $v_t(x) \in T_{(t,x)}E_3$.

The configuration space of a continuum mechanical system is therefore the group Diff (E_3) and the phase space is T^* Diff (E_3).

The mass of a subregion $B_t = \Psi_t(B_0) \subseteq D$ at the time t is defined by

$$M(t) = \int_{B_t} M_t(x)\, dV(x).$$

It has the property of being neither created nor destroyed; henceforth the mass density $m_t(x) \geq 0$ is the image of a function $m_0(X)$ defined on D_0

$$\Psi_t^*(m_t)J(\Psi_t) = m_0,$$

where $J(\Psi_t)$ is the Jacobian of Ψ_t. Using the change-of-variable formula, we have

$$\int_{B_t} m_t\, dV = \int_{B_0} \Psi_t^*(m_t)J(\Psi_t)\, dV_0 = \int_{B_0} m_0 dV_0, \qquad B_0 \subseteq D.$$

The conservation of M is in turn equivalent to

$$\frac{d}{dt}(\Psi_t^*(m_t)J(\Psi_t)\, dV_0) = (div\ (m_t v_t) + \partial_t m_t)\, dV(x)) = 0$$

thus

$$\partial_t m_t + \text{div}\,(m_t v_t) = 0 \tag{1}$$

is the differential form of the law of conservation of mass, known as the continuity equation.

3. Thermodynamics

In addition to the phase space of mechanical system, there exists in thermodynamics a state function of a special kind, the entropy. It can be defined at each time t as the extensive functional

$$S(t) = \int_{B_t} s_t(x)\, dV(x),$$

where $s_t(x) = s(t, x)$ is a density function.

The set of all states of a thermodynamical system is therefore a manifold $E = D(E_3) \times T^* \operatorname{Diff}(E_3)$, where $D(E_3)$ is the space of differentiable functions defined on E_3. Recall that if

$$P = \int_{E_3} p_i(x)\delta\phi^i(x)\,dV(x) \in T^*_\phi \operatorname{Diff}(E_3)$$

and

$$V = \int_{E_3} v^k(y)\frac{\partial}{\partial\phi^k(y)}\,dV(y) \in T_\phi \operatorname{Diff}(E_3),$$

the duality is defined by

$$\begin{aligned}\langle P, V\rangle &= \int_{E_3}\int_{E_3} p_j(x)v^k(y)\delta\phi^j(x)\Big(\frac{\partial}{\partial\phi^k(y)}\Big)\,dV(x)\,dV(y)\\ &= \int_{E_3}\int_{E_3} p_j(x)v^k(y)\delta^j_k\delta(x-y)\,dV(x)\,dV(y)\\ &= \int_{E_3} p_k(x)v^k(x)\,dV(x).\end{aligned}$$

We assume that a closed system is a dynamical system. In other words, the evolution is defined by a semi-flow

$$U : \Re_+ \times E \to E$$

generated by the vector field

$$Z(s, \zeta) = \frac{d}{dt}U(t, s, \zeta)\big|_{t=0}, \qquad (s, \zeta) \in E.$$

The vector field does not depend explicitly on time, which means that the system is autonomous or equivalently is closed.

Moreover, this dynamical system must satisfies the two principles of thermodynamics. Beginning with the first law, let us assume that there exists an extensive energy state functional defined on a bounded region $B \subseteq E_3$

$$H[s, p, \phi] = \int_B h(s(x), p(x), \phi(x), D\phi(x))\,dV(x),$$

where $D\phi$ is the covariant derivative of ϕ and $h(s, p, \phi, D\phi)$ is an energy density function, such that

$$i_z\delta H = \frac{\partial}{\partial t}H_t = \frac{\partial}{\partial t}\int_{B_t} h(s_t, p_t, \phi_t, D\phi_t)\,dV = 0. \tag{2}$$

Let us briefly recall that an extensive functional

$$F(t) = \int_{B_t} f(t, x)\,dV(x)$$

has a time derivative (see [4])

$$\frac{\partial}{\partial t} F(t) = \frac{\partial}{\partial t} \int_{B_t} f(t, x)\, dV(x) = \int_{B_t} \Delta_t f(t, x)\, dV(x)$$
$$= \int_{B_t} (d_t f + \operatorname{div}(f v_t))(x)\, dV(x),$$

where

$$\Delta_t f(t, x) = (\partial_t f + \operatorname{div}(v_t f)(x, t). \tag{3}$$

it is determined by a source density ρ_F and to an influx $-j_F$ through ∂B_t

$$\frac{\partial}{\partial t} F(t) = \int_{B_t} \rho_F(x)\, dV(x) - \int_{\partial B_t} j_F(x) \times n(x)\, dS(x).$$

Gauss' theorem leads to the inhomogeneous continuity equation

$$\partial_t f + \operatorname{div}(f v_t + j_F) = \rho_F \tag{4}$$

The first law can easily be extended to systems interacting with the outside world by means of work and heat.

The work will be defined by a 1-form possibly depending on time

$$W_t = \int_{B_t} \omega_t(x)\, dV(x),$$

where $\omega_t(x)$ is a density of 1-form. We can write the first law for an adiabatic closed system

$$i_{Z_t} \delta H = i_{X_t} W_t$$

with X_t denoting the part of Z_t acting on T Diff (E_3).

The heat entering the system is an extensive functional, possibly time dependent,

$$Q_t = \int_{B_t} q_t(x)\, dV(x).$$

The first law takes the general form

$$i_{Z_t} \delta H = i_{X_t} W_t + Q_t.$$

The second law states the principle of evolution of the entropy. In its equation of continuity (see 4)

$$\partial_t s + \operatorname{div}(s v_t + js) = i \geq 0, \tag{5}$$

the source density $i(x)$ (= density of irreversibility) is non negative. For a system adiabatically closed ($j_S(x) = 0$), we obtain

$$\frac{\partial}{\partial t} S_t = \int_{B_t} i(x)\, dV(x) \geq 0.$$

4. Equations of motion.

For a functional H depending on $s(x), p(x)$ and $\phi(x)$, the functional differential can be written

$$\delta H[s, p, \phi] = \int_B \left(\frac{\delta H[s, p, \phi]}{\delta s(x)} \delta s(x) + \frac{\delta H[s, p, \phi]}{\delta p(x)} \delta p(x) + \frac{\delta H[s, p, \phi]}{\delta \phi(x)} \delta \phi(x) \right) dV(x)$$

where

$$\frac{\delta H[s, p, \phi]}{\delta s(x)}, \frac{\delta H[s, p, \phi]}{\delta p(x)} \quad \text{and} \quad \frac{\delta H[s, p, \phi]}{\delta \phi(x)}$$

are the functional derivatives of $H[s, p, \phi]$. In the case where we have an extensive functional

$$F|\xi| = \int_B f(\xi(x), D\xi(x))\, dV(x),$$

the functional derivative is given by

$$\frac{\delta F[\xi]}{\delta \xi(x)} = \partial_\xi f - D_i(\partial_{D_i\xi} f),$$

where $D_i\xi$ is a component of the covariant derivative of ξ.

We frequently put

$$\delta H[s, p, \psi] = \int_B \frac{\delta H[s, p, \phi]}{\delta s(x)} \delta s(x)\, dV(x) + \delta_0 H[s, p, \phi],$$

where

$$\frac{\delta H[s, p, \phi]}{\delta s(x)} = \partial_s h(s(x), p(x), \phi(x), D\phi(x)) = T(x) \geq 0$$

is the local temperature and

$$\delta_0 H[s, p, \psi] = \int_B \left(\frac{\delta H[s, p, \psi]}{\delta p(x)} \delta p(x) + \frac{\delta H[s, p, \psi]}{\delta \phi(x)} \delta \phi(x) \right) dV(x).$$

By the extensive character of $H[s, p, \psi]$, we can set

$$\begin{aligned} \delta_0 h(s(x), p(x), \phi(x), D\phi(x)) = {} & \partial_p h(s(x), p(x), \phi(x), D\phi(x)) \delta p(x) \\ & + \partial_\phi h(s(x), p(x), \phi(x), D\phi(x)) \delta \phi(x) \\ & - \operatorname{div} (\partial_{D\phi} h(s(x), p(x), \phi(x), D\phi(x))) \delta \phi(x) \end{aligned}$$

and consequently

$$\delta_0 H[s, p, \phi] = \int_B \delta_0 h(s(x), p(x), \phi(x), D\phi(x))\, dV(x).$$

The corresponding decomposition of a vector field Z_t will be

$$Z_t = \int_{B_t} \left(\Delta_t s_t(x) \frac{\partial}{\partial s(x)} + X_t(x) \right) dV(x),$$

where $X_t(x)$ acts on $T_{(t,x)}E_3$.

The first law for a closed system:

$$i_{Z_t} \delta H = 0,$$

can be written

$$\int_{B_t} (T_t(x)\Delta_t s_t(x) + i_{X_t(x)} \delta_0 h(s_t, p_t, \phi_t, D\phi_t))\, dV(x) = 0$$

hence

$$\Delta_t s_t(x) = -\frac{1}{T_t(x)} i_{X_t(x)} \delta_0 h(s_t(x), p_t(x), \phi_t(x), D\phi_t(x)).$$

The second law imposes $\Delta_t s_t \geq 0$. In order to satisfy this inequality, we will adopt the Onsager hypothesis by writing

$$\Delta_t s_t(x) = \frac{1}{T_t(x)} \Delta_s(x)(X_t(x), X_t(x)), \tag{7}$$

where $\Lambda_s(x)(X(x), X(x))$ is a positive semidefinite quadratic form. (The subscript s indicates that $\Lambda_s(x)$ is symmetric). Consequently, we have

$$i_{X_t(x)} \delta_0 h(s_t(x), p_t(x), \phi_t(x), D\phi_t(x)) = -\Lambda(x)(X_t(x), X_t(x)).$$

Actually, we shall postulate that the vector field X_t is defined by

$$\delta_0 h(s_t(x), p_t(x), \phi_t(x), D\phi_t(x)) = -\Lambda(x)(X_t(x)), \tag{8}$$

where $\Lambda(x)$ is a positive semidefinite regular bilinear form on $T_{(t,x)}E_3$.

Since a bilinear form $\Lambda(x)$ can always be decomposed into the sum of a symmetric part $\Lambda_s(x)$ and an antisymmetric part $\Lambda_a(x)$ form, we recover equation (7)

If the system is not isolated, the vector field $X_t(x)$ is simply defined by

$$i_{X_t(x)} \delta_0 h(s_t(x), p_t(x), \phi_t(x), D\phi_t(x)) = -\Lambda(x)(X_t(x)) + \omega_t(x). \tag{8'}$$

The first principle

$$i_{Z_t} \delta H = \int_{B_t} (\Delta_t s_t(x) + i_{X_t(x)} \delta_0 h(s_t(x), p_t(x), \phi_t(x), D\phi_t(x))\, dV(x)$$
$$= \int_{B_t} \omega_t(X_t(x) + q_t(x))\, dV(x)$$

gives

$$\Delta_t s_t(x) = \frac{1}{T_i(x)} (\Delta_s(x)(X_t(x), X_t(x)) + q_t(x)). \tag{8''}$$

Equations (7) and (8″) defined the vector field Z completely and satisfy the two principle of thermodynamics by construction.

In symplectic mechanics, where the state space reduces to the manifold $T^*\,\mathrm{Diff}\,(E_3)$, the vector field X is defined by

$$\delta_0 H = -\Omega(X),$$

where Ω is a symplectic 2-form. From this point of view, our model appears as an extension of mechanics if we put in place of Ω

$$\Lambda = \int_{B_t} \Lambda(x)\, dV(x).$$

Indeed, the symmetric part, absent in mechanics, has been introduced to take dissipation into account.

Example. A finite dimensional case [1]

A damped harmonic oscillator is defined by

$$E = \Re \times M = \Re \times \Re^2 = \{(S, p, q)\}$$

with

$$H(S, p, q) = \frac{1}{2M} p^2 + \frac{1}{2} k(S) q^2 + f(S).$$

$$\Lambda(S, p, q) = dp \wedge dq + \Lambda_s(S, q)\, dq \otimes dq,$$

where m is the mass, $k(S)$ the spring constant, $f(S)$ a purely thermal energy and $\Lambda_s(S, q) \geq 0$ is interpreted as the friction coefficient.

With the notation $Z = (\dot{S}, \dot{p}, \dot{q})$ the equation $\mathbf{d}_0 H = -\Lambda(X)$ gives

$$\frac{1}{M} p dp + k(S) q dq = \dot{q} dp - \dot{p} dq - \Lambda_s(S, q) \dot{q} dq,$$

from which we get

$$M\dot{q} = p$$
$$\dot{p} = -\Lambda_s(S, q)\dot{q} - k(S)q.$$

The equation for the entropy, $\dot{S} = (1/T)\Lambda_s(X, X)$ reduces to

$$\dot{S} = \frac{1}{T} \Lambda_s(S, q) \dot{q}^2 \geq 0.$$

Now let us examine the case of a fluid with friction. The energy density is given by,

$$h = \frac{1}{2m(x)} |p(x)|^2 + u[s(x), D\phi(x)],$$

the bilinear form

$$\begin{aligned}\Lambda(x) &= \delta p(x) \wedge \delta\phi(x) + \Theta(x)\delta D\phi(x) \otimes \delta D\phi(x)\\ &= \delta p_i(x) \wedge \delta\phi^i(x) + \Theta_{jl}^{ik}(x)\delta D_i\phi^j(x) \otimes \delta D_k\phi^l(x)\end{aligned} \tag{9}$$

and the work density

$$\omega_t(x) = k_i(x)\delta\phi^i(x).$$

The vector field $X_t(x)$ takes the form

$$X_t(x) = \dot{\phi}_t^i(x)\frac{\partial}{\partial\phi^i(x)} + \Delta_t p_{tj}(x)\frac{\partial}{\partial p_j(x)}.$$

Then we obtain

$$\Lambda(x)(X_t(x)) = \Delta_t p_{ti}(x)\delta\phi^i(x) - \dot{\phi}_t^i(x)\delta p_i(x) + \Theta_{jl}^{ik}(x)D_i\dot{\phi}_t^j(x)\delta D_k\phi^l(x),$$

but $\sigma D\phi = D\delta\phi$. Supposing $\delta\phi(x)$ is of compact support in B_t and applying Green's theorem

$$\int_{B_t}\Theta_{jl}^{ik}(x)D_i\dot{\phi}_t^j(x)D_k\delta\phi^l(x)\,dV(x) - \int_{B_t}D_k(\Theta_{jl}^{ik}(x)D_i\dot{\phi}_t^j(x))\delta\phi^l(x)\,dV(x),$$

by setting

$$\tau_l^{(f)k}(x) = \Theta_{jl}^{ik}(x)\dot{\phi}_{t,i}^j(x), \tag{10}$$

the frictional part of the stress tensor, we obtain

$$\Lambda(x)(X_t(x)) = \Delta_t p_{ti}(x)\delta\phi^i(x) - \dot{\phi}_t^i(x)\delta p_i(x) - \operatorname{div}(\tau_l^{(f)})\delta\phi^l(x).$$

On the other hand, we have

$$\delta_0 h = \frac{1}{m_t(x)}p_t^i(x)\delta p_i(x) - D_k(\partial_{D_i\phi^i}u[s_t(x), D\phi_t(x)])\delta\phi^i(x),$$

where $p_t^i(x) = g^{ik}(x)p_{tk}(x)$. Here $(g^{ik}(x))$ is the Riemannian metric.

By definition, the elastic part $\tau_l^{(e)k}(x)$ of the stress tensor is

$$\tau_l^{(e)k}(x) = \partial_{D_k\phi^l}u[s_t(x), D\phi_t(x)]. \tag{11}$$

Generally, we have

$$\tau_l^{(e)k} = -p\delta_l^k,$$

is the scalar pressure. Hence, we obtain

$$\delta_0 h = \frac{1}{m_t(x)}p_t^i(x)\delta p_i(x) - \operatorname{div}(\tau_t^{(e)})\delta\phi^l(x).$$

The equation $\delta_0 h = -\Lambda(X_t) + \omega_t$ gives

$$\begin{aligned}&\frac{1}{m_t(x)}p_t^i(x)\delta p_i(x) - \operatorname{div}(\tau_i^{(e)})\delta\phi^i(x)\\ &\qquad = \dot{\phi}_t^i(x)\delta p_i(x) - (\Delta_t p_{ti}(x) - \operatorname{div}(\tau_i^{(f)}))\delta\phi^i(x) + k_i(x)\delta\phi^i(x).\end{aligned}$$

Thus we get Hamilton's equations

$$m_t(x)\dot{\phi}_t^i(x) = m_t(x)v_t^i(x) = p_t^i(x)$$

$$\Delta_t p_{ti}(x) = \operatorname{div}(\tau_i^{(e)}(x)) + \operatorname{div}(\tau_i^{(f)}(x)) + k_i(x).$$

By putting

$$\tau_l^k(x) = \tau_l^{(e)k}(x) + (\tau_l^{(f)k}(x) \tag{12}$$

and using (1), the second equation is equivalent to

$$\partial_t m_t(x) + \operatorname{div}(m_t(x)v_t(x)) = 0 \tag{1}$$

and

$$m_t(x)(\partial_t v_{ti}(x) + v_t^k D_k v_{ti}(x)) = \operatorname{div}(\tau_i)(x) + k(x), \tag{13}$$

which we recognize as Cauchy's equation.

The equation (8″)

$$m_t(x)\Delta_t s_t(x) = \frac{1}{T_t(x)}(\Delta_s(x)(X_t(x), X_t(x)) + q_t(x))$$

becomes

$$\Delta_t s_t(x) = \frac{1}{T_t(x)}(\tau_l^{(f)k}(x)D_k v_t^l(x) + q_t(x)).$$

We define the components of the tensor Θ to be $\Theta_{jl}^{ik} = 2\eta[s, \phi]\delta_l^i\delta_j^k + (\xi[s, \phi] - \frac{2}{3}\eta[s, \phi])\delta_j^i\delta_l^k$, where $\eta[s, \phi]$ is the transversal viscosity coefficient and $\xi[s, \phi]$ is the longitudinal viscosity coefficient.

Thus, we obtain

$$\tau_l^{(f)k} = 2\eta[s, \phi]D_l v_t^k + (\xi[s, \phi] - \frac{2}{3}\eta[s, \phi])D_l v_t^j \delta_j^k.$$

By introducing the decomposition of Dv_t in its trace $D_i v_t^i$ and its trace-less irreducible part with respect to the Galilei group $D_j v_t^{i(0)}$

$$D_j v_t^i = D_j v_t^{k(0)} + \frac{1}{3}D_l v_t^l \delta_j^i,$$

the term $\Lambda_t(x)(X_t(x), X_t(x))$ can be written

$$\Lambda_t(x)(X_t(x), X_t(x)) = 2\eta D_l v_t^{k(0)}(x)D_k v_t^{l(0)}(x) + \xi(D_l v_t^l(x))^2. \tag{14}$$

The heat density $q_t(x)$ is given, in the Fourier conduction law, by

$$q_t(x) = -\operatorname{div}(T_t(x)j_s(x))$$

with

$$j_s(x) = -\frac{\kappa}{T_t(x)}\operatorname{grad} T_t(x),$$

where $\kappa[s] \geq 0$ is the thermal condictivity coefficient.

Consequently,

$$\Delta_t s_t(x) = \frac{1}{T_t(x)} (\Lambda_s(x)(X_t(x), X_t(x)) - T_t(x) \operatorname{div} j_S + \frac{\kappa}{T_t(x)} |\operatorname{grad} T_t(x)|^2).$$

In this way, we have obtained the inhomogeneous equation for the entropy

$$\partial_t s_t(x) + \operatorname{div} (s_t(x) v_t(x)) + \operatorname{div} j_S(x) = i(x) \tag{15}$$

with

$$i(x) = \frac{1}{T_i(x)} \left(2\eta D_t v_t^{k(0)}(x) D_k v_t^{l(0)}(x) + \xi (D_l v_t^l(x))^2 + \frac{\kappa}{T_i(x)} |\operatorname{grad} T_t(x)|^2 \right). \tag{16}$$

Conclusion

The set of equations (1), (13), (15) and (16) describes a fluid with friction. It would be easy to adapt these results to elasticity, in the Lagrangian description.

In general, the literature devoted to ideal fluids considers the Hamiltonian structure by means of the Poisson bracket.

In our model, we would have to consider a "metriplectic" structure [5], where the Poisson bracket does not satisfy the Jacobi identity.

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