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Analysis of the generating-functions for a two-peaked spectrum

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Abstract. Identifying the reasons for the inadequacies of the available forms of the generating function (g.f.) for a *symmetric* two-peaked spectrum, we show how a unique analytic g.f. describing this spectrum can be arrived at by considering the physics of the photon-counting-statistics (PCS) of the Gaussian light.

1. Introduction

Like in any area of theoretical physics, the question of ‘analyticity’ is of central importance in the field of Quantum Optics as well. In a special area, namely the ‘inverse scattering problems in Quantum Optics’, Ross and Fiddy [1] have presented a very interesting study of the analytic nature of the scattered fields. In the present paper, we take up the study of the analytical aspects of Photon Counting Statistics (PCS) of Gaussian light with the specific case of a *symmetric* two-peaked spectrum.

A study of the various aspects of the PCS of Gaussian light of a given spectral shape is facilitated via the generating-function (g.f.) which is obtained on solving the Fredholm integral equation of the second kind and then rendering the resulting Fredholm-determinant (F-D) to be analytic in nature. However, as the analyticity of a given g.f. does not automatically ensure its suitability to describe the PCS of the spectrum in question (see Section 2), we need to find a *unique* analytic g.f. as the mathematical rigor would require. For this purpose, as we had formulated in [2], it would suffice if we find an analytic g.f. which correctly follows the probability bounds on it (equations (1) and (2)) and its immediate variation (the first differentiation) gives us the correct mean count (equation (3)):

$$0 \leqslant Q(s)\left(= \langle e^{-Es} \rangle = \int_0^T \bar{e}^{Es} P(E)\, dE \right) \leqslant 1 \tag{1}$$

where $E(t) = \int_0^t I(t')\, dt'$ and $I(t)$ is the instantaneous intensity of the incident radiation.

$$Q(s)|_{s=0} = \sum_{n=0}^{\infty} (1-s)^n P(n) = \sum_{n=0}^{\infty} P(n) = 1 \tag{2}$$

$$(-1)\partial Q(s)/\partial s|_{s=0} = \langle n \rangle \tag{3}$$

Having successfully shown the efficacy of the conditions stated in equations (1)–(3) in the context of the Higher Order PCS of Gaussian–Lorentzian light in [2] – in the present paper we further establish the utility of equations (1)–(3), first by showing how to decide whether the earlier g.f.'s given by Mehta and Gupta [3] and Singh and Srinivasan [4] are suitable or not to describe the PCS of a symmetric two-peaked spectrum (Section 2) and then by obtaining the *unique* form of the analytic g.f. pertaining to this spectrum (Section 3).

2. Analysis of the g.f.'s for a symmetric two-peaked spectrum

A multiple-peaked spectrum is obtained on scattering the laser light through a polydisperse medium [5]. The auto-correlation of the scattered field is given by [5]:

$$g(\tau) = \int_0^\infty e^{-\Gamma\tau} G(\Gamma)\, d\Gamma \tag{4}$$

where $G(\Gamma)$ is the distribution function. Recently Ostrowsky et al. [6] have considered the solutions of equation (4) for $G(\Gamma)$ by using the Laplace transform techniques.

It is easy to see from equation (4) that the auto-correlation for a sum of Lorentzian-spectral profiles is given by,

$$g(\tau) = \int_0^\infty e^{-\Gamma\tau}\left(\sum_{j=1}^{N} \alpha_j \delta(\Gamma - \Gamma_j)\right) d\Gamma = \sum_{j=1}^{N} \alpha_j e^{-\Gamma_j \tau} \tag{5}$$

where $\sum_{j=1}^{N} \alpha_j = 1$.

We shall confine our interest to the case where $N = 2$. Blake and Barakat [7] were the first to study the PCS of a symmetric multiple-peaked spectrum, namely the Brillouin-spectrum via the Gauss-quadrature method. Mehta and Gupta [3] have considered the following form of a symmetric two-peaked spectrum with equal half-widths and intensities:

$$I(\omega) = \sigma\langle I\rangle\{(\omega - \omega_0)^2 + \sigma^2]^{-1} + [(\omega + \omega_0)^2 + \sigma^2]^{-1}\} \tag{6}$$

for which the auto-correlation is given by,

$$g(\tau) = \langle I\rangle e^{-\sigma T} \cos \omega_0 \tau \tag{7}$$

The g.f. is obtained on solving the following Fredholm-integral equation of the second kind:

$$s \int_0^T g\, |t - t'|\, \phi(t')\, dt' = \lambda \phi(t) \tag{8}$$

The g.f. for the Gaussian-light can be expressed as the following infiniteproduct [8]:

$$Q(s) = \prod_k [1 + \langle I \rangle \lambda_k]^{-1} \tag{9}$$

where λ_ks are the eigenvalues of equation (8).

Mehta and Gupta [3] solve equation (8) by converting it to a differential equation as demonstrated by Slepian in [9]. By employing the contour-integration techniques, the g.f. given by these authors is as follows:

$$Q(s) = 4\sigma^2(\sigma^2 + \omega_0^2)e^{2\sigma T}[\Delta(z_1)]^{-1} \tag{10}$$

where

$$\Delta(z_1) = \Delta_1(z_1)\Delta_2(z_1) \tag{11}$$

and

$$\Delta_1(z_1) = \begin{vmatrix} (\sigma^2 + \omega_0^2 + z_1)\cosh\frac{1}{2}\sqrt{z_1}\,T & (\sigma^2 + \omega_0^2 + z_2)\cosh\frac{1}{2}\sqrt{z_2}\,T \\ \quad + 2\sigma\sqrt{z_1}\sinh\frac{1}{2}\sqrt{z_1}\,T & \quad + 2\sigma\sqrt{z_2}\sinh\frac{1}{2}\sqrt{z_2}\,T \\ \sqrt{z_1}(\sigma^2 + \omega_0^2 + z_1)\sinh\frac{1}{2}\sqrt{z_1}\,T & \sqrt{z_1}(\sigma^2 + \omega_0^2 + z_2)\sinh\frac{1}{2}\sqrt{z_2}\,T \\ \quad + 2\sigma z_1\cosh\frac{1}{2}\sqrt{z_1}\,T & \quad + 2\sigma z_2\cosh\frac{1}{2}\sqrt{z_2}\,T \end{vmatrix} (z_1 - z_2)^{-1} \tag{12}$$

$$\Delta_2(z_1) = \begin{vmatrix} (\sigma^2 + \omega_0^2 + z_1)\left(\sinh\sqrt{z_1}\frac{T}{2}\right)\Big/\sqrt{z_1} & (\sigma^2 + \omega_0^2 + z_2)\left(\sinh\sqrt{z_2}\frac{T}{2}\right)\Big/\sqrt{z_2}\,T \\ \quad + 2\sigma\cosh\frac{1}{2}\sqrt{z_1}\,T & \quad + 2\sigma\cosh\frac{1}{2}\sqrt{z_2}\,T \\ (\sigma^2 + \omega_0^2 + z_1)\cosh\frac{1}{2}\sqrt{z_1}\,T & (\sigma^2 + \omega_0^2 + z_2)\cosh\frac{1}{2}\sqrt{z_2}\,T \\ \quad + 2\sigma\sqrt{z_1}\sinh\frac{1}{2}\sqrt{z_1}\,T & \quad + 2\sigma\sqrt{z_2}\sinh\frac{1}{2}\sqrt{z_2}\,T \end{vmatrix} (z_1 - z_2)^{-1} \tag{13}$$

with

$$z_1 = R + (R^2 - S)^{1/2}, \qquad z_2 = R - (R^2 - S)^{1/2}$$

and

$$R = (\sigma^2 - \omega_0^2) + s\sigma\langle I \rangle, \qquad S = (\sigma^2 + \omega_0^2)(\sigma^2 + \omega_0^2 + 2s\sigma\langle I \rangle) \tag{14}$$

For comparison sake, it is instructive to have the explicit functional form of this g.f. due to Mehta and Gupta [3] and it is given by,

$$\begin{aligned} Q(s) = 4\sigma^2(\sigma^2 + \omega_0^2)e^{2\sigma T}[&M(z_1, z_2)N(z_1, z_2) + (L_+(z_1, z_2) \\ &+ L_-(z_1, z_2))M(z_1, z_2) + (L_+(z_1, z_2) - L_-(z_1, z_2))N(z_1, z_2) \\ &+ (L_+^2(z_1, z_2) - L_-^2(z_1, z_2))]^{-1}\sqrt{z_1 z_2} \end{aligned} \tag{15}$$

where

$$L_{\pm}(z_1, z_2) = [2\sigma^2\sqrt{z_1 z_2} \pm 1/2(\sigma^2 + \omega_0^2 + z_1)(\sigma^2 + \omega_0^2 + z_2)] \times \sinh \tfrac{1}{2}(\sqrt{z_1} \pm \sqrt{z_2})T/(\sqrt{z_1} \pm \sqrt{z_2}),$$

$$M(z_1, z_2) = 2\sigma[\sqrt{z_1 z_2} \sinh \tfrac{1}{2}\sqrt{z_1}\, T \sinh \tfrac{1}{2}\sqrt{z_2}\, T + (\sigma^2 + \omega_0^2) \cosh \tfrac{1}{2}\sqrt{z_1}\, T \cosh \tfrac{1}{2}\sqrt{z_2}\, T],$$

$$N(z_1, z_2) = 2\sigma[(\sigma^2 + \omega_0^2) \sinh \tfrac{1}{2}\sqrt{z_1}\, T \sinh \tfrac{1}{2}\sqrt{z_2}\, T + \sqrt{z_1 z_2} \cosh \tfrac{1}{2}\sqrt{z_1}\, T \cosh \tfrac{1}{2}\sqrt{z_2}\, T] \tag{16}$$

We notice from equations (15) and (16) above that for short counting time approximation the correct form of the g.f. should have been,

$$Q(s) = (1 + 2\sigma T)\{1 + \sigma T + 1/4\sigma^{-1}T[(\sigma^2 + \omega_0^2) + (z_1 + z_2) + z_1 z_2(\sigma^2 + \omega_0^2)^{-1}]\}^{-1} \tag{17}$$

rather than the following form arrived at by Mehta and Gupta [3] where the basic parameter characterising the symmetric two-peaked spectrum namely ω_0, has been completely sacrificed (!),

$$Q(s) = 1/(1 + s\langle I \rangle T) \tag{17'}$$

Now to test the single-valuedness of the g.f. given by equations (10)–(17), let us write,

$$z_1 = re^{i\theta} \quad \text{and} \quad z_2 = z_1^* = re^{-i\theta} \tag{18}$$

to account for the situation when $R^2 < S$ in equation (14). If we go around a complete circle about the origin in the complex-plane, we find that both $\Delta_1(z_1)$ (equation (12)) and $\Delta_2(z_1)$ (equation (13)) remain unchanged as $\sqrt{z_1} \to -\sqrt{z_1}$ and $\sqrt{z_2} \to -\sqrt{z_2}$. Also we notice that the product $\Delta_1(z_1)\Delta_2(z_1)$ is symmetric with respect to the variables z_1 and z_2. Consequently, this g.f. given by Mehta and Gupta [3] promises to be an analytic function in the entire complex-plane. 'However, the analyticity of a given g.f. does not automatically ensure its suitability to describe the physics of the PCS stated in equations (1)–(3)'. To prove this fact, let us see whether the g.f. given by equation (17) gives us the correct mean count or not. On differentiation we find from equation (17) for short counting time,

$$(-1)\frac{\partial Q(s)}{\partial s}\bigg|_{s=0} = \langle n \rangle/(1 + 2\sigma T) \tag{19}$$

We notice from equation (19) above that for the Gaussian field where the half width 'σ' is invariably large, the g.f. function due to Mehta and Gupta [3] does not even give us the correct mean count.

Thus, we find that this g.f. due to Mehta and Gupta [3] is a *suspect function* for describing the PCS of Gaussian light with a symmetric two-peaked spectrum satisfactorily.

Next, we analyse the form of the g.f. given by Singh and Srinivasan in [4]. Using the notations that of Blake and Barakat [7], we write the auto-correlation as follows:

$$g(\tau) = \alpha_1 e^{-\beta\tau} + \alpha_2 e^{-\beta^*\tau}$$

where

$$\beta = \delta - i\Delta, \qquad \beta^* = \delta + i\Delta, \qquad \alpha_1 = \alpha_2 = 1/2. \tag{20}$$

Using Slepian's method [9] of solving the Fredholm integral equation for a *rational-spectrum*, we get the following differential equation for the two-peaked spectrum defined by equation (20):

$$D^4\phi - AD^2\phi + B\phi = 0 \qquad (D \equiv d/dt) \tag{21}$$

where

$$A = (\beta^2 + \beta^{*2}) - 2s\xi(\alpha_1\beta + \alpha_2\beta^*),$$
$$B = \beta\beta^*[\beta\beta^* - 2s\xi(\alpha_1\beta^* + \alpha_2\beta)],$$

and

$$\xi = 1/\lambda. \tag{22}$$

Extending the method of [2] for obtaining the F-Ds, we get the following form of the F-D for a two-peaked spectrum given by equation (20):

$$D(\xi) = \alpha_1^2\alpha_2^2 \begin{vmatrix} R(p_1) & R(-p_1)E(p_1) & R^*(p_1) & R^*(-p_1)E^*(p_1) \\ R(-p_1) & R(p_1)E(-p_1) & R^*(-p_1) & R^*(p_1)E^*(-p_1) \\ R(p_2) & R(-p_2)E(p_2) & R^*(p_2) & R^*(-p_2)E^*(p_2) \\ R(-p_2) & R(p_2)E(-p_2) & R^*(-p_2) & R^*(p_2)E^*(-p_2) \end{vmatrix} \tag{23}$$

where

$$\begin{aligned} &R(p) = (\beta + p)Z^*(p),\ R^*(p) = (\beta^* + p)Z(p), \\ &E(p) = \exp[-(\beta + p)T],\ E^*(p) = \exp[-(\beta^* + p)T], \\ &Z(p) = (\beta^2 - p^2) \text{ and } Z^*(p) = (\beta^{*2} - p^2) \end{aligned} \tag{24}$$

The values of 'p' can be easily determined by equations (21) and (22) and are given by

$$p = \pm[A \pm \sqrt{A^2 - 4B}]^{1/2}/\sqrt{2} \tag{25}$$

A function

$$P(\xi) = D(\xi)/p_1p_2 \tag{26}$$

was obtained earlier in [4] for the symmetric spectrum defined by equation (20).

The function $P(\xi)$ has the following explicit functional form:

$$P(\xi) = e^{-(\beta+\beta^*)T}\{\beta\beta^*[Z^{*2}(p_1)Z^2(p_2)F(p_1)F^*(p_2) + Z^2(p_1)Z^{*2}(p_2)F^*(p_1)F(p_2)] - 2[\delta^2 G(p_1)1)G(p_2) + \Delta^2] \times Z(p_1)Z(p_2)Z^*(p_1)Z^*(p_2)\} \tag{27}$$

where

$$F(p) = [\cosh pT + \tfrac{1}{2}(\beta/p + p/\beta)\sinh pT],$$

$$F^*(p) = [\cosh pT + \tfrac{1}{2}(\beta^*/p + p/\beta^*)\sinh pT],$$

and

$$G(p) = \cosh pT + [(\beta\beta^* + p^2)/p(\beta + \beta^*)]\sinh pT \tag{28}$$

Let's see whether the g.f. obtained from this $P(\xi)$ in [4]

$$Q_0(s) = P(0)/P(-\langle I\rangle s) \qquad (\text{'}o\text{' implies old}) \tag{29}$$

satisfies equations (1)–(3) or not. We first note from equations (26)–(29) the value of the g.f. $Q_0(s)$ for $T = 0$, is given by

$$Q_0(s)|_{T=0} = (\beta^{*2} - \beta^2)^2/(p_1^2 - p_2^2)^2 \tag{30}$$

whereas equation (1) suggests that we should have got $Q_0(s)|_{T=0} = 1$! Next, on differentiating once we get,

$$(-1)\partial Q(s)/\partial s|_{S=0} = \langle n\rangle + 2\langle I\rangle(\beta^* + \beta)^{-1} \tag{31}$$

thus violating equation (3) as $2\langle I\rangle(\beta^* + \beta)^{-1}$ is always positive.

Thus, as discussed in [2, p. 1295], this g.f. $Q_0(s)$ could as well be a *non-analytic* function and this is indicated by the denominator in equation (30) which goes to zero whenever $p_1 = \pm p_2$ in the complex-plane. We investigate these facts below.

It can be easily seen from equations (26)–(28) that the function $P(\xi)$ remains unchanged under the following conditions:

(i) $p_1 \leftrightarrow p_2$ and (ii) $p_1 \to -p_2$ or $p_2 \to -p_1$

So far so good, but if we were to allow for the change $p_1 = \pm p_2$, *the F-D* $D(\xi)$ *in equation* (23) *becomes zero*! The points at which this $D(\xi)$ becomes zero are determined by the following equation,

$$A^2 - 4B = 0 \tag{32}$$

Equation (32) leads to the solution of the following equation for ξ:

$$(\beta + \beta^*)\xi^2 - 2(\beta + \beta^*)^2\xi + (\beta + \beta^*)(\beta^2 - \beta^{*2}) = 0 \tag{33}$$

Thus, whenever,

$$\xi = 2\frac{\Delta}{\delta}(\Delta + \sqrt{(\delta^2 + \Delta^2)}), \quad \text{or} \quad 2\frac{\Delta}{\delta}(\Delta - \sqrt{(\delta^2 + \Delta^2)}), \tag{34}$$

the F-D D(ξ) in equation (23) becomes zero.

3. The unique analytic g.f.

In this section, we shall show how the conditions stated in equations (1)–(3) help us in modifying the $P(\xi)$ (equation (28)) so as to make it the unique analytic function in the entire complex $-\xi$ plane. To counter the zero values of the function $P(\xi)$, it suffices to divide it by the factor $(p_1^2-p_2^2)$. But we see that even $P(\xi)/(p_1^2-p_2^2)$ does not lead to satisfying equation (3). Thus, we need to divide the function $P(\xi)$ by some function $f(\xi)$ which not only removes its zeros but also makes it satisfy equations (1)–(3). For this, let us define,

$$M(\xi)=P(\xi)/f(\xi) \tag{35}$$

to be an entire function of order $\leqslant 1$ so that the Hadamard's Theorem [10] could be applied. The corresponding g.f. is now given by,

$$Q_N(s)=M(o)/M(-\langle I\rangle) \tag{36}$$

where the suffix 'N' implies the new. On differentiation, we should get,

$$\left\{(-1)\frac{\partial Q_N(s)}{\partial s}=\frac{1}{P(o)}\frac{\partial P}{\partial s}-\frac{1}{f(o)}\frac{\partial f}{\partial s}\right\}_{|s=0}=\langle n\rangle \tag{37}$$

as required by the condition in equation (3). But we have from equation (30),

$$\left.\frac{1}{P(o)}\frac{\partial P}{\partial s}\right|_{s=0}=\langle n\rangle+2\langle I\rangle/(\beta+\beta^*) \tag{38}$$

Thus, the function $f(\xi)$ must satisfy the following *partial differential equation*:

$$\{\partial f/\partial p_1+\partial f/\partial p_2\}_{\substack{|p_1=\beta\\ |p_2=\beta^*}}=4f(o)/(\beta+\beta^*), \tag{39}$$

as can be easily seen from equations (25), (37) and (38). 'It is interesting to observe that the requirement of the F-D to be analytic in the entire ξ-plane, leads to solving a partial differential equation in the case of a two-peaked spectrum.'

Equation (39) coupled with the condition that

$$\{f(p_1,p_2)_{|p_1=\pm p_2}\}=0 \tag{40}$$

will provide us with the complete solution. There are several functions which satisfy equation (39) or equation (40) but we have the following functions which satisfy both the equations (39) and (40):

$$f(p_1,p_2)=[(p_1^2-p_2^2)/(\beta^2-\beta^{*2})]\exp[(p_1^2-p_2^2)/(\beta^2-\beta^{*2})] \qquad [f(o)=e^1] \tag{41}$$

and

$$f(p_1,p_2)=(p_1^2-p_2^2)^2/(\beta^2-\beta^{*2})^2 \qquad [f(o)=1] \tag{42}$$

Though the function given by equation (41) gives the correct mean count, however, we find that it is the choice in equation (42) which satisfies all the other requirements, namely,

(i) it does not change sign whenever $p_1 \leftrightarrow p_2$, or $p_1 \rightarrow -p_1$, or $p_2 \rightarrow -p_2$; (ii) it satisfies equations (1)–(3). The choice of equation (41) clearly does not satisfy the condition (i) above and also equation (1).

Thus, the new g.f. $Q_N(s)$ must read as follows:

$$Q_N(s) = Q_0(s)(p_1^2 - p_2^2)^2/(\beta^2 - \beta^{*2})^2 \tag{43}$$

where $Q_0(s)$ is given by equations (27)–(29).

A direct comparison of equations (13)–(14) with equations (27)–(29) and (43) clearly shows that the functional form of the g.f. given by Mehta and Gupta [3] is in fact quite different from the correct one derived here. As an important measure of the g.f., we evaluate the probability of zero counts defined as,

$$P(o, T) = Q_N(s)_{|s=1} \tag{44}$$

A comparative study of the values of the probability of zero counts $P(o, T)$,

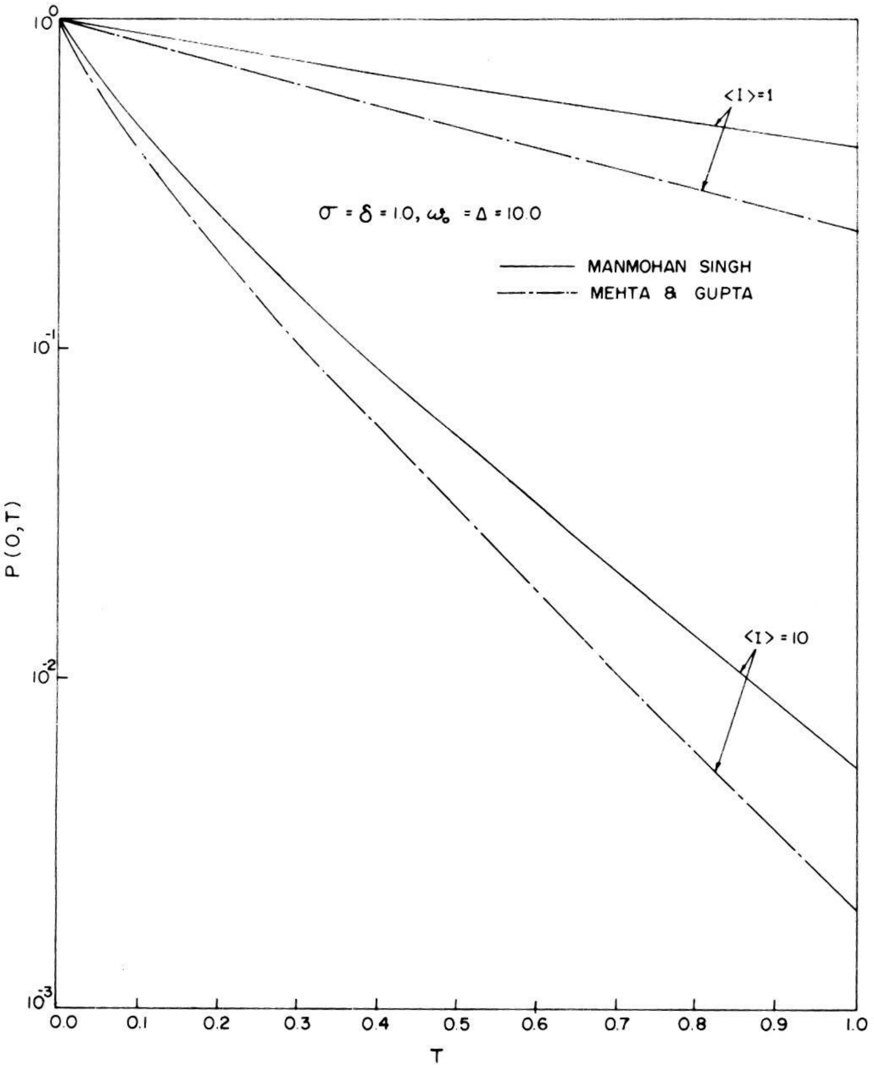

Figure 1

A comparative study of the probability of zero counts $P(o, T)$ as a function of the parameters characterizing the symmetric two-peaked spectrum. The average error due to the formulation of Mehta and Gupta [3] increases rapidly as the counting time T and the mean count rate $\langle I \rangle$ increase.

as obtained from the formulation of Mehta and Gupta [3] and the one given in the present paper, is provided in Fig. 1. We notice an average difference of ~26% for $\langle I \rangle = 1$, ~41% for $\langle I \rangle = 10$, when the counting time T varies from 0 to 1. Thus in the light of the results given above and the equations (17) and (19) earlier, it is fairly evident that this g.f. due to Mehta and Gupta [3] holds very little promise to correctly describe the PCS of a symmetric two-peaked spectrum.

An interesting account of the various experimental techniques employed to study the Brillouin frequencies, is provided in the review article by Borovik-Romanov and Kreines [11]. The recent paper due to Simonsohn [12] which provides an interesting data on the time-interval-distribution in a Brillouin scattering experiment, is particularly relevant to the present work.

In a future publication, we will provide a detailed account of the time-interval-statistics based on the g.f. (equations (27)–(29)) derived here.

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