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Time-delay operator for a class of singular potentials

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Abstract. We prove the existence of the time-delay operator defined by taking the large space limit of the approximate sojourn times for a class of singular potentials: $V = V_1 + V_2$, where V_1 is a smooth short range potential and V_2 and $x \cdot \nabla V_2$ are both bounded from H^2 to $L^{2,2+\varepsilon_0}$ for some $\varepsilon_0 > 0$.

1. Introduction

In [8], we proved the finiteness of time-delay defined by taking the space limit of sojourn times and established its equivalence with Eisenbud–Wigner's time-delay in scattering theory for smooth short range potentials. In this work we will show that our method developed there can be also applied to a class of singular potentials.

Let $H_0 = -\Delta$ and $H = H_0 + V$ in $L^2(\mathbb{R}^n)$. We suppose that the short range potential V can be decomposed as: $V(x) = V_1(x) + V_2(x)$ where V_1 is C^{∞} on \mathbb{R}^n and for some $\varepsilon_0 > 0$

$$\left|\partial_x^{\alpha} V_1(x)\right| \le c_{\alpha} \langle x \rangle^{-1-\varepsilon_0 - |\alpha|} \tag{1.1}$$

and the multiplication by V_2 is bounded as operator from H^2 to $L^{2,2+\varepsilon_0}$ and so is the distributional derivative $x\nabla_x \cdot V_2$. Here $\langle x \rangle = (1+|x|)$ and H^s is the usual Sobolev space of order s; $L^{2,s}$ is the weighted L^2 space with the norm: $||f||_s = ||\langle x \rangle^s f||$. This assumption will be made throughout this work. Under this condition on V, it is well known that the wave operators W_{\pm} defined by:

$$W_{\pm} = s - \lim_{t \to \pm \infty} e^{itH} e^{-itH_0} \quad \text{in } L^2$$

exist and are complete. Let \tilde{P}_R denote the multiplication by the characteristic function for the ball $\{|x| < R\}$. Then the local time-delay of f in $\{|x| < R\}$ is defined as the difference of the sojourn times:

$$\langle f, T_R f \rangle = \int_{-\infty}^{\infty} (\|\tilde{P}_R e^{-itH} W_- f\|^2 - \|\tilde{P}_R e^{-itH_0} f\|^2) dt$$
 (1.2)

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Notice that (1.2) is well defined for $f \in L^2$ such that the Fourier transform \hat{f} has suitable compact support in $\mathbb{R}^n|_{\{0\}}$. Finally the time-delay operator T is defined by:

$$\langle f, Tf \rangle = \lim_{R \to +\infty} \langle f, T_R f \rangle$$
 (1.3)

whenever the limit exists. Surely the existence of time-delay operator depends on how much such f's we can find. As in [8], we consider here a similar question. Let P_R be the multiplication by P(x/R), where P(.) is a smooth, spherically symmetrical function such that P(x) = 1 for $|x| \le 1$ and P(x) = 0 for $|x| \ge 2$. It is clear that P_R can be regarded as an approximation of \tilde{P}_R . In the following we denote still T_R the operator defined by (1.2) with \tilde{P}_R replaced by P_R .

Then for smooth short range potentials we proved in [8] that the limit (1.3) exists for a dense subset in L^2 and in the spectral representation of H_0 , the time-delay operator T is given by a family of operators $T(\lambda)$, $\lambda > 0$, where

$$T(\lambda) = -iS(\lambda)^* \frac{d}{d\lambda} S(\lambda)$$
(1.4)

 $S(\lambda)$ being the scattering matrix. (1.4) is the Eisenbud-Wigner's formula for time-delay. It reveals that the method and techniques used in [8] are powerful enough. It can be applied to treat time-delay in other scattering theories (see [6]) and to include a class of singular potentials.

Let $A = -i(x \cdot \nabla_x + \nabla_x x)/2$ be the generator of dilation group. We define the set \mathcal{D} by:

$$\mathcal{D} = \{ f \in L^2; f \in D(\langle x \rangle) \cap D(A^2) \text{ and } \exists \chi \in C_0^{\infty}(\mathbb{R}_+ \setminus \sigma_p(H)), \, \chi(H_0)f = f \}$$

$$(1.5)$$

In this work we want to prove the following result.

Theorem 1. Under the above assumption on V, the limit (1.3) exists for T_R defined by (1.2) with \tilde{P}_R replaced by P_R and for $f \in \mathcal{D}$. We have:

$$\langle f, Tf \rangle = \langle f, -S^*[A, S]f_1 \rangle$$

where f_1 is determined by $2H_0f_1 = f$ and $S = W_+^*W_-$ is scattering operator. The time-delay operator T is essentially selfadjoint with core \mathcal{D} and the Eisenbud–Wigner formula (1.4) is true for $\lambda \in \mathbb{R}_+/\sigma_p(H)$.

The proof of this result consists in regarding V_2 as a perturbation of the Hamiltonian $H_1 = H_0 + V_1$. In §2, we give some technical preparations, which were mostly proved in [8]. In §3, we achieve the main step of the proof, reducing the existence of the limit (1.3) to that of $\lim_{R\to+\infty} \int_0^\infty \langle U_0(t)f, S^*P_RS - P_R)f \rangle dt$. We finish the proof of Theorem 1 in §4 by the method of [8]. Very recently Nakamura ([12]) considered the similar problem by a different method. His proof is in the spirit of Lavine [4], while ours is in that of Martin [5].

2. Some preparations

Let U(t) (resp. $U_0(t)$, $U_1(t)$) denote the unitary group associated to H (resp. H_0 , H_1). Let $E_{ac}(H_1)$ denote the spectral projector onto the absolute continuous space of H_1 . The wave operators W_{\pm}^1 , W_{\pm}^2 are defined by:

$$W_{\pm}^{1} = s - \lim_{t \to \pm \infty} U_{1}(t)^{*} U_{0}(t)$$
$$W_{\pm}^{2} = s - \lim_{t \to \pm \infty} U(t)^{*} U_{1}(t) E_{ac}(H_{1})$$

in $L^2(\mathbb{R}^n)$. By chain rule, $W_{\pm} = W_{\pm}^2 W_{\pm}^1$ (see [1]). Put: $\mathbb{R}_+ =]0, +\infty[$.

Lemma 2.1. Let
$$f \in C_0^{\infty}(\mathbb{R}_+/\sigma_p(H))$$
. Then for every $0 \le \mu \le 1$, one has:
 $\|\langle A \rangle^{-\mu} f(H) U(t) W_{\pm} \langle A \rangle^{-\mu} \| \le C(1+|t|)^{-\mu}$ (2.1)

for $t \in \mathbb{R}$. For every $\mu > 1$, there exists $\rho > 1$ such that

$$\|\langle A \rangle^{-\mu} f(H) U(t) W_{\pm} \langle A \rangle^{-\mu} \| \le c (1+|t|)^{-\rho} \quad \text{for } t \in \mathbb{R}$$

$$(2.2)$$

Note that this result is proved in [8] for $V = V_1(V_2 = 0)$. But the proof can be carried over, because we used only the short range properties of V and $x \cdot \nabla V$.

Recall that if $V = V_1 + V_2$ with V_1 satisfying (1.1) and V_2 bounded from H^2 to $L^{2,2+\varepsilon_0}$ it is proved in [3] that $S(\lambda)$ is continuously differentiable in $\mathcal{L}(L^2(S^{n-1}))$ for $\lambda \in \mathbb{R}_+/\sigma_p(H)$. Since under the assumptions of Theorem 1, $x \cdot \nabla V$ satisfies still the above conditions, it should be clear that by exterior scaling method, we can easily prove that $S(\lambda)$ is two times differentiable for $\lambda \in \mathbb{R}_+/\sigma_p(H)$. For reader's convenience we give the details of the proof.

Let $\lambda \in \mathbb{R}_+ / \sigma_p(H)$. Then we have the following representation for the scattering matrix $S(\lambda)$ (= $S(\lambda, V)$) ([11]):

$$S(\lambda, V) = 1 - i\pi \mathcal{F}(\lambda)(V - VR(\lambda + i0, V)V)\mathcal{F}(\lambda)^*$$

where $R(\lambda \pm i0, V)$ is the boundary values of the resolvent $(H_0 + V - z)^{-1}$ and $\mathscr{F}(.): L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}_+, L^2(S^{n-1}))$ is a spectral representation for the free Hamiltonian H_0 . Take a > 0 to be sufficiently small. Put: I =] -a, a[. We can prove that:

$$S(e^{2\theta}\lambda, V) = S(\lambda, V(\theta)) \text{ for } \theta \in I$$
 (S)

where $V(\theta) = e^{-2\theta}U(\theta)^*VU(\theta)$ and $U(\theta)$ is the unitary group generated by A. Now we check the derivability of $V(\theta)$ and $R(\lambda + i0, V(\theta))$ for $\theta \in I$. Let $H^{s,m}$ denote the weighted Sobolev space with the norm $||\langle x \rangle^m (1-\Delta)^{s/2} f||$. Put: $\rho = 1 + \varepsilon_0 > 1$. Then the assumptions on V say that $i[A, V_2]$ defines a bounded operator from $H^{s,r}$ to $H^{s-2,r+\rho+1}$ for $0 \le s \le 2$ and $r \in \mathbb{R}$. From this we derive that $A[A, V_2]$ and $[A, V_2]A$ are both bounded from $H^{s,r}$ to $H^{s-4,r+\rho}$ for $0 \le s \le 3$ and $r \in \mathbb{R}$. Since V_1 satisfies (1.1), we conclude easily from the above remarks that the operator valued function $\theta \mapsto V(\theta)$ is in the class

$$C^{1}(I; \mathscr{L}(H^{s,r}; H^{s-2,r+\rho+1})) \cap C^{2}(I; \mathscr{L}(H^{s+1,r}; H^{s-3,r+\rho}))$$

Since $R(\lambda \pm i0, V(\theta))$ is in $\mathcal{L}(H^{0,r}; H^{2,-r})$ if $r > \frac{1}{2}$ ([10]), we can also prove that the map $\theta \mapsto R(\lambda + i0, V(\theta))$ belongs to the class:

$$C^{1}(I; \mathscr{L}(H^{0,r}; H^{2,-r})) \cap C^{2}(I; \mathscr{L}(H^{0,r}; H^{0,-r}))$$

for $r > \frac{1}{2}$. This means that the map

$$I \ni \theta \mapsto \langle H_0 \rangle^{-2} V(\theta) (1 - R(\lambda + i0, V(\theta)) V(\theta)) \langle H_0 \rangle^{-2}$$

is in $C^2(I; \mathscr{L}(L^{2,-s}; L^{2,s}))$ for $s \in]\frac{1}{2}$, $\rho/2[$. Making use of the relation: $\mathscr{F}(\lambda)\langle H_0 \rangle^{-1} = \langle \lambda \rangle^{-1} \mathscr{F}(\lambda)$, we derive from (S) that $S(\lambda, V)$ is twice continuously differentiable in $\mathscr{L}(L^2(S^{n-1}))$ for $\lambda \in \mathbb{R}_+/\sigma_p(H)$. This proves our assertion.

Since in the spectral representation of H_0 , A is given by a family of operators $A(\lambda) = i(\lambda d/d\lambda + d/d\lambda \cdot \lambda)$, we conclude that the domain of A^2 is invariant by $Sf(H_0)$ for $f \in C_0^{\infty}(\mathbb{R}_+/\sigma_p(H))$. Now we can easily prove the following lemma which is important in this work.

Lemma 2.2. Let \mathcal{D} be defined by (1.5). Then \mathcal{D} is invariant by S. In particular if f belongs to \mathcal{D} , $\langle x \rangle Sf$ and A^2Sf are both in $L^2(\mathbb{R}^n)$.

Proof. It remains to show that $\langle x \rangle Sf$ is in L^2 . We can use the same commutator method as in the proof of Prop. 4.2 in [8]. The details are omitted here.

In order to regard V_2 as a perturbation to H_1 , we need some continuity of wave operators W_{\pm}^1 .

Lemma 2.3. Let $f \in C_0^{\infty}(\mathbb{R}_+)$. Under the condition (1.1) on V_1 , the four operators $A^2W_{\pm}^1f(H_0)\langle A \rangle^{-2}$ and $A^2W_{\pm}^{1*}f(H_1)\langle A \rangle^{-2}$ are all bounded on L^2 .

Proof. We prove only the result for $A^2 W^{1*}_+ f(H_1) \langle A \rangle^{-2}$. The other cases can be treated in the same way. Put $W(t) = U_0(t)^* U_1(t)$. We can write, as forms on $D(A) \times D(A)$,

$$AW(t)f(H_1) = W(t)f(H_1)A + U_0(t)^*[A, f(H_1)]U_1(t) + 2tU_0(t)V_1U_1(t)f(H_1) + W(t)\int_0^t U_1(s)^*f(H_1)\tilde{V}U_1(s) \, ds \qquad (2.3)$$

where $\tilde{V} = i[A, V_1] - 2V_1$. Notice that $i[A, f(H_1)] = 2f'(H_1) + Q$ with Q bounded from L^2 to $L^{2,1+\varepsilon_0}$ (see [8]). Now we need the following result due to Jensen et al.:

$$\|\langle A \rangle^{-r} f(H_1) U_1(t) \langle A \rangle^{-r} \| \le c(1+|t|)^{-r+\varepsilon} \quad t \in \mathbb{R}$$

$$(2.4)$$

for every r > 0 and $0 < \varepsilon < r$. Take $g \in C_0^{\infty}(\mathbb{R}_+)$ such that g = 1 on supp f. Multiplying (2.3) by $g(H_1)\langle A \rangle^{-2}$ and taking the limit $t \to +\infty$, applying (2.4), we get:

$$AW_{+}^{1*}f(H_{1})\langle A \rangle^{-2} = W_{+}^{1*}f(H_{1})Ag(H_{1})\langle A \rangle^{-2} + W^{1*}h(H_{1})\langle A \rangle^{-2} + W_{+}^{1*}\int_{0}^{+\infty} f(H_{1})U(-s)\tilde{V}U(s)g(H_{1})\langle A \rangle^{-2} dt$$
(2.5)

where h = 2if'g. Since $AW_{+}^{1*}f(H_1)\langle A \rangle^{-1}$ is bounded on L^2 , in order to prove the desired result by (2.5), it is sufficient to show that

$$\int_{0}^{+\infty} Af(H_1) U_1(-s) \tilde{V} U_1(s) g(H_1) \langle A \rangle^{-2} \, ds \tag{2.6}$$

is bounded on L^2 . To simplify notations, we denote f, g the operators $f(H_1)$, $g(H_1)$ respectively. We have the following relation:

$$[A, U_{1}(-s)g\tilde{V}gU_{1}(s)] = -2sU_{1}(-s)g[H_{0}, \tilde{V}]gU_{1}(s) + U_{1}(-s)[A, g\tilde{V}g]U_{1}(s) - \int_{0}^{s} U_{1}(t-s)\tilde{V}gU_{1}(-t)\tilde{V}U_{1}(s)g dt + \int_{0}^{s} U_{1}(-s)g\tilde{V}U_{1}(s-t)g\tilde{V}U_{1}(t) dt$$
(2.7)

Since $g[H_0, \tilde{V}]$ is continuous from $L^{2,r}$ to $L^{2,r+2+\varepsilon_0}$, we can prove as in [8] (Prop. 4.2) that $\int_0^{+\infty} 2sf U_1(-s)[H_0, \tilde{V}]U_1(s)g\langle A \rangle^{-1} ds$ is bounded on $L^2(\mathbb{R}^n)$. Since $[A, g\tilde{V}g]$ is bounded as operator from $L^{2,r}$ to $L^{2,r+1+\varepsilon_0}$, it follows from (2.4) that:

$$\left\| [A, g\tilde{V}g] U_1(s)g_1\langle A \rangle^{-2} \right\| \leq C(1+|s|)^{-1-\varepsilon_0/2}$$

for $s \in \mathbb{R}$. Here $g_1 = g_1(H_1)$ is chosen so that $g_1g = g$. Therefore the integral $\int_0^{\infty} fU_1(-s)[A, g\tilde{V}g]U_1(s)g_1\langle A \rangle^{-2} ds$ defines a bounded operator on L^2 . To treat the last two terms in (2.7), we use the local H_1 -smoothness of $\langle x \rangle^{-1/2-\varepsilon}$, which implies that the operator $\int_0^s fU_1(t)\tilde{V}U_1(-t)g dt$ is uniformly bounded with respect to $s \in \mathbb{R}$. Applying (2.4), we get the estimate over the third term in (2.7):

$$\left\|\int_{0}^{s} fU_{1}(t-s)\tilde{V}gU_{1}(-t)\tilde{V}U_{1}(s)g_{1}\langle A\rangle^{-2} dt\right\| \leq C(1+|s|)^{-1-\varepsilon_{0}/2}$$

The last term in (2.7) can be estimated in the same way. Since the commutator [A, f] is bounded, we derive from (2.7) that (2.6) is a bounded operator on L^2 . This proves that $A^2W_+^{1*}f(H_1)\langle A \rangle^{-2}$ is bounded. The lemma is proved.

3. Reduction of the problem

In the proof of the finiteness of time-delay, an important step is to show that

$$\lim_{R \to +\infty} \left(\langle f, T_R f \rangle - \int_0^{+\infty} \langle U_0(t) f, (S^* P_R S - P_R) U_0(t) f \rangle dt \right) = 0$$
(3.1)

(3.1) makes also clear the close relationship between time-delay operator T and scattering operator S. In this section we will prove the following result which implies (3.1).

Theorem 3.1. Let $f \in \mathcal{D}$. Put g = Sf. Then we have: $\lim_{R \to +\infty} \int_{-\infty}^{0} (\langle U_0(t)f, (W_-^* P_R W_- - P_R) U_0(t)f \rangle) dt = 0 \qquad (3.2)$ $\lim_{R \to +\infty} \int_{0}^{\infty} (\langle U_0(t)g, (W_+^* P_R W_+ - P_R) U_0(t)g \rangle) dt = 0 \qquad (3.3)$

Proof. Since the set \mathcal{D} is invariant by S, it suffices to prove (3.2). Put $h = W_{-}^{1}f$, which is in $D(A^{2})$ by Lemma 2.3. The integrand in (3.2) can be written as: $\langle U_{1}(t)h, (W_{-}^{2*}P_{R}W_{-}^{2} - P_{R})U_{1}(t)h \rangle + \langle U_{0}(t)f, (W_{-}^{1*}P_{R}W_{-}^{1} - P_{R})U_{0}(t)f \rangle$. It is proved in [8] by method of pseudo-differential operators that for $f \in \mathcal{D}$, we have:

$$\lim_{R \to +\infty} \int_{-\infty}^{0} \langle U_0(t)f, (W_-^{1*}P_R W_-^1 - P_R) U_0(t)f \rangle dt = 0$$

Therefore we have to prove

$$\lim_{R \to +\infty} \int_{-\infty}^{0} \langle U_1(t)h, (W_-^{2*}P_R W_-^2 - P_R)U_1(t)h \rangle dt = 0$$
(3.4)

The integrand in (3.4) can be written as:

$$\langle P_R U(t) W_{-}f, (U(t) W_{-}^2 - U_1(t))h \rangle$$

+ $\langle (U(t) W_{-}^2 - U_1(t))h, P_R U_1(t)h \rangle$ (3.5)

Take $\chi \in C_0^{\infty}(\mathbb{R}_+/\sigma_p(H))$ such that $\chi(H_0)f = f$. We get:

$$(U(t)W_{-}^{2} - U_{1}(t))h = -\int_{-\infty}^{t} \chi(H)U(t-s)V_{2}U_{1}(s)\chi(H_{1})h \, ds$$
$$+ (\chi(H) - \chi(H_{1}))U_{1}(t)h$$

By the assumption, $V_2\chi(H_1)$ is continuous from $L^{2,-\varepsilon_0}$ to $L^{2,2}$. We can easily prove that $\chi(H) - \chi(H_1)$ is continuous from $L^{2,-1-\varepsilon_0}$ to L^2 . Thus the first term in (3.5) can be estimated as:

$$\begin{aligned} \left| \left\langle P_{R}U(t)W_{-}f, \left(U(t)W_{-}^{2}-U_{1}(t)\right)h \right\rangle \right| \\ &\leq c \int_{-\infty}^{t} (1+|s|)^{-2+\varepsilon} \left\| \left\langle x \right\rangle^{-\varepsilon_{0}} \chi(H)U(s-t)P_{R}U(t)\chi(H)W_{-}f \right\| \left\| \left\langle A \right\rangle^{2} f \right\| ds \\ &+ c (1+|t|)^{-1-\varepsilon_{0}} \left\| \left\langle A \right\rangle^{2} f \right\|^{2}, \quad \text{for any} \quad \varepsilon > 0 \end{aligned}$$

$$(3.6)$$

Here we have used (2.4) and Lemma 2.3. Before going on with the proof of Theorem 3.1, we need still a lemma.

Lemma 3.2. For every $0 \le \mu \le 1$, we have:

$$\|\langle x \rangle^{-\mu} \chi(H) U(s-t) P_R U(t) W_{-f} \| \leq C (1+|s|)^{-\mu} \|\langle A \rangle W_{-f} \|$$
(3.7)

uniformly in $t \in \mathbb{R}$ and $R \ge 1$.

Proof. Observe first that $|\partial_x^{\alpha} P_R(x)| \leq c \langle x \rangle^{-|\alpha|}$ uniformly in $R \geq 1$. By the arguments used in the proof of Lemma 2.3, we can show that:

$$\|A\chi(H)U(-t)P_RU(t)\chi(H)\langle A\rangle^{-1}\| \leq C$$

uniformly in $t \in \mathbb{R}$ and $R \ge 1$. (3.7) follows from (2.1) and the fact that $\langle A \rangle \chi(H) \langle x \rangle^{-1}$ is bounded on L^2 .

Now return to the proof of Theorem 3.1. By (3.6) and (3.7) we obtain, for $\varepsilon > 0$ sufficiently small and for t < 0,

$$|\langle P_R U(t) W_{-}f, (U(t) W_{-}^2 - U_1(t))h\rangle| \le c(1+|t|)^{-1-\varepsilon_0/2} ||\langle A \rangle^2 f||^2$$

uniformly in $R \ge 1$. The same estimate is also true for the second term in (3.5). Since the integrand in (3.4) tends to 0 as R tends to $+\infty$, by the dominated convergence theorem, (3.4) is proved. This finishes the proof of Theorem 3.1.

We remark that the various constants c appeared in the proof of Theorem 3.1 depend on the function χ , but not $f \in \mathcal{D}$ such that $\chi(H_0)f = f$.

4. Existence of time-delay operator

In this section we prove Theorem 1. Let $\chi \in C_0^{\infty}(\mathbb{R}_+/\sigma_p(H))$. We put:

 $\mathcal{D}_{\chi} = \{ f \in \mathcal{D}; \chi(H_0) f = f \}$

Lemma 4.1. Let $f \in \mathcal{D}_{\gamma}$. Then,

$$\int_{R^{5/2}}^{+\infty} |\langle U_0(t)f, (S^*P_RS - P_R)U_0(t)f\rangle| dt \le CR^{-1/2} ||\langle A\rangle f||^2$$

Proof. It follows easily from the estimate:

 $||P_R^{1/2}U_0(t)f|| \leq C_{\chi}R(1+|t|)^{-1} ||\langle A\rangle f||$

for $t \in \mathbb{R}$, $R \ge 1$ and $f \in \mathcal{D}_{\chi}$.

Lemma 4.2. For $f \in \mathcal{D}_{\chi}$, we have the asymptotic expansion:

$$\int_{0}^{R^{3/2}} \langle U_0(t)f, P_R U_0(t)f \rangle dt = R \langle f, a(D)f \rangle$$

- $\langle f, b^w(x, D; \chi)f \rangle + O(R^{-1/2})$ (4.1)

where $a(\xi) = c_0 |\xi|^{-1} \chi(|\xi|^2)$ with $c_0 = \frac{1}{2} \int_0^\infty P(s) ds$, $b^w(x, D; \chi)$ is a Weyl pseudodifferential operator with symbol $\frac{1}{2}x\xi |\xi|^{-2} \chi(|\xi|^2)$. The remainder $O(R^{-1/2})$ can be estimated by

$$|O(R^{-1/2})| \le C_{\chi}R^{-1/2}(||\langle x\rangle f|| + ||\langle A\rangle^2 f||)^2 \quad R \ge 1$$

Proof. We use the fact that $U_0(-t)P_RU_0(t)$ is a Weyl pseudo-differential operator with symbol $P((x + 2t\xi)/R)$ and develop the symbol around $2t\xi/R$. Since this result is proved in [8] (Proposition 4.3) for $f \in D(\langle x \rangle) \cap D(\langle A \rangle^3)$, we indicate only the difference and omit the details. Checking the proof of Proposition 4.3([8]), we see that the condition $f \in D(\langle A \rangle^3)$ is only used to get the estimate (see (4.14)[8]):

$$\frac{t}{R^3} \|Q^w(\tau x/R, D)U_0(t/\tau)f\| \le CR^{-1/2}t^{-3/2} \|\langle A \rangle^3 f\|$$

for $t \ge 1$, $\tau \in [0, 1]$ and $R \ge 1$, where $Q(x, \xi) = P_1(x)\chi(|\xi|^2)$, $P_1(x)$ is some derivative of P(x). Hence it is supported in $\{1 \le |x| \le 2\}$. But this term can be equally estimated as follows: Since all the derivatives of $\tau^2 R^{-2}Q(\tau x/R, \xi)$ are bounded by a constant times $\langle x \rangle^{-2}$ uniformly with respect to $R \ge 1$ and $\tau \in [0, 1]$, we have, by continuity result of pseudo-differential operators ([2]),

$$\frac{t}{R^3} \| Q^w(\tau x/R, D) U_0(t/\tau) f \|$$

$$\leq c R^{-1} \tau^{-2} t \| \langle x \rangle^{-2} U_0(t/\tau) f \| \leq c_{\chi} R^{-1} t^{-1} \| \langle A \rangle^2 f \|$$
(4.2)

for $t \ge 1$, $R \ge 1$ and $\tau \in (0, 1)$. Integrating (4.2) over $[1, R^{5/2}]$, we get:

$$\int_0^{R^{5/2}} tR^{-3} \|Q^w(\tau x/R, D)U_0(t/\tau)f\| dt \le C_{\chi}R^{-1/2} \|\langle A \rangle^2 f\|$$

This gives the desired result. The lemma is proved.

Now we are able to give the proof of Theorem 1.

Proof of Theorem 1. Let f be in \mathcal{D} . Then Sf is also in \mathcal{D} . Take $\chi \in C_0^{\infty}(\mathbb{R}_+/\sigma_p(H))$ such that $\chi(H_0)f = f$. Since S commutes with a(D), we derive from Lemmas 4.1 and 4.2 that

$$\int_0^\infty \langle U_0(t)f, S^*[P_R, S]U_0(t)f \rangle dt + \langle f, S^*[b^w(x, D; \chi), S]f \rangle \bigg|$$

$$\leq C_f R^{-1/2}$$
(4.3)

By a simple calculus of Weyl pseudo-differential operators ([2]), we get:

$$\langle f, S^*[b^w(x, D; \chi), S]f \rangle = \langle f_1, S^*[A, S]f \rangle$$
(4.4)

where $f_1 = (2H_0)^{-1}\chi(H_0)f$. Theorem 1 is a consequence of (4.3) and (4.4). See also [8].

REFERENCES

[1] W. O. AMREIN, J. M. JAUCH and K. D. SINHA, Scattering Theory in Quantum Mechanics, Reading, Benjamin Inc., 1977.

- [2] L. HÖRMANDER, The Weyl calculus of pseudo-differential operators, Comm. Pure Appl. Math., 32 (1979), 359-443.
- [3] A. JENSEN, A stationary proof of Lavine's formula for time-delay, Lett. in Math. Phys., 7 (1983), 137-143.
- [4] R. LAVINE, Commutators and local decay, in *Scattering Theory in Mathematical Physics*, pp. 144–156, ed. J. A. La Vita, J. P. Marchand; Reidel, Dordrecht, 1974.
- [5] PH. MARTIN, On the time-delay of simple scattering systems, Commun. Math. Phys., 47 (1976), 221-227.
- [6] D. ROBERT, X. P. WANG, Time-delay and spectral density for Stark Hamiltonians, in preparation.
- [7] X. P. WANG, Opérateurs de temps-retard dans la théorie de la diffusion, C.R. Acad. Sci. Paris, Sér. I, 301 (1985), 789–791.
- [8] X. P. WANG, Phase-space description of time-delay in scattering theory, to appear.
- [9] X. P. WANG, Low energy resolvent estimates and continuity of time-delay operators, to appear in Proc. Royal Soc. Edinburgh.
- [10] S. AGMON, Spectral properties of Schrödinger operators and scattering theory, Ann. Scuol. Norm. Pisa, 4 (1975), 151–218.
- [11] S. T. KURODA, Scattering theory for differential operators, I, operator theory, J. Math. Soc. Japan, 25 (1973), 75-104.
- [12] S. NAKAMURA, Time-delay and Lavine's formula, preprint.