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On the viscous damping of the magneto-acoustic oscillations

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Abstract. Radial magneto-acoustic oscillations in cold, bounded, cylindrical plasmas are investigated, taking account for the viscous losses. It is shown, that strong viscous stresses may arise in such plasmas nearby the wall, where the radial component of the mass velocity drops quickly to zero. Viscous dissipation connected with this boundary effect proves to be more pronounced than in the infinite plasmas with similar parameters. Some discrepancies between theory and experiment, described in the earlier papers, are interpreted as due to neglect of the viscous losses.

1. Introduction

Magneto-acoustic oscillations in bounded, cylindrical plasmas were investigated in a detailed way in a number of papers (see e.g. Cantieni, [3]; Elmiger, [5]). However, it seems that the viscous mechanism of their damping was never accounted for. (Some estimates of its role, without consideration of the boundary effects, were given recently for a particular case of the high-temperature, fusion-plasmas by Kapitza, [6]).

For a plane, low-frequency magneto-acoustic wave propagating in an infinite medium the attenuation coefficient due to both Joule and viscous losses is (Landau, [9])

$$\kappa = \frac{\omega^2}{2c_A^3}\left(\frac{4}{3}\nu_h + \nu_m\right), \tag{1}$$

where $\nu_h = \eta/\rho$, $\nu_m = c^2/4\pi\sigma$, c_A Alfvén velocity, c light velocity, η viscosity coefficient, σ plasma conductivity, ρ mass density and ω is the oscillation frequency. Thus, the ratio of the attenuation coefficients connected with viscous and Joule losses is given by $4\nu_h/3\nu_m$ or, for a strongly ionized plasma, by

$$\frac{4}{3}\frac{\nu_h}{\nu_m} = \frac{3 \cdot 10^{13} T^4}{A^{1/2} n \Lambda^2}, \tag{2}$$

where T is the temperature (expressed here and in the following in eV), n is the ion density, Λ is the Coulomb logarithm, $A = m_i/m_p$, m_i, m_p are the ion and proton masses. (We use here the transport coefficients for a strongly ionized plasma as defined by Braginskii, [2]). For not too hot and not too thin plasmas the ratio ν_h/ν_m is small. (It is equal to $5 \cdot 10^{-3}$ for $A = 1$, $T = 1$ eV, $n = 10^{14}$ cm^{-3}).

Thus, for infinite plasmas with similar parameters the viscous damping would be a negligible one.

However, in a plasma surrounded by a solid boundary the viscous effects may lead to the appearance of a boundary layer with quickly changing plasma velocity, where the viscous force $\eta(d^2v/dx^2)$ (v is the radial plasma velocity, $x-a$ coordinate counted normally to the boundary) can reach big values, thus influencing the damping mechanism. The investigation of such effects described below leads, indeed, to the conclusion, that the viscous damping proves to be more important in the bounded plasmas than in the infinite ones, and can manifest itself there even under conditions when the parameter ν_h/ν_m is small. (A similar phenomenon is known to exist in acoustics: a sound wave reflected from a wall experiences a strong absorption in result of the boundary viscous effects (Landau, [9], §77)).

2. Basic equations

The linearized fluid equations for many-component, cold plasmas in the magnetic field $\vec{B}_0$, we will study here, can be written as follows (see e.g. Spitzer, [11]):

$$m_\alpha n_\alpha\left(\frac{\partial \vec{v}_\alpha}{\partial t}\right)_i = e_\alpha n_\alpha\left(\vec{E}+\frac{1}{c}[v_\alpha B_0]\right)+\vec{R}_{\alpha i}-\frac{\partial \pi^\alpha_{ik}}{\partial x_k}, \tag{3}$$

where n_α, m_α, $\vec{v}_\alpha$, e_α are respectively the density, mass, velocity and charge corresponding to an α-particle, π^α_{ik} is the viscous stress-tensor, and $\vec{R}_{\alpha i}$ is a friction force acting on an α-particle. In the following we will restrict ourselves with homogeneous, quasineutral, isothermal plasmas consisting of neutral particles (density n_a, mass m_a, velocity $\vec{v}_a$), ions (density n_i, mass m_i, velocity $\vec{v}_i$) and electrons (density n_e, mass m_e, velocity $\vec{v}_e$). Introducing in the usual way the mass velocity $\vec{v}=(m_i n_a \vec{v}_a + m_i n_i v_i + m_e \vec{v}_e)/\rho$, current density $\vec{j}=e(n_e\vec{v}_e - n_i\vec{v}_i)$, neglecting the members of the order of m_e/m_i and of the order of $(m_e/m_i)^{1/2}$, using an assumption of quasineutrality ($n_e=n_i=n$) and an assumption $\nu_{\text{in}}/\omega \gg 1$ (ν_{in} is the collision frequency of ions and neutrals) we reduce (3) to a set of two equations:

$$\rho\,\frac{\partial \vec{v}}{\partial t}=\frac{1}{c}[\vec{j}\vec{B}_0]+\vec{f}, \qquad \rho=m_i n_0, \qquad n_0=n_a+n, \tag{4}$$

$$\frac{\partial \vec{j}}{\partial t}=-\nu\vec{j}+\frac{\omega_p^2}{4\pi}\left(\vec{E}+\frac{1}{c}[\vec{v}\vec{B}_0]\right)-\omega_{ce}[\vec{j}\vec{h}], \tag{5}$$

where $\omega_p^2=4\pi e^2 n/m_e$, $\omega_{ce}=eB_0/m_e c$, ν is an electron-ion collision frequency, $\vec{h}$ is a unit vector in the direction of $\vec{B}_0$ and $\vec{f}$ is the viscous force. Under an additional assumption, we will use in the following, $\omega_{ci}\tau_i \ll 1$ ($\omega_{ci}=eB_0/m_i c$ and τ_i is ion-ion collision time) $\vec{f}$ can be represented in the form: $\vec{f}=\eta\ \Delta\vec{v}+(\zeta-\frac{1}{3}\eta)$ graddiv $\vec{v}$, where ζ is the coefficient of second viscosity.

The inequality $\nu_{\text{in}}/\omega \gg 1$ (meaning that the neutral particles are carried along with ions and, hence, that $\vec{v}_i \approx \vec{v}_a$) holds if the density of neutrals n_a is big enough. On the other hand, for a completely ionized plasma we arrive at the same system (4)–(5) neglecting the presence of neutrals from the very beginning.

The equations (4)–(5) must be complemented by the Maxwell equations:

$$\operatorname{rot} \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j}, \qquad \operatorname{rot} \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}, \tag{6}$$

which together with (4), (5) form a complete system. One must satisfy at a solid boundary S, besides usual electrodynamical conditions, some constraints imposed on the mass velocity $\vec{v}$. The hydrodynamical condition of adherence of gas particles to the boundary $\vec{v}(S) = 0$ can be used as such constraint in case the mean free paths of ions and neutrals are small compared to the relevant distances (including a boundary layer thickness). Otherwise (in a rarefied plasma) a well known slip flow phenomenon can arise (see, e.g. Kennard, 1938). To account for this phenomenon the condition $v(S) = 0$ is modified for neutral gases as follows (see, e.g. Serrin [13]): $v_n(S) = 0$, $v_t(S) = K\pi_t(S)$, where $v_n(S)$, $v_t(S)$ are the normal and tangential components of the velocity $\vec{v}$ at S, $\pi_t(S)$ is a tangential component at S of the vector $\pi_{ik}h_k$ ($\vec{h}$ is a unit vector normal to S) and K is a constant dependent of the gas properties.

A similar boundary condition, which is applicable for the cases when the mean free paths of the particles are not so small compared to the boundary layer thickness, can be formulated also for the rarefied plasmas. However, restricting in the following to purely radial magneto-acoustic oscillations of an infinite cylindrical plasma column surrounded by a solid boundary, we use only one of the above mentioned conditions: $v_n(S) = 0$. This condition, which means simply that a gas particle can not penetrate into the boundary, is just sufficient to obtain uniquely the solution of the system (4)–(6). Nevertheless, it must be kept in mind that the hydrodynamical description used here is correct only if the mean free paths of the plasma particles do not exceed the boundary layer thickness.

3. Radial magneto-acoustic oscillations

We will assume as usual (see, e.g. Elmiger, [5]) that the excitation of the magneto-acoustic oscillations of frequency ω is performed by a coil of radius b, while a homogenous plasma is situated in an axial magnetic field B_0 within a cylinder of radius $a < b$. The space between the coil and the plasma is assumed to be filled with an opaque dielectric. The non-zero components of the vectors $\vec{E}$, $\vec{H}$, $\vec{v}$, $\vec{j}$ now are: azimuthal component E_Φ, radial component E_r, axial component $H_z = H$, radial component of the mass velocity $v_r = v$ and azimuthal component of the current density j_Φ. (The disappearance of the radial component $j_r = 0$ is a consequence of the fact, that all the quantities $\vec{E}$, $\vec{H}$, $\vec{v}$, $\vec{j}$ can be considered as functions of the radius r counted from the system axis only; in combination with an additional assumption, we use in the following, that the displacement current $(1/c)(\partial\vec{E}/\partial t)$ in (6) can be neglected, it leads to the conclusion that

$$0 = (\operatorname{rotrot} \vec{E})_r = \frac{4\pi\omega i j_r}{c^2} \text{ i.e. } j_r = 0.$$

Accounting now for the fact that all the quantities in (4)–(6) depend upon the time as $e^{-i\omega t}$, we can reduce this system to the form

$$-i\rho\omega v = \frac{1}{c}\, j_\phi B_0 + \frac{4}{3}\eta\, \Delta_i v, \qquad \Delta_1 = \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r \tag{7}$$

$$(\nu - i\omega) J_\phi = \frac{\omega_p^2}{4\pi}\left(E_\phi - \frac{1}{c}\, vB_0\right) \tag{8}$$

$$\frac{\partial H}{\partial r} = -\frac{4\pi}{c}\, j_\phi, \qquad \frac{1}{r}\frac{\partial}{\partial r} rE_\phi = i\frac{\omega}{c} H \tag{9}$$

$$0 = (\nu - i\omega) j_r = \frac{\omega_p^2}{4\pi} E_r + \omega_{ce} j_\phi \tag{10}$$

It is necessary to indicate here, that for this particular case the equations (7)–(9) form a complete system, which, once solved, allows to find from (10) the radial component E_r.

As boundary conditions we will use

$$v(a) = 0. \qquad H(a) = H_0 \tag{11}$$

and conditions of regularity inside of the plasma column. In (11) H_0 means the field generated by the coil at its internal surface. This boundary condition (formally allowing to eliminate the layer of dielectric $a < r < b$) is applicable if $(\omega/c)(b-a) \gg 1$.

Introducing a dimensionless variable $r \to r/a$ one can reduce (7)–(9) to an equation for the function $v = v(r)$, satisfying the condition $v(1) = 0$:

$$\mu\, \Delta_1\, \Delta_1 v - \alpha\, \Delta_1 v - q^2 v = 0, \tag{12}$$

where

$$\mu = \frac{\nu_h \nu_m}{c_A^2 a^2}, \quad \nu_h = \frac{4\eta}{3\rho}, \quad \nu_m = \frac{c^2}{\omega_p^2}(\nu - i\omega), \quad q = \frac{\omega a}{c_A}, \quad c_A = \frac{B_0^2}{4\pi m_i n},$$

$$\alpha = 1 - i(\varepsilon_1 + \varepsilon_2), \quad \varepsilon_1 = \frac{\omega \nu_m}{c_A^2}, \quad \varepsilon_2 = \frac{\omega \nu_h}{c_A^2}$$

A solution of (12) can be easily obtained owing to the fact that solutions of the equation $\Delta_1 f + k^2 f = 0$, i.e. $f_1 = J_1(kr)$ (J_1 is a Bessel function), satisfy also (12), provided the parameter k is defined by

$$\mu k^4 + \alpha k^2 - q^2 = 0. \tag{13}$$

We get, hence, two independent solutions of (12): $v_{1,2} = C_{1,2} J_1(k_{1,2} r)$, where $C_{1,2}$ are some constants and, in virtue of (13)

$$k_{1,2}^2 = \frac{-\alpha \pm \sqrt{a^2 + 4\mu q^2}}{2\mu}. \tag{14}$$

It must be indicated here, that the parameter μ in (12) can be usually considered to be small, e.g. for strongly ionized, monoatomic plasmas, when $\eta = \eta_i$, $\eta_i =$

$0.96Tn\tau_i$ (Braginskii, 1965) and for

$$\nu \gg \omega,\ \mu = \frac{5 \cdot 10^2 A^{1/2} T}{B_0^2 a^2} \text{ (i.e. } \mu = 1.3 \cdot 10^{-6} \text{ for } T = 1\,\text{eV}, B_0 = 10^4\,\text{G}, A = 40, a = 5\,\text{cm}).$$

According to (13) we have in a senior approximation over μ:

$$k_1 = \frac{q}{\sqrt{\alpha}}, \qquad k_2 = i\sqrt{\frac{\alpha}{\mu}}. \tag{15}$$

The boundary condition $v(1) = 0$ leads, evidently, to the following definition of the constants: $C_{1,2} = \pm CJ(k_{2,1})$, i.e. the solution of (12), we look for, reads as follows

$$v = C[J_1(k_2)J_1(k_1 r) - J_1(k_1)J_1(k_2 r)], \tag{16}$$

where C is a constant defined by the condition $H(1) = H_0$.

Particular solutions for the magnetic field-vector H, as defined by (7)–(9), are: $H_{1,2} = D_{1,2}J_0(k_{1,2}r)$. The constants $D_{1,2}$ are connected with $C_{1,2}$ by the relation

$$D_{1,2} = B_0 k_{1,2} C_{1,2}/\nu_m\left(i\frac{\omega}{\nu_m} - k_{1,2}^2\right).$$

Thus, the magnetic field distribution is given by

$$H = \frac{CB_0}{\nu_m}\left(\frac{k_1 J_1(k_2)}{i\dfrac{\omega}{\nu_m} - k_1^2} J_0(k_1 r) - \frac{k_2 J_1(k_1)}{i\dfrac{\omega}{\nu_m} - k_2^2} J_0(k_2 r)\right). \tag{17}$$

As the argument k_2 in (17) possesses a big imaginary part, one can show, with the help of the asymptotic representation of the Bessel functions, that $(J_1(k_2)/J_0(k_2)) = i$. We get hence, a final expression for $H = H(r)$:

$$H = H_0[J_0(k_1 r) - iDJ_1(k_1)(J_0(k_2 r)/J_1(k_2))]/F \tag{18}$$

$$D = i\sqrt{\frac{\nu_h}{\nu_m}} \cdot \frac{1 + i(\varepsilon_1/\alpha)}{1 + i(\varepsilon_2/\alpha)}, \qquad F = J_0(k_1) - DJ_1(k_1). \tag{19}$$

Our formulas can be further simplified, if $|\varepsilon| \ll 1$, $|\varepsilon_2| \ll 1$, $|\varepsilon_2| \ll |\varepsilon_1|$. These inequalities are often realized nearby the first magneto-acoustic resonance (MAR) and they ensure, that it is a narrow one. We can now represent k_1 and $J_1(k_1)$ by the developments

$$k_1 = \frac{q}{\sqrt{\alpha}} = q(1 + \tfrac{1}{2}i\varepsilon_1), \qquad J_0(k_1) = -J_1(q_0)(q - q_0) - J_1(q_0)i\tfrac{1}{2}q_0\varepsilon_1,$$

where $q_0 = \omega_0 a/c_A = 2.4$ is the first root of the equation $J_0(q) = 0$. The coefficient F in (19) become now:

$$F = -J_1(q_0)q_0\left(\frac{\omega - \omega_0}{\omega_0} + i\gamma\right), \qquad \gamma = \tfrac{1}{2}\varepsilon_1 + \frac{1}{q_0}\sqrt{\frac{\nu_h}{\nu_m}}.$$

A quantity of interest is the modulus of the ratio of the magnetic field at the axis $H(0)$ and at the plasma boundary $H_0: N=|H(0)/H_0|$. As follows from (10), it is given nearby the first MAR (for $\omega \ll \nu$) by the formula

$$N=\frac{1}{q_0J_1(q_0)}\frac{1}{\sqrt{\left(\frac{\omega-\omega_0}{\omega_0}\right)^2+\gamma^2}}. \qquad \omega_0=\frac{2.4c_A}{a} \tag{20}$$

where

$$\frac{1}{q_0J_1(q_0)}=0.8$$

and

$$\gamma=\tfrac{1}{2}\varepsilon_1+\frac{1}{q_0}\sqrt{\frac{\nu_h}{\nu_m}}=\frac{q_0\nu_m}{2c_Aa}+\frac{1}{q_0}\sqrt{\frac{\nu_h}{\nu_m}}. \tag{21}$$

The parameter $2\gamma\omega_0$ represents evidently the width of the first MAR, while the quantity $1/\gamma$ represents, as usual, the quality factor. It must be stressed here, that the influence of viscosity is characterized by $\sqrt{\nu_h/\nu_m}$ in (20), i.e. it is, indeed, more pronounced than in infinite plasmas, where these effects are of the order of ν_h/ν_m. For strongly ionized plasmas and for $\omega \ll \nu$ the magnetic ν_m and kinematic ν_h viscosities are given by

$$\nu_m=\frac{c^2}{\omega_p^2\tau_e}=\frac{8.3\cdot 10^5\Lambda}{T^{3/2}} \text{ (for } \omega_{ce}\tau_e \gg 1)$$

$$\nu_h=\frac{1.3T\tau_i}{m_i}=\frac{2.6\cdot 10^{19}T^{5/2}}{A^{1/2}\Lambda n},$$

where τ_e, τ_i are the electron-ion and ion-ion collision times defined in (Braginskii, [2]). The parameter γ is defined now by

$$\gamma=\frac{10^6\Lambda}{c_AaT^{3/2}}+\frac{2.4\cdot 10^6T^2}{A^{1/4}\Lambda n^{1/2}}, \qquad c_A=\frac{B_0}{\sqrt{4\pi m_in}}. \tag{22}$$

The influence of viscosity effects becomes appreciable, according to (22), if

$$\frac{1}{q_0}\sqrt{\frac{\nu_h}{\nu_m}}>\frac{q_0\nu_m}{2c_Aa}.$$

It leads to a condition specifying a range of densities where these effects must be accounted for : $n<5\cdot 10^{11}B_0T^{7/2}aA^{-3/4}\Lambda^{-2}$. It is interesting to indicate here that the quality factor $1/\gamma$ (or the value of the maximal ratio $N_{max}=0.8/\gamma$, corresponding to $\omega=\omega_0$) possess as a function of temperature T (with all the other parameters fixed) a pronounced maximum (Fig. 1).

Formula (22) for γ is valid only for a high degree of ionization satisfying the condition

$$\frac{1}{\tau_i}=\frac{\Lambda n}{2.1\cdot 10^7T^{3/2}A^{1/2}} \ll \nu_{in}\left(\frac{n}{n_a}\right)^2.$$

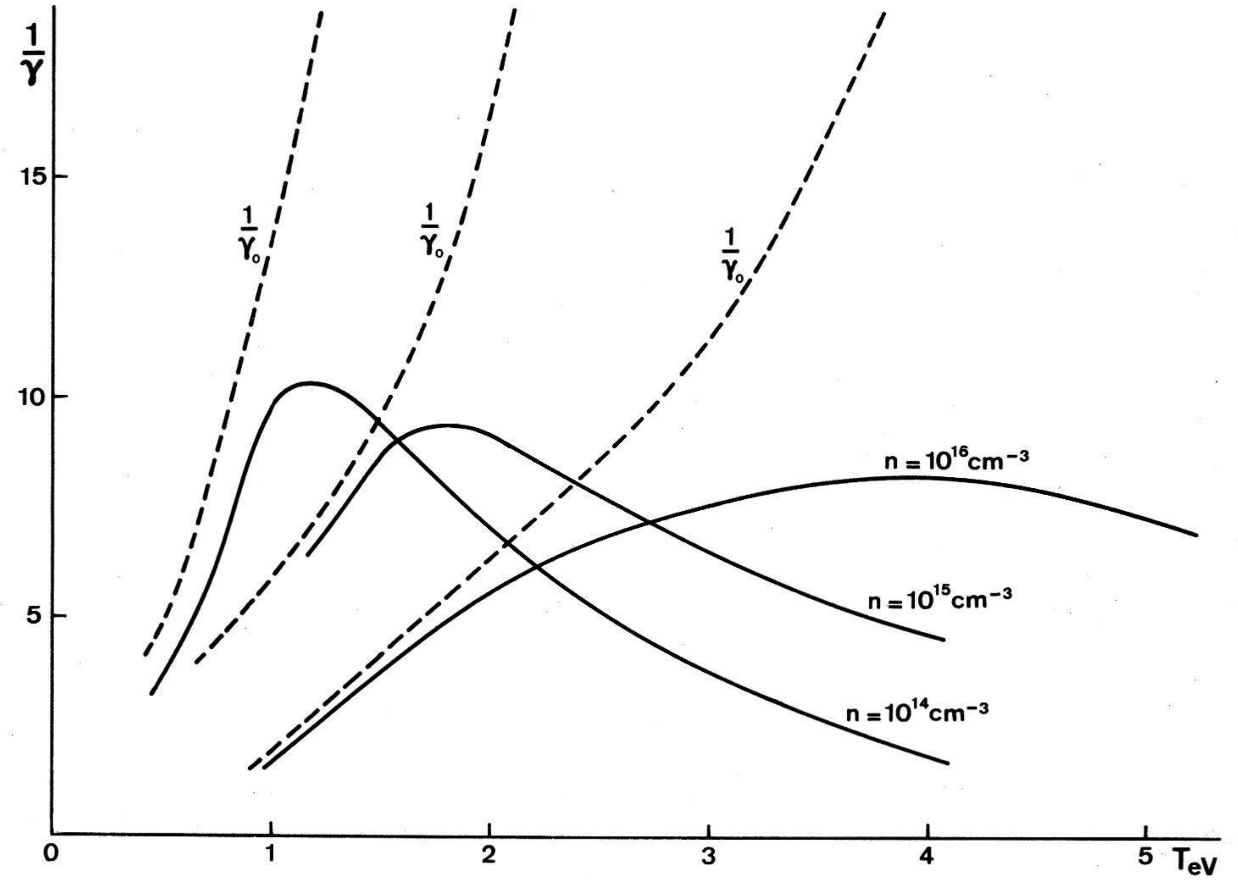

Figure 1

Parameter $1/\gamma$ for strongly ionized plasmas (formula (22)) as a function of temperature T for $B_0 = 10^3$ G, $a = 5$ cm, $A = 1$ (Hydrogen) and for different electron densities n. For comparison the corresponding values of the parameter $1/\gamma_0 = 1(\omega_0 \nu_m / 2c_A^2)$ (i.e. with viscosity effects neglected) are indicated by dashed curves. The value of the maximal ratio $N = |H(0)/H_0|$ for $\omega = \omega_0$ differs from $1/\gamma$ by the factor $1/[q_0 J_1(q_0)] = 0.8$

(For Ar-plasmas ν_{in} is given by a semiempirical expression (Appert, [1]) $\nu_{in} = 2.0 \cdot 10^{-9} T^{1/2} n_a$). For plasmas with comparable concentrations of ions and neutrals $1/\tau_i \gg \nu_{in}$ and viscosity is due mainly to ion-atom collisions. An approximate formula for the viscosity coefficient, $\eta = \frac{1}{3}\rho u l$, can be used in this case. (Here u is a mean thermal velocity and l is the mean free path of atoms connected with their collision frequency by the relation $l = u/\nu_{in}$). For Ar-plasma we get, in virtue of the expression for ν_{in} mentioned above $\eta = 1.0 \cdot 10^{-3} T^{1/2}$ poise, in a reasonable agreement with measurements (Schreiber, ([12]) of η for Ar-plasmas. A corresponding expression for the parameter γ in (20) valid for a particular case of the Ar-plasmas is

$$\gamma = \frac{10^6 \Lambda}{c_A a T^{3/2}} + 1.8 \cdot 10^6 T \sqrt{\frac{1}{\Lambda n_0}}, \qquad c_A = \frac{B_0}{\sqrt{4\pi m_i n_0}}. \tag{23}$$

The dependence of γ from T as given by (23) is reproduced in Fig. 2. Using the assumptions $|\varepsilon_{1,2}| \ll 1$, we can also simplify the formulas (16), (18) for $v = v(r)$,

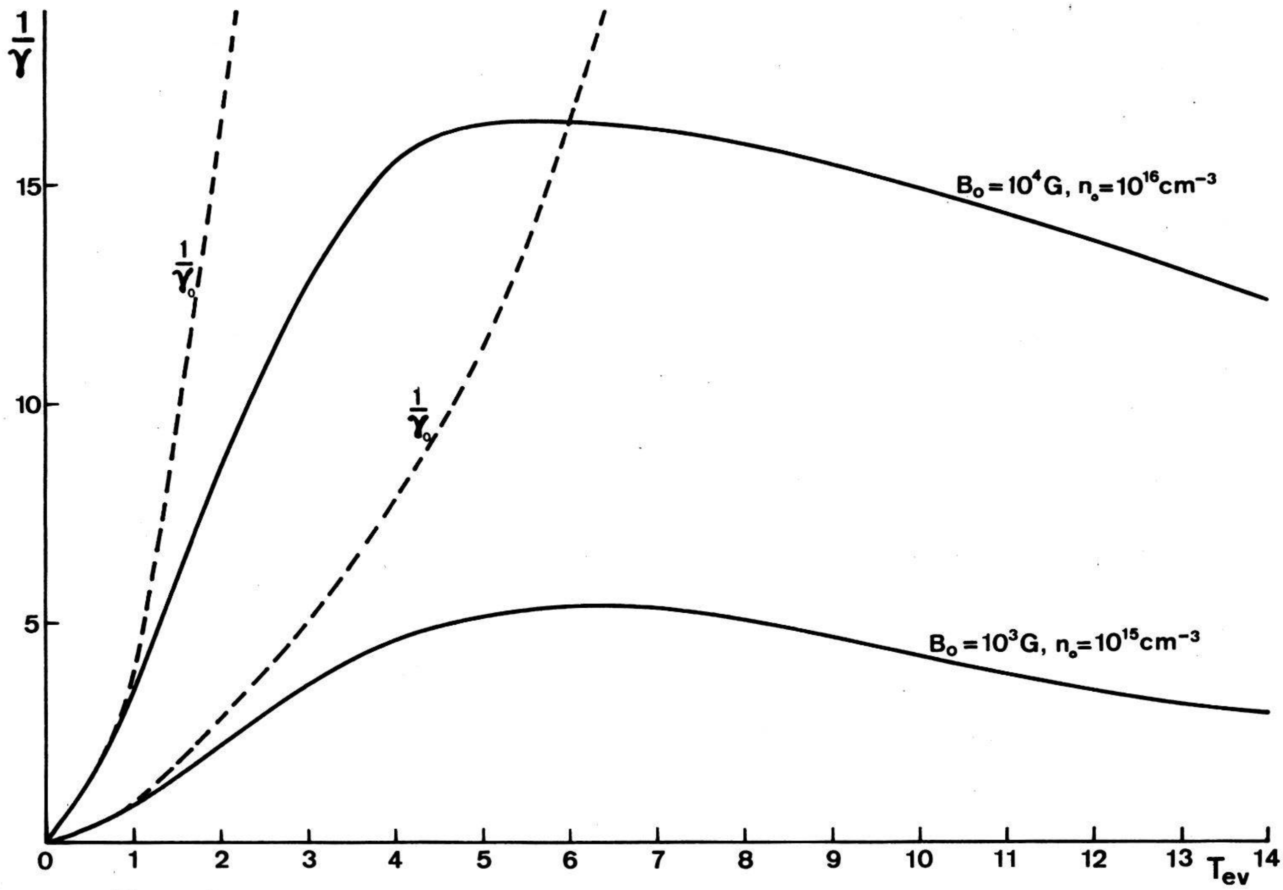

Figure 2

Parameter $1/\gamma$ for Ar-plasmas ($A = 40$) as a function of temperature (formula (23)) for $a = 5$ cm and $B_0 = 10^4$ G, $n_0 = 10^{16}$ cm^{-3} or $B_0 = 10^3$ G, $n_0 = 10^{15}$ cm^{-3}. For comparison the corresponding values of the parameter $1/\gamma_0 = 1/(\omega_0 \nu_m / 2c_A^2)$ are indicated by dashed curves.

$H = (r)$. They are given now by

$$H = H_0\left[J_0(k_1 r) - iJ_1(k_1)(\nu_h/\nu_m)^{1/2} \exp\left(-\frac{1-r}{\sqrt{\mu}}\right)\right]\Big/F \tag{24}$$

$$v = c_A \frac{H_0}{B_0}\left[J_1(k_1 r) - J_1(k_1) \exp\left(-\frac{1-r}{\sqrt{\mu}}\right)\right]\Big/F, \tag{25}$$

where

$$F = J_0(k_1) - i\sqrt{\frac{\nu_h}{\nu_m}}\, J_1(k_1)$$

or, nearby the first MAR,

$$F = -J_1(q_0)q_0\left(\frac{\omega - \omega_0}{\omega_0} + i\gamma\right).$$

From (24), (6) we have also to the same approximation:

$$E_\phi = iH_0 \frac{c_A}{c} J_1(k_1 r)/F \tag{26}$$

(In (24)–(26) the ratios $J_1(k_2r)/J_0((k_2r)$, $J_1(k_2r)/(J_1(k_2r)$ were replaced by their asymptotic expressions valid nearby $r=1$).

The formulas (24)–(26) exhibit the presence of the boundary layer of thickness

$$\delta = a\sqrt{\mu} = \frac{\sqrt{\nu_h \nu_m}}{c_A},$$

due to both electromagnetic and viscous effects.

It is instructive to rewrite the expression (21) for γ, introducing the parameter δ, as follows:

$$\gamma = \frac{q_o}{2}\frac{\nu_m}{c_A a} + \frac{1}{q_0}\frac{\nu_h}{c_A \delta}.$$

As it is clear from this expression, viscosity begins to play an important role if

$$\frac{\nu_h}{\nu_m} \gtrsim \frac{q_0^2}{2}\frac{\delta}{a} = 2.9\frac{\delta}{a} \ll 1,$$

whereas for the infinite plasmas the corresponding condition is much more stringent: $\nu_h/\nu_m > 1$. The foregoing discussion was restricted to the relatively thick plasmas satisfying the condition $\nu_{in}/\omega \gg 1$. The investigation of the viscous effects for arbitrary values of the parameter ν_{in}/ω could be, in principle, performed by replacing the system (4)–(5) by a set of equations accounting for the separate motion of neutrals

$$\rho_i \frac{\partial \vec{v}_i}{\partial t} = \frac{1}{c}[\vec{j}\vec{B}_0] - \rho_i \nu_{in}(\vec{v}_i - \vec{v}_a) + \vec{f}_i \tag{27}$$

$$\rho_a \frac{\partial \vec{v}_i}{\partial t} = \rho_i \nu_{in}(\vec{v}_i - \vec{v}_a) + \vec{f}_a, \qquad \rho_i = m_i n_i \tag{28}$$

$$\frac{\partial \vec{j}}{\partial t} = -\nu \vec{j} + \frac{\omega_p^2}{4\pi}\left(\vec{E} + \frac{1}{c}[\vec{v}_i \vec{B}_0]\right) - \omega_{ce}[\vec{j}\vec{h}] \tag{29}$$

where $\vec{f}_a = \eta_a \Delta \vec{v}_a + (\zeta_a + \frac{1}{3}\eta_a)$ graddiv $\vec{v}_a$, $\vec{f}_i = \eta_i \Delta \vec{v}_i + (\zeta_i + \frac{1}{3}\eta_i)$ graddiv $\vec{v}_i$ and η_i, η_a are the coefficients of viscosity for ions and neutrals. (System (27)–(29) differs from a system (2)–(4) in Hoegger et al. [15] by inclusion of the viscous forces $\vec{f}_a$, $\vec{f}_i$). The analysis of the general system (27)–(29) is quite complicated. However, in case $\nu_{in}/\omega \ll 1$ (opposite to that investigated above), the velocity of neutrals $\vec{v}_a$ appears to be small compared to $\vec{v}_i$, as follows from (27)–(29). (The neutrals form now a kind of 'background', not participating in the motion of ions). We have, hence in this case a relation $\vec{v}_i \approx (1/x)\vec{v}$ between the mass velocity $\vec{v}$ and the ion velocity $\vec{v}_i$, $x = n/n_0$ being the degree of ionization. Adding now the equations (27) and (28) and inserting into (29) $\vec{v}_i \doteq (1/x)\vec{v}$, we reduce (27)–(29) to a set of two equations

$$\rho \frac{\partial \vec{v}}{\partial t} = \frac{1}{c}[\vec{j}\vec{B}_0] + \vec{f}_i \tag{30}$$

$$\frac{\partial \vec{j}}{\partial t} = -\nu \vec{j} + \frac{\omega p^2}{4\pi}\left(\vec{E} + \frac{1}{cx}[\vec{v}\vec{B}_0]\right) - \omega_{ce}[\vec{j}\vec{h}] \tag{31}$$

differing from that used in Skipping et al. [14] by accounting for the viscous force

$$\vec{f}_i = \frac{1}{x}\eta_i\,\Delta\vec{v} + \frac{1}{x}(\zeta_i + \tfrac{1}{3}\eta_i)\,\text{graddiv}\,\vec{v}$$

in (30). Repeating the calculations described above, we get the formulas coinciding with (20)–(24)–(26) with the sole difference that the Alfven velocity must be replaced by

$$c_A = \frac{B_0}{\sqrt{4\pi m_i n}}$$

and the viscous coefficient η by η/x.

4. Propagation of the magneto-acoustic waves, MAW, in cylindrical channels

It was assumed in the foregoing discussion, that the excitation of MAW is performed by a coil having the same length as the plasma column. However, a question may arise about the role of the viscous damping if MAW are generated by a short coil and propagate in a long plasma channel. The boundary effect investigated above for a standing radial wave proves to be even more important here. This can be shown easily under the following assumptions: (1) Excitation of an infinitely long cylindrical plasma column of radius $r = a$ is performed by a coil situated symmetrically relative to a middle plane $z = 0$ on the surface of a thin dielectric surrounding this column. It means, that the distribution of an azimuthal external current-density is given by

$$j(z) = \int_{-\infty}^{\infty} j(k_z)e^{ik_z z}\,dk_z.$$

Accordingly, we can assume, using in the following the homogeneous equations (4)–(6), that the axial magnetic field $H(r, z)$ satisfies for $r = q$ the boundary condition

$$H(a, z) = \frac{4\pi}{c}\int_{-\infty}^{\infty} j(k_z)e^{ik_z z}\,dk_z \tag{32}$$

where z is a cylindrical coordinate counted along cylinder axis; (2) The Ohm's law will be taken in its scalar form $j = \sigma(\vec{E} + (1/c)[\vec{v}\vec{B}_0])$, which implies

$$\nu \gg \omega \quad \text{and} \quad \nu \gg \omega_{ce}.$$

The basic equations (4)–(6) read now as follows

$$-i\omega\rho v_r = \frac{1}{c}B_0\sigma\left(E_\phi - \frac{1}{c}v_r B_0\right) + \eta(\Delta v_r + \tfrac{1}{3}\Delta_1 v_r) \tag{33}$$

$$\Delta E_\phi = -\frac{4\pi\sigma\omega i}{c^2}\left(E_\phi - \frac{1}{c}v_r B_0\right), \qquad H = \frac{ic}{\omega}\Delta_1 E_\phi \tag{34}$$

where $\Delta = \Delta_1 + \partial^2/\partial z^2$. (The axial component of the mass velocity $v_z = 0$ since

$(1/c)[\vec{j}\vec{B}_0]_z = 0)$. System (33)–(34) possess the partial solutions

$$v_r(k_z) = AJ_1(k_r r)e^{ik_z z}, \qquad E_\phi(k_z) = BJ_1(k_r z)e^{ik_z z},$$

where A, B are some constants. Inserting these solutions into (33), (34) we get a dispersion equation coinciding (for frequencies not too far from the first MAR) with (13), the sole difference being that k^2 in (13) must be now replaced by $k^2 = k_r^2 + k_z^2$. Repeating the calculations described in Section 3, assuming $\varepsilon_1 \ll 1$, $\varepsilon_2 \ll 1$, $\varepsilon_2 \ll \varepsilon_1$ and introducing the dimensionless variables $r \to (r/a)$, $z \to (z/a)$ we find approximate expressions for the two roots of this dispersion equation.

$$k_1^2 = k_{r1}^2 + k_z^2 = q^2(1 + i\varepsilon_1), \qquad k_2^2 = k_{r2}^2 + k_z^2 = -\frac{\nu_h \nu_m}{c_A^2 a^2} = -\frac{1}{\mu},$$

$$q = \frac{\omega a}{c_A}. \tag{35}$$

We can take in the following $k_z < q$ (since $|k_2| \gg |k_1|) k_2 \cong i\sqrt{1/\mu} \cong k_{r2}$. The partial solution $v_r(k_z)$ satisfying the boundary conditions $v_r(a) = 0$ can be written now in the form

$$v_r(k_z) = C[J_1(k_{r1}r)J_1(k_2) - J_1(k_2 r)J_1(k_{1r})]e^{ik_z z} \tag{36}$$

where C is a constant.

For the magnetic component $H(k_z)$ we find an expression coinciding with (24), where H_0 must be replaced by $(4\pi/c)j(k_z)$ and k_1 by $\sqrt{k_1^2 - k_z^2}$. Integrating this expression over k_z taking, for sake of definiteness $j(k_z) = J/2\pi$, i.e. $j(z) = J\delta(z)$ (excitation by a single loop), we get finally (for $1 - r \gg \sqrt{\mu}$):

$$H(r, z) = \frac{2J}{c} \int_{-\infty}^{\infty} \frac{J_0(\sqrt{k_1^2 - k_z^2}\, r)e^{ik_z z}\, dk_z}{J_0(\sqrt{k_1^2 - k_z^2}) - i\lambda J_1(\sqrt{k_1^2 - k_z^2})} \tag{37}$$

where $\lambda = (\nu_h/\nu_m)^{1/2}$. Considering in (37) k_z as a complex variable, we conclude that the integrand (37) becomes exponentially small in the upper half-plane of k_z for $|k_z| \to \infty$. We can, hence, replace the integral in (37) by a sum of residues corresponding to this domain. Accounting for the fact that zeroes k_z^s of the denominator in (27) are given by the equation

$$k_r^s = \sqrt{k_1^2 - (k_z^s)^2} = q_s - i\lambda, \quad \text{i.e.} \quad k_z^s = [q^2 - q_s^2 + i(q^2\varepsilon_1 + 2\lambda q_s)]^{1/2}, \tag{38}$$

where q_s are roots of the equation $J_0(q) = 0$, we arrive at the expression

$$\frac{H}{H_0} = \sum_{s=0}^{\infty} \frac{k_r^s J_0(k_r^s r)e^{ik_z z}}{k_z^s J_1(q_s)}, \qquad H_0 = \frac{4\pi J}{ca}. \tag{39}$$

It is seen immediately from (39) that for $z \to \infty$ the first term in this sum dominates. Thus, accounting for the fact that $\varepsilon_i \ll 1$, $\lambda \ll 1$, i.e.

$$k_z^s \cong \sqrt{q^2 - q_s^2} + (q^2\varepsilon_1 + 2\lambda q_s)/2\sqrt{q^2 - q_s^2},$$

one can represent $|H(r, z)/H_0|$ for $z \to \infty$ in the form:

$$\left|\frac{H}{H_0}\right| = \frac{q_0 J_0(q_0 r)e^{-\kappa z}}{J_1(q_0)\sqrt{(q^2 - q_0^2)^2 + (q^2\varepsilon_1 + 2\lambda q_0)^2}}, \qquad \kappa = \frac{q^2\varepsilon_1 + 2\lambda q_0}{2\sqrt{q^2 - q_0^2}} \tag{40}$$

where κ plays a role of an effective attenuation coefficient replacing in this respect expression (1) valid for a plane wave. For the frequencies very close to the first MAR (40) must be rewritten as follows:

$$\left|\frac{H}{H_0}\right| = \frac{J_0(q_0 r)e^{-\kappa z}}{2q_0 J_1(q_0)\sqrt{\left(\frac{\omega-\omega_0}{\omega_0}\right)^2+\gamma^2}}, \qquad \kappa = q_0\sqrt{\gamma} \tag{41}$$

where γ is defined by (21). It is clear from (40) that viscous effects lead to an additional damping with distance z, described by a factor

$$\exp\left(-\frac{\lambda q_0 z}{\sqrt{q^2-q_0^2}}\right)$$

(formula (40)), which can be very important for not too small values of the parameter λ. It must be stressed that the attenuation coefficient defined by (40) or (41) can possess, as a function of temperature, a pronounced minimum (see Fig. 1 and Fig. 2).

As follows from formulas (39)–(41), the results obtained here for an infinitely long cylindrical channel can be extended also for a finite plasma column (e.g. for a column closed by some conducting ends), provided the distances between the coil and the ends are big compared with the attenuation length $1/k$. In this case the boundary conditions at these ends become irrelevant, since the field amplitudes are exponentially small there.

5. Comparison with experiment

In several experiments (Cantieni, [3]; Elmiger, [5]) a considerable reduction of the magnetic field amplitude at the axis, compared with that calculated without accounting for the viscous dissipation, was observed. This fact can be explained in a natural way with a help of our formula (20). As follows from (20) the reduction of the normalized amplitude N compared with that (N^0) corresponding to neglect of the viscous term $(\nu_h/\nu_m)^{1/2}$, for $\omega = \omega_0$, is given by

$$\frac{N_{\max}}{N^0_{\max}} = \frac{1}{1+\frac{2}{q_0\varepsilon_1}\sqrt{\frac{\nu_h}{\nu_m}}}. \tag{42}$$

For an Ar-plasma studied in (Cantieni, [3], Fig. 12), for: $t = 100\ \mu$s, $B = 4\cdot 10^3$ G, $a = 4.7$ cm, $T = 2$ eV (this value of T was taken by analogy with Fig. 10) the ratio of experimental and theoretical (computed without accounting for the viscous losses) amplitudes for $\omega = \omega_0$ is equal to about 0.66. On the other hand, we get from (42) for this ratio $N_{\max}/N^0_{\max} = 0.75$. (In this case $c_A = B_0/\sqrt{4\pi\rho}$).

For plasma with parameters $t = 150\ \mu$s, $x = 0.32$ (Fig. 12 in Cantieni, [3]) we find, taking $T = 1.6$ eV, from (42) $N_{\max}/N^0_{\max} = 0.9$, also in accordance with experiment. Other curves in (Cantieni, 1963, Fig. 12) showing discrepancies with a theory neglecting the viscous losses, can be interpreted in a similar way. Thus, it seems plausible, that reduction of the magnetic field amplitudes observed in this paper could be due to the viscous effect. For the plasma studied in (Elmiger, [5],

Fig. 10) the ratio of experimental N and theoretical N^0_{max} amplitudes for $\omega = \omega_0$ is equal to about 0.3. We find from (42) in this case, using the experimental parameter listed in (Elmiger, [5]), the ratio $N_{max}/N^0_{max} = 0.4$.

However, for this plasma the mean free paths of the plasma particles exceed the boundary layer thickness δ and, hence, the application of the hydrodynamical theory for interpretation of the experiment may appear doubtful. Nevertheless, it seems possible, that an additional dissipation due to viscous boundary effects could take place also in such rarefied plasmas. Of course, only a consequent kinetic theory could justify this conjecture.

In conclusion, it is necessary to mention, that the problem of viscous losses discussed here is of interest also in connection with the possibility of a HF heating of the bounded cylindrical plasmas. The corresponding calculations were performed e.g. in (Cross, [4]) without accounting of the viscous effects. In fact, as follows from our theory, under conditions accepted in (Cross, [4]) the viscous losses may compete with Joule losses, thus influencing the HF power absorption in the plasma.

The considerable discrepancies between theoretical (computed without accounting for the viscosity effects) and experimental values of the impedance of the plasma column in an axial magnetic field (they are stressed e.g. in Lammers [10]) could be interpreted in a similar way. Investigation of these and related problems is in progress.

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