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Autor(en): **Maugin, Gérard A.**

Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **52 (1979)**

Heft 2

PDF erstellt am: **27.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-115023

Helvetica Physica Acta, Vol. 52 (1979), Birkhäuser Verlag, Basel

Nonlinear wave propagation in relativistic continuum mechanics[1]

by **Gérard A. Maugin**

Université Pierre-et-Marie Curie
Laboratoire de Mécanique Théorique associé
au C.N.R.S., Tour 66, 4 Place Jussieu,
75230 Paris Cedex 05, France

(9. X. 1978)

Summary. We present a unified approach to the foundations of nonlinear wave propagation in relativistic continuum mechanics. The material descriptions of interest are elasticity and magnetoelasticity and the limiting cases of relativistic hydrodynamics and magnetohydrodynamics. The interest is focused on the propagation properties of infinitesimal discontinuities in finite initial states, the properties of weak shocks and the thermodynamics of strong discontinuities (shocks). The study is made for a particular type of motion, so-called one-dimensional relativistic motions.

Contents

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[1]) This paper was completed while the author was a visitor at Princeton University. The author wishes to express his gratitude to Prof. A. C. Eringen and the Authorities of the Department for their kind hospitality.

1. Introduction

Unless one is interested in very weak signals, which occur in the detection problem (Cf. [1]–[2] and the contributions of Pallattino and Pizella in [3]), either physical circumstances (the dense solid-like matter of certain astrophysical objects [4], the large velocities involved in galactic motions, the effects of strong magnetic fields [4]) or the very nonlinearity of the field equations forces upon us the consideration of *nonlinear wave propagation* in relativistic continuous matter. Though sketchy as it is, this paper offers an attempt at a unified approach to this problem for the states of continuous matter which may prove of interest in various applications of relativity theory. We primarily consider relativistic elasticity and magnetoelasticity, the cases of relativistic hydrodynamics and magnetohydrodynamics being obtainable by some adequate adjustments. In Section 2, a general notion of deformation of matter in space-time is first given. In contrast to previous papers of the author [5]–[9], a specialization is given to so-called *one-dimensional* relativistic motions. Field equations and various simple descriptions of relativistic elastic matter are also given. Section 3 is devoted to recalling the precise mathematical definition of the two main types of singular manifolds of interest (infinitesimal discontinuities and shock waves)[2]. Infinitesimal discontinuities in nonlinear elastic bodies and hydrodynamics are dealt with in Section 4. Section 5 offers a short (but probably the first one) treatment of *shock* waves in relativistic elasticity. Section 6 is devoted to extending the above results to the case of relativistic magnetoelasticity and magnetohydrodynamics. In particular, the Hugoniot equation is constructed for such schemes once a plausible model has been constructed and infinitesimal discontinuities have been considered.

Since relativistic continuum mechanics is as old as relativity theory itself (Einstein introduced the energy-momentum tensor for perfect fluids in his pioneering papers), we think that combining this field of research with one of the building blocks of twentieth-century applied mathematics, nonlinear waves, with our modest means, is appropriate to a celebration of the genius of Einstein on the occasion of the centennial anniversary of his birth.

2. Preliminaries

2.1. *Notation*

Let $M = (V^4, \mathbf{g})$ be a space-time of general relativity equipped with a normal

[2]) *Simple waves*, of which the study constitutes a formidable task in the relativistic framework, are not considered.

hyperbolic metric $g_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3, 4$; index 4 timelike; Lorentzian signature $+, +, +, -$). $\mathbf{u}$ is the four-velocity such that $g_{\alpha\beta}u^\alpha u^\beta + 1 = 0$ ($c = 1$ for notational convenience). ∂_α and ∇_α denote the partial and covariant derivatives in a local chart x^α of M. $D_{\mathbf{A}}$ in general indicates the gradient operator in the direction of a vector field $\mathbf{A}$. Thus $D_u = u^\alpha \nabla_\alpha \cdot P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ is the spatial projector which is systematically used in the following development to write down the local canonical space-time decomposition of any tensor field defined on M. The local spatial projection of any geometrical object $\mathbf{A}$ is noted $\mathbf{A}_\perp$ and admits $\mathbf{u}$ as zero vector for all its indices in a local chart. Objects such that $\mathbf{A} \equiv \mathbf{A}_\perp$ are said to be spatial. The transverse or spatial covariant derivative is defined by $\overset{\perp}{\nabla}_\alpha \equiv P_\alpha^{\cdot\beta} \nabla_\beta \cdot R^\alpha_{\cdot\beta\gamma\delta}$ is the curvature tensor $\mathbf{R}$ of M in a local chart.

2.2. *Deformation of matter in space-time*

Following previous works (e.g., [5]–[9]), we admit that the motion of a relativistic continuum is described either by means of a canonical differentiable projection $\mathcal{P}$ such that $\mathcal{P}: \mathcal{T}[B] \to \mathcal{M}$ or with the aid of the space-time parametrized congruence of world lines $\mathcal{C}: \mathbf{x} = \mathcal{X}(\mathbf{X}, \tau)$, $\mathbf{X} \in B$, $\tau \in \mathbb{R}$. Here $\mathcal{T}[B]$ is the open tube of V^4 which is swept out by the material body B (whose constituent parts are the material "particles" $\mathbf{X}$) and $\mathcal{M} = (V^3, G_{KL})$, $K, L = 1, 2, 3$, is the three-dimensional "material" manifold which serves to describe the material continuum. B is an open region of $\mathcal{M}$. τ is the proper time of $\mathbf{X}$ along $\mathcal{C}$. $\mathcal{M}$ is equipped with the local background metric G_{KL} and local charts X^K, $K = 1, 2, 3$. We have thus

$$\mathcal{P}: X^K = \tilde{X}^K(x^\alpha), \qquad \tau = \tau(x^\alpha), \qquad \mathcal{C}: x^\alpha = \mathcal{X}^\alpha(X^K, \tau). \tag{2.1}$$

These relations are in general assumed to possess a sufficient degree of continuity and differentiability in their arguments so as to allow for the forthcoming manipulations. For instance, one can define the inverse motion gradient X^K_α by

$$X^K_\alpha \equiv \partial_\alpha \tilde{X}^K \qquad (u^\alpha X^K_\alpha = D_u \tilde{X}^K = 0). \tag{2.2}$$

The Jacobian determinant of $(2.1)_3$ is defined by

$$J = \det \|\mathbb{F}\|, \tag{2.3}$$

where

$$\mathbb{F} = \left\{ x^\alpha_K = \left(\frac{\partial \mathcal{X}^\alpha}{\partial X^K} \right)_\perp; \; x^\alpha_K u_\alpha = 0 \right\} \tag{2.4}$$

is the *direct motion gradient* two-point tensor field. J is assumed to keep the same sign (say, plus) in the course of the relativistic motion of $\mathbf{X}$. The chain rule of differentiation yields

$$X^K_\alpha x^\alpha_L = \delta^K_L, \qquad X^K_\alpha x^\beta_K = P^{\cdot\beta}_\alpha. \tag{2.5}$$

It is easily shown that

$$(D_u x^\alpha_K)_\perp = e^\alpha_{\cdot\lambda} x^\lambda_K, \tag{2.6}$$

$$2d_{\alpha\beta} = (\pounds_u P_{\alpha\beta})_\perp \equiv \pounds_u P_{\alpha\beta}, \tag{2.7}$$

where

$$e_{\alpha\beta} \equiv \overset{\perp}{\nabla}_\beta u_\alpha, \tag{2.8}$$

$$d_{\alpha\beta} \equiv e_{(\alpha\beta)} = \tfrac{1}{2}(\overset{\perp}{\nabla}_\beta u_\alpha + \overset{\perp}{\nabla}_\alpha u_\beta), \tag{2.9}$$

$$\left(\underset{u}{\pounds} A_{\alpha\beta}\right)_\perp = (D_u A_{\alpha\beta})_\perp + A_{\gamma\beta}\,\overset{\perp}{\nabla}_\alpha u^\gamma + A_{\alpha\gamma}\overset{\perp}{\nabla}_\beta u^\gamma, \qquad \forall \mathbf{A} \equiv \mathbf{A}_\perp, \tag{2.10}$$

where $\underset{\mathbf{V}}{\pounds}$ indicates the Lie derivative with respect to a vector field $\mathbf{V}$. In terms of the differentiable projection $\mathscr{P}$, we have

$$\left(\underset{u}{\pounds} A\right)_\perp (\mathbf{x}) = \mathscr{P}^{-1}\left[\frac{\partial}{\partial \tau}\mathscr{P}(\mathbf{A})(\mathbf{X}, \tau)\right](\mathbf{x}), \qquad \forall \mathbf{A}(\mathbf{x}) = \mathbf{A}_\perp(\mathbf{x}). \tag{2.11}$$

2.3. *One-dimensional relativistic motions*

In the present work we focus our attention on the case of so-called one-dimensional relativistic motions (cf. [10]). That is, we set forth the

Definition. A one-dimensional relativistic motion is a mapping $(2{,}1)_3$ which, at fixed τ, depends only on a scalar coordinate defined along a given curve C in $\mathcal{M}$.

This definition is covariantly expressed as follows. Let Λ^K be the components of the unit oriented tangent to C on $\mathcal{M}$. Then we call X the scalar coordinate such that

$$\partial_X = D_\Lambda = \Lambda^K\,\partial_K. \tag{2.12}$$

It follows from this and $(2.1)_3$ that (2.4) reads

$$\mathbb{F} = \mathbf{f} \otimes \mathbf{\Lambda} \quad \text{or} \quad x^\alpha_K = f^\alpha \Lambda_K, \quad \Lambda_K = G_{KL}\Lambda^L, \qquad f^\alpha \equiv (\partial_X \mathscr{X}^\alpha)_\perp. \tag{2.13}$$

That is, the strain field is entirely defined by the (spatial) *strain vector* $\mathbf{f}$. Let $D_f = f^\alpha \nabla_\alpha = f^\alpha\,\overset{\perp}{\nabla}_\alpha$. Then for futher use we evaluate the commutator $[D_u, D_f]$. An easy calculation leads to

$$[D_u, D_f] = u^\alpha f^\beta [\nabla_\alpha, \nabla_\beta] - \mathscr{D}_f, \tag{2.14}$$

where

$$\mathscr{D}_f \equiv D_{\dot{f}} + e^\alpha_{\cdot\beta} f^\beta\,\overset{\perp}{\nabla}_\alpha, \qquad D_{\dot{f}} \equiv D_{(D_u f)}. \tag{2.15}$$

Therefore, the commutator $[D_u, D_f]$ in general involves the curvature of space-time. When applied to a scalar field or for vanishing curvature (special relativity), (2.24) takes on the operator form

$$[\partial_\tau, \partial_X] + \mathscr{D}_f = 0, \tag{2.14'}$$

if $\partial_\tau \equiv D_u$, and since

$$D_f \equiv \partial_X = D_\Lambda \tag{2.16}$$

as is readily checked. For instance, for flat space-time

$$D_u f^\alpha = D_u D_f \mathscr{X}^\alpha = \partial_\tau \partial_X \mathscr{X}^\alpha = \partial_X \partial_\tau \mathscr{X}^\alpha - \mathscr{D}_f \mathscr{X}^\alpha \tag{2.17}$$

on account of (2.14′). Hence

$$(\partial_\tau f^\alpha)_\perp = (\partial_X u^\alpha)_\perp + a^\alpha (u_\beta \, \partial_X \mathscr{X}^\beta), \tag{2.18}$$

where $a^\alpha \equiv \partial_\tau u^\alpha = (\partial_\tau u^\alpha)_\perp$ is the four-acceleration of $\mathbf{X}$. The second term in the right-hand side of (2.18) represents a purely relativistic epect. Equation (2.18) might be called the *kinematic compatibility condition* for one-dimensional relativistic motions.[3])

We finally note the following demonstrable result:

$$\overset{\perp}{\nabla}_\alpha (J^{-1} x^\alpha_K) \equiv 0, \quad \text{i.e.,} \quad \overset{\perp}{\nabla}_\alpha (J^{-1} f^\alpha) \equiv 0, \tag{2.19}$$

and the fact that the proper density of matter, ρ, is defined as being the image by $\mathscr{P}$ of the invariant density ρ_0 in a reference configuration of the material (for which $J = 1$). That is,

$$\rho(\mathbf{x} \in \mathscr{C}) = \rho_0(\mathbf{X}) J^{-1}, \tag{2.20}$$

where $\mathbf{x}$ and $\mathbf{X}$ are related by $(2.1)_3$.

2.4. *Field equations*

In addition to Einstein's equations

$$R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R = k T^{\alpha\beta}, \qquad R^{\alpha\beta} \equiv R^{\mu}_{\cdot\gamma\mu\delta} g^{\gamma\alpha} g^{\delta\beta}, \qquad R \equiv R^{\alpha}_{\cdot\alpha}, \tag{2.21}$$

where $T^{\alpha\beta}$ is the total energy-momentum tensor, we have

$$\nabla_\alpha (\rho u^\alpha) = 0 \text{ (continuity)}, \tag{2.22}$$

$$\nabla_\alpha T^{\alpha\beta} = 0, \qquad T^{[\alpha\beta]} \equiv \tfrac{1}{2}(T^{\alpha\beta} - T^{\beta\alpha}) = 0. \tag{2.23}$$

In absence of electromagnetic fields and spins $T^{\alpha\beta}$ admits the following simplecanonical space-time decomposition:

$$T^{\alpha\beta} = \rho(1+\varepsilon) u^\alpha u^\beta - t^{\alpha\beta}, \tag{2.24}$$

where $t^{\alpha\beta} = t^{\beta\alpha} = (t^{\alpha\beta})_\perp$ is the spatial relativistic stress tensor, and ε is the internal energy per unit of proper mass. Projecting $(2.23)_1$ along $\mathbf{u}$ and orthogonally to it, we obtain

$$\rho D_u \varepsilon = t^{\alpha\beta} \, d_{\alpha\beta} \quad \text{(energy equation)}, \tag{2.25}$$

$$\rho F^{\alpha}_{\cdot\beta} a^\beta = P^{\alpha}_{\cdot\gamma} \overset{\perp}{\nabla}_\beta t^{\gamma\beta} \quad \text{(Euler–Cauchy equations)}, \tag{2.26}$$

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[3]) Compare the nonrelativistic case in Bland [11].

where

$$F_{\alpha\beta} \equiv (1+\varepsilon)P_{\alpha\beta} - \rho^{-1}t_{\alpha\beta} = F_{\beta\alpha} \equiv (F_{\alpha\beta})_{\perp} \tag{2.27}$$

is the *tensorial index* of the continuum, cf. [5].

In absence of dissipative processes, the local statement of the second principle of thermodynamics reduces to the equation

$$D_u\eta = \partial_\tau \eta = 0, \tag{2.28}$$

where η is the entropy per unit of proper mass.

For one-dimensional motions, (2.25) and (2.26) transform as follows. Introduce the object with components $T^{\gamma K}$ and the spatial vector field T^{γ} (the spatial *stress vector*) in such a way that

$$T^{\gamma K} \equiv JX^K_\beta t^{\gamma\beta}, \qquad T^{\gamma} \equiv T^{\gamma K}\Lambda_K. \tag{2.29}$$

Then

$$t^{\gamma\beta} = J^{-1}x^\beta_K T^{\gamma K} = J^{-1}T^{\gamma}f^{\beta}, \quad \text{i.e.,} \quad \mathbf{t} = J^{-1}\mathbf{T}\otimes\mathbf{f}. \tag{2.30}$$

On account of (2.19) and (2.20), (2.26) takes on the form

$$\rho_0 F^{\alpha}_{\cdot\beta}\, \partial_\tau u^{\beta} = (\partial_X u^{\alpha})_{\perp}. \tag{2.31}$$

By the same token (2.25) transforms to

$$\rho_0\, \partial_\tau \varepsilon = T_\alpha\, (\partial_X u^{\alpha})_{\perp}. \tag{2.32}$$

2.5. *Constitutive equations for elastic solids*

For nonlinear elastic solids we may consider

$$\varepsilon = \tilde{\varepsilon}(\mathbf{f}, \eta). \tag{2.33}$$

Then we have the constitutive equations

$$T_\alpha = \rho_0\left(\frac{\partial\tilde{\varepsilon}}{\partial f^{\alpha}}\right)_{\perp}, \qquad \theta = \frac{\partial\tilde{\varepsilon}}{\partial\eta} \tag{2.34}$$

from Gibb's equation

$$\rho_0\theta\, d\eta = \rho_0\, d\varepsilon + T_\alpha\, df^{\alpha}, \tag{2.35}$$

where θ is the proper thermodynamical temperature ($\theta > 0$, $\inf \theta = 0$), and (2.32) is satisfied on account of (2.28) and (2.18) if and only if

$$u_\beta\, \partial_X \mathscr{X}^{\beta} = 0. \tag{2.36}$$

By use of $\mathcal{P}$ we can associate with $\mathbf{\Lambda}$ a *unit* spatial vector field on M by the relation

$$\lambda^{\alpha} = L^{-1}x^{\alpha}_K\Lambda^K, \; \Lambda^K = LX^K_\alpha\lambda^{\alpha}, \tag{2.37}$$

where

$$L \equiv (P_{\alpha\beta}x^{\alpha}_K\Lambda^K x^{\beta}_L\Lambda^L)^{1/2} = (P_{\alpha\beta}f^{\alpha}f^{\beta})^{1/2} = |\mathbf{f}| \neq 0. \tag{2.38}$$

These definitions will prove useful in the sequel.

2.6. *Isotropic materials*

According to previous works [5]–[6], ε depends on the components of $\mathbf{f}$ through a dependence on the strain tensor $E_{KL}=\frac{1}{2}(P_{\alpha\beta}x^{\alpha}_{K}x^{\beta}_{L}-G_{KL})$. That is, $\mathbb{E}=\frac{1}{2}[(\mathbf{f})^2\mathbf{\Lambda}\otimes\mathbf{\Lambda}-\mathbf{G}]$. If the material body is *isotropic*, then ε depends on $\mathbf{f}$ only through the three elementary invariants of $\mathbb{E}$. Following Bland [11], it then is a simple matter to show that $\varepsilon(\mathbf{f},\eta)$ reduces necessarily to

$$\varepsilon=\phi(f,N,\eta), \tag{2.39}$$

where

$$f=f_{\parallel}=f^{\alpha}\lambda_{\alpha},\qquad N=\mathbf{f}^2-f^2=\mathbf{f}_{\perp}^2=S_{\alpha\beta}f^{\alpha}f^{\beta},\qquad f^{\alpha}_{\perp}=S^{\alpha}_{\cdot\beta}f^{\beta}, \tag{2.40}$$

with

$$S^{\alpha}_{\cdot\beta}=P^{\alpha}_{\cdot\beta}-\lambda^{\alpha}\lambda_{\beta},\qquad S^{\alpha}_{\cdot\beta}u_{\alpha}=S^{\alpha}_{\cdot\beta}\lambda_{\alpha}=0,\qquad S^{\alpha}_{\cdot\alpha}=2. \tag{2.41}$$

It is assumed that $\eta=0$ is the entropy of the homogeneous, undeformed natural state of the body. For the sake of simplicity, ϕ is supposed to be an *analytic* function of its arguments, but additional conditions will be imposed as the need arises. In particular, the following derivatives are always meaningful:

$$\phi_N=\frac{\partial\phi}{\partial N},\qquad \phi_{ff}=\frac{\partial^2\phi}{\partial f^2},\qquad \phi_{fN}=\frac{\partial^2\phi}{\partial f\,\partial N},\qquad \phi_{NN}=\frac{\partial^2\phi}{\partial N^2}, \tag{2.42}$$

$$\phi_{f\eta}=\frac{\partial^2\phi}{\partial f\,\partial\eta},\qquad \phi_{\eta N}=\frac{\partial^2\phi}{\partial\eta\,\partial N}.$$

2.7. *Neo-Hookean materials*

If we expand $\phi(f,N,\eta)$ about the natural undeformed state $(f,N,\eta)=(0,0,0)$ and retain terms in η and in the components of $\mathbf{f}$ up to the second order inclusive, we obtain

$$\varepsilon=\theta_0\eta+\tfrac{1}{2}c_L^2f^2+\tfrac{1}{2}c_T^2N+\frac{C}{2}\eta^2-\kappa\eta f, \tag{2.43}$$

where c_L, c_T, C, κ and $\theta_0(>0)$ are suitable constants. c_L and c_T will be recognized as the longitudinal and transverse disturbance speeds of conventional elasticity. It is assumed that

$$c_T^2>0,\qquad c_L^2>2c_T^2. \tag{2.44}$$

The latter condition holds good for all known materials. The positive definiteness of ε and the fact that $\theta>0$ require that

$$\theta=\theta_0+C_\eta-\kappa f>0,\qquad C>3\kappa^2/(3c_L^2-4c_T^2). \tag{2.45}$$

C is always positive, but κ may be either positive or negative according as the solid expands or contracts on heating. The classical Lamé moduli λ_1 and λ_2 are defined by $\lambda_2 = \rho_0 c_T^2 > 0$ and $\lambda_1 = \rho_0(c_L^2 - 2c_T^2) > 0$. For neo-Hookean materials, (2.42) reduce to

$$\phi_N = \tfrac{1}{2}c_T^2, \qquad \phi_{ff} = c_L^2, \qquad \phi_{fN} = \phi_{nn} = \phi_{\eta N} = 0, \qquad \phi_{f\eta} = -\kappa. \tag{2.46}$$

3. Definition of singular surfaces (Cf. [12]–[13])

3.1. *Infinitesimal discontinuities*

Let $W(x^\alpha) = 0$ be the timelike regular hypersurface that represents a discontinuity front which propagates in V^4, and thus separates $\mathcal{B} = \mathcal{T}[B]$ in two subregions $\mathcal{B}^+$ and $\mathcal{B}^-$ at each time. We set

$$l_\alpha = \partial_\alpha W = L(\lambda_\alpha - \mathcal{U}u_\alpha), \qquad \mathcal{U} = L^{-1}(u^\sigma l_\sigma), \tag{3.1}$$

where λ_α and L are given in (2.38) and $\mathcal{U}$ is the (nondimensional) speed of the singular surface measured with respect to the moving matter. l_α is oriented from the "minus" to the "plus" face of W. $\mathbf{A}^+$ and $\mathbf{A}^-$ being the uniform limits in approaching W on its two faces of a field $\mathbf{A}$, we note $[\![\mathbf{A}]\!] = \mathbf{A}^+ - \mathbf{A}^-$ the jump of $\mathbf{A}$ across W. If $\mathbf{A}$, $\mathbf{g}$ and $\mathbf{u}$ are continuous across W and if $\bar{\delta}$ denotes the Dirac distribution with compact support on W, then we can write

$$\bar{\delta}[\![\nabla_\alpha \mathbf{A}]\!] = l_\alpha\, \delta\mathbf{A} \tag{3.2}$$

and

$$\bar{\delta}[\![\overset{\perp}{\nabla}_\alpha \mathbf{A}]\!] = L\lambda_\alpha\, \delta\mathbf{A}, \qquad \bar{\delta}[\![D_u \mathbf{A}]\!] = L\mathcal{U}\, \delta\mathbf{A}, \tag{3.3}$$

where the field $\delta\mathbf{A}$ is called the *infinitesimal discontinuity* of $\mathbf{A}$ across W. We call π_λ the two-plane orthogonal to the unit spatial vector λ_α. Then $S_{\alpha\beta}$ is the covariant projector onto π_λ. The canonical decomposition of any *spatial* geometrical object along the direction of $\boldsymbol{\lambda}$ and onto π_λ is effected by applying the operator $\mathbf{S}$, e.g., wit an obvious notation and obvious properties for the elements of decomposition thus introduced,

$$\delta u^\alpha = \delta u_\perp^\alpha + \lambda^\alpha\, \delta u, \qquad \lambda_\alpha\, \delta u_\perp^\alpha = 0, \qquad \delta u \equiv \lambda_\alpha\, \delta u^\alpha, \tag{3.4}$$

$$F_{\alpha\beta} = \bar{F}_{\alpha\beta} + 2\bar{F}_{(\alpha}\lambda_{\beta)} + \bar{F}\lambda_\alpha\lambda_\beta = F_{\beta\alpha} \tag{3.5}$$

We call $\mathfrak{M}_0(\mathbf{x} \in \mathcal{T}[B] \subset M) = \{\rho, \varepsilon, \eta, u^\alpha, t^{\alpha\beta}, f^\alpha, g_{\alpha\beta}\}$ a solution of the system of equations formed by eqs. (2.21), (2.22), (2.31), (2.32) and (2.34) – provided such a solution exists; this difficult problem of existence is not approached in this paper)[4]. Then weak (or infinitesimal) discontinuities are defined by the following set of hypotheses: h_1: any typical solution $\mathfrak{M}_0(\mathbf{x})$ is continuous across W; h_2: except for the metric $\mathbf{g}$, all space-time derivatives of the first order of the fields of the solution $\mathfrak{M}_0(\mathbf{x})$ suffer discontinuities across W (the case where $[\![\partial_\gamma g_{\alpha\beta}]\!] \neq 0$ requires a special study); h_3: W is not a gravitational front, i.e., $\mathcal{U}^2 = 1$ is excluded; h_4: W is not a *material* wave front or, in other words, since $D_u\eta = 0$ implies $\mathcal{U}\,\delta\eta = 0$ in agreement with $(3.3)_2$, W is not an entropy front, i.e., $\mathcal{U} = 0$ is excluded, so that $\delta\eta = 0$ necessarily.

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[4]) See the early work of Pichon [14].

In virtue of h_1, W is not a shock wave since $[\![u^\alpha]\!]=0$. In virtue of h_3 and h_4 the admissible range for $\mathcal{U}$ is limited to the open interval $]0, 1[\subset \mathbb{R}$ if $\mathcal{U}$ is to be real and less than the light velocity in vacuum (relativistic causality).

We call *principal* wave fronts those wave fronts for which λ_α coincides with an eigenvector of the initial state of stress $t^{\alpha}_{\cdot\beta}\in\mathfrak{M}_0$. According to previous studies [6], if W is such a wave front, then the corresponding λ_α coincides also with an eigenvector of the initial state of strains in the case of *isotropic* (even though nonlinear) elastic bodies. *Longitudinal* wave fronts are those wave fronts for which $\delta u\neq 0$, $\delta u^\alpha_\perp=0$, and *transverse* wave fronts for which $\delta u=0$, $|\delta u^\alpha_\perp|\neq 0$. We shall not consider general wave fronts which may be called *mixed* wave fronts (Cf. [5]).

3.2. *Strong discontinuities*

For strong discontinuities, or *shocks*, one must replace the system of conservation laws by a system of *jump relations* at the point of discontinuity (which is a mathematical idealization). Let Y^- and Y^+ be the Heaviside (characteristic) functions of $\mathcal{B}^-$ and $\mathcal{B}^+$, respectively. Let $T^{\alpha\beta}$ be a tensor-valued function on $\mathcal{B}\subset M$, which is of class $C^1_{(M)}(\mathcal{B})$ – i.e., piecewise continuous. Then with $T^{\alpha\beta}$ we can associate a distribution-tensor ${}^DT^{\alpha\beta}$ with compact support on $\mathcal{B}$ (in the sense of distributions) such that

$$^DT^{\alpha\beta}=Y^-T^{\alpha\beta}+Y^+T^{\alpha\beta}. \tag{3.6}$$

Then (cf. Lichnerowicz [12])

$$\nabla_\alpha{}^DT^{\alpha\beta}=l_\alpha\,\bar{\delta}[\![T^{\alpha\beta}]\!]_+{}^D(\nabla_\alpha T^{\alpha\beta}), \tag{3.7}$$

so that with the balance law $\nabla_\alpha T^{\alpha\beta}=0$ in $\mathcal{B}-W$, there is associated the jump relation

$$l_\alpha[\![T^{\alpha\beta}]\!]=0 \text{ across } W. \tag{3.8}$$

4. Infinitesimal discontinuities and characteristic manifolds

4.1. *Wave speeds*

For one-dimensional motions eqs. (3.3) are shown to take the form

$$\bar{\delta}[\![\partial_X\mathbf{A}]\!]=|\mathbf{f}|^2\,\delta\mathbf{A},\qquad \bar{\delta}[\![\partial_\tau\mathbf{A}]\!]=|\mathbf{f}|\,\mathcal{U}\,\delta\mathbf{A}. \tag{4.1}$$

Then we take the infinitesimal discontinuity of eqs. (2.18), (2.22), (2.31) and (2.28) on account (2.36), (3.3) and (4.1), and obtain the following system by noting that the operator of infinitesimal discontinuity is a "derivative":

$$\mathcal{U}\,\delta\rho+\rho\,\delta u=0, \tag{4.2a}$$

$$\mathcal{U}\,\delta f^\beta-|\mathbf{f}|\,\delta u^\beta=0, \tag{4.2b}$$

$$\begin{aligned}&\mathcal{U}\,|\mathbf{f}|^{-1}F^{\alpha}_{\cdot\beta}\,\delta u^\beta-[\phi_{ff}\lambda^\alpha+2\phi_{fN}(\mathbf{f}_\perp)^\alpha]\,\delta f\\&\quad-[2\phi_{fN}\lambda^\alpha(\mathbf{f}_\perp)_\beta+4\phi_{NN}(\mathbf{f}_\perp)^\alpha(\mathbf{f}_\perp)_\beta]\,\delta f^\beta_\perp-[\phi_{f\eta}\lambda^\alpha+2\phi_{N\eta}(\mathbf{f}_\perp)^\alpha]\,\delta\eta=0,\end{aligned} \tag{4.2c}$$

$$\mathcal{U}\,\delta\eta=0. \tag{4.2d}$$

This is a linear system of eight equations for the unknowns δu, $\delta u_{\perp}^{\alpha}$, $(\delta f^{\alpha})_{\perp}$ and $\delta\eta$. Since we discard entropy fronts, $\delta\eta = 0$. Substituting then for $(\delta u^{\beta})_{\perp}$ from (4.2b) into (4.2c) we are led to a linear system of three equations for $(\delta f^{\alpha})_{\perp}$ in the form

$$(\mathcal{U}/|\mathbf{f}|)^2 F^{\alpha}_{\cdot\beta}\, \delta f^{\beta} - [\phi_{ff}\lambda^{\alpha} + 2\phi_{fN}(\mathbf{f}_{\perp})^{\alpha}]\,\delta f \\ - [2\phi_{fN}\lambda^{\alpha}(\mathbf{f}_{\perp})_{\beta} + 4\phi_{NN}(\mathbf{f}_{\perp})^{\alpha}(\mathbf{f}_{\perp})_{\beta}]\,\delta f_{\perp}^{\beta} = 0. \quad (4.3)$$

This we can decompose by using the canonical decomposition along λ^{α} and onto π_{λ} by using the operator $\mathbf{S}$ and (3.5). Considering the case of principal wave fronts, λ^{α} is an eigenvector of $F^{\alpha}_{\cdot\beta}$ according to (2.27) if λ^{α} is an eigenvector of $t^{\alpha}_{\cdot\beta}$. Thus $\bar{F}^{\alpha} \equiv 0$. Let $(\mathbf{d}_2, \mathbf{d}_3)$ be two unit space-like vectors which form a nonholonomic orthonormal basis in π_{λ}. Let $\bar{F}_2 = \bar{F}^{2}_{\cdot 2}$ and $\bar{F}_3 = \bar{F}^{3}_{\cdot 3}$ be the diagonal components of $\bar{F}^{\alpha}_{\cdot\beta}$ with $\mathbf{d}_2$ and $\mathbf{d}_3$ chosen along the principal directions of $\bar{F}^{\alpha}_{\cdot\beta}$. Call f_2 and f_3 the corresponding components of $\mathbf{f}_{\perp}$. Then (4.3) admits a nontrivial solution if and only if the determinant of the 3×3 matrix

$$\mathbf{M} = \mathbf{A}(\mathfrak{M}_0, \mathcal{U}) - \mathbf{\Phi}(\mathfrak{M}_0) \quad (4.4)$$

vanishes, where

$$\mathbf{A}(\mathfrak{M}_0, \mathcal{U}) = \operatorname{diag}\left(\bar{F}\left(\frac{\mathcal{U}}{|\mathbf{f}|}\right)^2, \bar{F}_2\left(\frac{\mathcal{U}}{|\mathbf{f}|}\right)^2, \bar{F}_3\left(\frac{\mathcal{U}}{|\mathbf{f}|}\right)^2\right) \quad (4.5)$$

and

$$\mathbf{\Phi}(\mathfrak{M}_0) = \begin{pmatrix} \phi_{ff} & 2\phi_{fN}f_2 & 2\phi_{fN}f_3 \\ 2\phi_{fN}f_2 & 2\phi_N + 4\phi_{NN}f_2^2 & 4\phi_{NN}f_2f_3 \\ 2\phi_{fN}f_3 & 4\phi_{NN}f_2f_3 & 2\phi_N + 4\phi_{NN}f_3^2 \end{pmatrix}. \quad (4.6)$$

Since $\bar{F}$, $\bar{F}_2$ and $\bar{F}_3$ are positive quantities, it follows that the system (4.3) will have real disturbance speeds $\mathcal{U}$ if and only if the matrix $\mathbf{\Phi}(\mathfrak{M}_0)$ is *positive-definite* [a fact expressed symbolically by $\mathbf{\Phi}(\mathfrak{M}_0) > 0$]. This is Hadamard's celebrated condition. Thus,

$$\mathbf{\Phi}(\mathfrak{M}_0) = \tilde{\mathbf{\Phi}}(\mathbf{f}) > 0 \quad \text{(Hadamard's hyperbolicity condition)} \quad (4.7)$$

Considering the case $f_3 \equiv 0$, this condition yields the inequalities (the principal minors of $\mathbf{\Phi}$ must be positive)

$$\phi_{ff} > 0, \qquad \phi_{ff}(2\phi_N + 4\phi_{NN}f_2^2) - (2\phi_{fN}f_2)^2 > 0, \qquad \phi_N > 0. \quad (4.8)$$

Then the characteristic speeds of the system (4.3) are found to be

$$\begin{aligned} \mathcal{U}_{(1)}^2 &= \frac{|\mathbf{f}|^2}{2}\left(\frac{\phi_{ff}}{\bar{F}} + \frac{2\phi_N + 4\phi_{NN}f_2^2}{\bar{F}_2} + \sqrt{D}\right), \\ \mathcal{U}_{(2)}^2 &= \frac{|\mathbf{f}|^2}{2}\left(\frac{\phi_{ff}}{\bar{F}} + \frac{2\phi_N + 4\phi_{NN}f_2^2}{\bar{F}_2} - \sqrt{D}\right), \\ \mathcal{U}_{(3)}^2 &= 2\frac{|\mathbf{f}|^2}{\bar{F}_3}\phi_N \end{aligned} \quad (4.9)$$

where

$$D = \left(\frac{\phi_{ff}}{\bar{F}} + \frac{2\phi_N + 4\phi_{NN}f_2^2}{\bar{F}_2}\right)^2 - \frac{4}{\bar{F}\bar{F}_2}[\phi_{ff}(2\phi_N + 4\phi_{NN}f_2^2) - 4\phi_{fN}^2 f_2^2] \quad (4.10)$$

If $f_2=0$, eqs. (4.9) reduce to

$$\mathcal{U}^2_{(1)}=\frac{|\mathbf{f}|^2}{2}\left[\frac{\phi_{\mathrm{ff}}}{\bar{F}}+\frac{2\phi_N}{\bar{F}_2}+\left(\frac{\phi_{\mathrm{ff}}}{\bar{F}}-\frac{2\phi_N}{\bar{F}_2}\right)\right]=\frac{|\mathbf{f}|^2}{\bar{F}}\phi_{\mathrm{ff}}=\mathcal{U}^2_L$$

$$\mathcal{U}^2_{(2)}=\frac{2\,|\mathbf{f}|^2}{\bar{F}_2}\phi_N=\mathcal{U}^2_{(3)}=\mathcal{U}^2_T \tag{4.11}$$

We have a single root and a double root. For a neo-Hookean material (4.9) reduce to (with $|\mathbf{f}|\simeq 1$)

$$\mathcal{U}^2_{(1)}=\mathcal{U}^2_L=c^2_L/\bar{F},$$

$$\mathcal{U}^2_{(2)}=\mathcal{U}^2_{(3)}=\mathcal{U}^2_T=c^2_T/\bar{F}_2. \tag{4.12}$$

That is, we have a single root that corresponds to longitudinal elastic waves and a double root that corresponds to transverse elastic waves.

From here one we shall consider only longitudinal waves. The study of the polarization of the various waves is a straightforward matter and will not be reproduced here (See [6] for a related study not limited to one-dimensional motions).

4.2. *The case of relativistic hydrodynamics*

For perfect hydrodynamics ε depends on f^α only through the determinant J or, equivalently, through the matter density ρ. The deformation field is isotropic so that $J=|\mathbf{f}|^3=\rho_0/\rho$, and the body must necessarily be compressible. Thus $\varepsilon=\phi(\rho,\eta)$. With $p=\rho^2(\partial\phi/\partial\rho)$, it is found that $t^{\alpha\beta}=-pP^{\alpha\beta}$, $\phi_N=0$, and $\phi_{\mathrm{ff}}=|\mathbf{f}|^{-2}(\partial p/\partial\rho)$ after some calculation. Hence the results (4.9) coalesce to provide the relativistic sound speed a by

$$a^2(\mathfrak{M}_0)=F^{-1}(\partial p/\partial\rho)_\eta,\ F^{-1}\equiv 1+\varepsilon+(p/\rho), \tag{4.13}$$

where F is Lichnerowicz' index [15] of relativistic hydrodynamics. Of course, only longitudinal (sound) waves can propagate in this case. Viscosity would be required to allow for transverse waves.

4.3. *Growth of infinitesimal discontinuities*

The system examined above is a special case of that examined in Ref. [6], and therefore is *quasi-linear hyperbolic*. This means that for certain initial conditions the corresponding infinitesimal discontinuities will grow to infinity in modulus after a finite interval of time along the corresponding ray. That is, taking δu as a typical magnitude, δu is governed along the ray of longitudinal elastic waves (for nonlinear elastic bodies) by an equation of the type (cf. [5], [9])

$$D_R(\delta u)-A(G_2^W,\mathfrak{M}_0)(\delta u)-B(\mathcal{U}_{\|},\mathfrak{M}_0)(\delta u)^2=0, \tag{4.14}$$

where D_R is the invariant derivative along the ray, A and B are scalars, and G_2^W symbolizes the second-order geometry of the wave front. For A and $B<0$, and for a compressive wave ($\delta u<0$), $|\delta u|\to\infty$ with a characteristic proper time $\tau^*=|A|^{-1}\log(\delta u^0/(\delta u^0-(A/B))]$, where δu^0 is an initial value. The infinitesimal-discontinuity solution breaks out and we are led to study shocks.

5. Longitudinal shock waves in relativistic elasticity

5.1. *The Hugoniot condition for relativistic elasticity*

We consider shock waves, i.e., singular surfaces across which $[\![u^\alpha]\!] \neq 0$, $[\![f^\alpha]\!] \neq 0$, *etc.* The space-time is assumed to be flat. On applying the formalism of Paragraph 3.2, we find that the balance laws (2.22) and (2.23) give

$$[\![\rho(u^\alpha l_\alpha)]\!] = 0 \quad \text{or} \quad m = \rho(l^\alpha u_\alpha) = \text{const. through } W, \tag{5.1}$$

and

$$m[\![(1+\varepsilon)u^\alpha]\!] - [\![t^{\alpha\beta} l_\beta]\!] = 0. \tag{5.2}$$

On using the decomposition (3.5) for $t^{\alpha\beta}$, (5.2) transforms to

$$m[\![\bar{F}u^\alpha]\!] - [\![L\bar{T}^\alpha + \bar{T}l^\alpha]\!] = 0, \tag{5.3}$$

where $\bar{F}$ is the index in the direction of propagation:

$$\bar{F} = 1 + \varepsilon - (\bar{T}/\rho). \tag{5.4}$$

Let $\langle u^\alpha \rangle \equiv \frac{1}{2}((u^\alpha)^+ + (u^\alpha)^-)$. Taking the inner product of (5.3) with $\langle u_\alpha \rangle$ yields, with $\tau \equiv \rho^{-1}$,

$$m[\![\varepsilon - \langle \bar{T} \rangle \tau]\!] - \langle L\bar{T}^\alpha - m\bar{F}u^\alpha \rangle [\![u_\alpha]\!] = 0. \tag{5.5}$$

This is the *Hugoniot jump condition* for relativistic continua)[5]. This can also be written as

$$m[\![\varepsilon]\!] + \langle \tilde{T}^\alpha \rangle [\![u_\alpha]\!] = 0, \qquad \tilde{T}^\alpha \equiv T^{\alpha\beta} l_\beta. \tag{5.5'}$$

The case $m = 0$, which corresponds to tangential shocks (so-called contact discontinuities), is not envisaged. Furthermore, realistic shocks must be such that $[\![\eta]\!] > 0$.

For one-dimensional relativistic motions (5.5) can be transformed further. Multiply (2.18) – on account of (2.36) – by ρ and take the jump of the resulting equation in accordance with (3.8) to obtain

$$m[\![f^\alpha]\!] = \rho_0 [\![u^\alpha]\!], \tag{5.6}$$

so that (5.5) or (5.5′) takes the form

$$\rho_0 [\![\varepsilon]\!] + \langle \tilde{T}^\alpha \rangle [\![f_\alpha]\!] = 0. \tag{5.7}$$

It follows from (2.30), (3.1) and (2.20) that

$$\tilde{T}^\alpha = m(1+\varepsilon)u^\alpha - \rho \frac{\partial \varepsilon}{\partial f_\alpha} Lf. \tag{5.8}$$

---

5) For *perfect hydrodynamics* the shock is principal both ahead and behind and $\bar{T}^\alpha = \bar{F}^\alpha = 0$, and $\bar{T} = -p$. Hence (5.5) takes on the form

$$m[\![\varepsilon + \langle p \rangle \tau]\!] + m\langle \bar{F}u^\alpha \rangle [\![u_\alpha]\!] = 0. \tag{a}$$

This can be shown to reduce to Lichnerowicz's Hugoniot equation for relativistic hydrodynamics. At the *nonrelativistic limit* (5.5) reduces to

$$m[\![\varepsilon]\!] = \langle \mathbf{T} \rangle \cdot [\![\mathbf{v}]\!], \tag{b}$$

where $\mathbf{T}$ is the stress vector in the direction of propagation (compare Duvaut [16]). In the same condition (a) reduces to $[\![\varepsilon + \langle p \rangle \tau]\!] = 0$ (compare Jeffrey and Taniuti [17], p. 138).

Hence (5.7) reads

$$\rho_0[\![\varepsilon]\!]+\left\langle m(1+\varepsilon)u^\alpha-\rho\frac{\partial\varepsilon}{\partial f_\alpha}Lf\right\rangle[\![f_\alpha]\!]=0. \tag{5.9}$$

This is the Hugoniot condition for one-dimensional relativistic motions in non-linear elastic bodies.

5.2. *Weak shocks in isotropic solids*

These are shocks for which $|[\![f^\alpha]\!]|$ and $[\![\eta]\!]$ are small. On expanding $\varepsilon=\phi(f,N,\eta)$ in terms of its arguments, we have

$$(\phi_\eta^-+\phi_{N\eta}^-(\mathbf{f}_\perp)_\alpha^-[\![(\mathbf{f}_\perp)_\alpha]\!]+\tfrac{1}{2}\phi_{f\eta}^-[\![f]\!])[\![\eta]\!]+\tfrac{1}{2}\phi_\eta^-[\![\eta]\!]^2+\cdots=0, \tag{5.10}$$

where "$+\cdots$" stands for analytic terms of third and higher orders in $[\![\mathbf{f}_\perp]\!]$, $[\![f]\!]$ and $[\![\eta]\!]$. It follows from the implicit function theorem, (5.9) and (5.10) that $[\![\eta]\!]$ is an analytic function in the components of $[\![f^\alpha]\!]$ and in fact is of the *third order* in the components of $[\![f^\alpha]\!]$. That is,

$$[\![\eta]\!]=0(|[\![\mathbf{f}]\!]|^3)\quad\text{for weak shocks.} \tag{5.11}$$

More precisely, it can be shown that, with $\phi_\eta=\theta$,

$$[\![\eta]\!]=\frac{1}{12\langle\theta\bar{F}\rangle}\langle\phi_{fff}\rangle[\![f]\!]^3+\cdots \tag{5.12}$$

for longitudinal shocks. The speed of these weak shocks is evaluated from (5.3) and is given by

$$\mathcal{U}_{WS}^2=\left(\frac{f^2\phi_{ff}}{\bar{F}}\right)^-+0([\![\eta]\!]), \tag{5.13}$$

which means that $\mathcal{U}_{WS}$ does not differ much from the speed of infinitesimal discontinuities ahead of the shock [compare eq. $(4.11)_1$].

The second law of thermodynamics imposes that $[\![\eta]\!]>0$ across the shock. According to (5.12), this requires that if $\phi_{fff}^->0$, $[\![f]\!]>0$ for realistic shocks which are therefore *tensive* shocks, and if $\phi_{fff}^-<0$, we must have $[\![f]\!]<0$, i.e., *compressive* shocks only are thermodynamically admissible. The general study of longitudinal shocks of finite strength mainly is a thermodynamical study concerned with the convexity and the connectedness of the so-called Hugoniot curve (Compare the classical elastic case in Duvaut [16] and the relativistic MHD case in Lichnerowicz [18]). This will not be done here.

5.3. *Compressive shocks in neo-Hookean materials*

In this case $[\![f]\!]<0$ and (2.43) holds good. We have

$$m[\![f^\alpha]\!]=[\![\rho f^2u^\alpha]\!], \tag{5.14}$$

while eq. (3.8) gives

$$[\![\rho(1+\varepsilon)f\mathcal{U}u^\alpha]\!]=[\![\rho f^2\left(c_L^2-\kappa\frac{\eta}{f}\right)f^\alpha]\!]. \tag{5.15}$$

Taking account of (5.14) in (5.15) and taking the inner product of the resulting equation with l_α, it is found that the invariant speed of the shock is approximately given by

$$\mathscr{U}^2 \simeq \frac{1}{(\bar{F})^-}\left(c_L^2 - \kappa \frac{[\![\eta]\!]}{[\![f]\!]}\right) \tag{5.16}$$

We see that if $\kappa < 0$, the right-hand side of this equation is positive for compressive shocks since $[\![f]\!] < 0$ and $[\![\eta]\!] > 0$. $(\bar{F})^-$ is the longitudinal index ahead of the shock, i.e.,

$$\begin{aligned}(\bar{F})^- &= 1 + \varepsilon^- - \left(\frac{\partial \varepsilon}{\partial f^\alpha} f^\beta \lambda^\alpha \lambda_\beta\right)^- \\ &= 1 + \theta_0 \eta^- - \tfrac{1}{2} c_L^2 (f^2)^- = 1 + \text{relativistic terms.}\end{aligned} \tag{5.17}$$

6. Relativistic magnetoelasticity

6.1. *Basic equations*

A simple realistic scheme for the relativistic magnetoelasticity of perfect conductors of electricity can be extracted from the general theory presented in Refs. [19]–[20]. First, as shown by Lichnerowicz, the whole of Maxwell's equations for perfect conductors is contained in the covariant equation

$$\nabla_\alpha (u^\alpha \mathscr{H}^\beta - \mathscr{H}^\alpha u^\beta) = 0, \tag{6.1}$$

where $\mathscr{H}$ is the spatial magnetic-field four-vector.

The continuity equation and the entropy balance, (2.22) and (2.28), are still valid, i.e.,

$$D_u \rho + \rho \nabla_\alpha u^\alpha = 0, \qquad D_u \eta = 0, \tag{6.2}$$

and eqs. (2.23) are replaced by

$$\nabla_\alpha T^{\alpha\beta}_{(\mathrm{tot})} = 0, \qquad T^{[\alpha\beta]}_{(\mathrm{tot})} = 0, \tag{6.3}$$

where the *total* energy-momentum tensor $T^{\alpha\beta}_{(\mathrm{tot})}$ contains all effects of matter, electromagnetic interactions, and electromagnetic free fields (spin effects are not considered). Since the spatial electric-field four-vector must vanish ($\boldsymbol{\varepsilon} = \mathbf{0}$) for perfect conductors, the formulas given in Ref. [20] yield

$$T^{\alpha\beta}_{(\mathrm{tot})} = [\rho(1+\varepsilon) u^\alpha u^\beta - t^{\beta\alpha}] + \{\tfrac{1}{2}\mathscr{B}^2 u^\alpha u^\beta - [\mathscr{B}^\beta \mathscr{H}^\alpha - \tfrac{1}{2}(\mathscr{B}^2 - 2\mathscr{M}.\mathscr{B}) P^{\alpha\beta}\} \tag{6.4}$$

and

$$\rho D_u \varepsilon = \rho \theta D_u \eta + t^{\beta\alpha} \overset{\perp}{\nabla}_\beta u_\alpha - \mathscr{M}^\alpha D_u \mathscr{B}_\alpha \quad \text{(Gibbs' equation)} \tag{6.5}$$

where $\mathscr{B}$ and $\mathscr{M}$ are the spatial magnetic-induction and magnetization four-vectors, respectively. For an elastic body undergoing one-dimensional relativistic motion we may take

$$\varepsilon = \tilde{\varepsilon}(\mathbf{f}, \mathscr{B}, \eta). \tag{6.6}$$

Then (6.5) may be rewritten as

$$\rho_0 D_u \varepsilon = \rho_0 \theta D_u \eta + T_\alpha D_u f^\alpha - \rho_0 \mu^\alpha D_u \mathcal{B}_\alpha, \qquad \mu^\alpha \equiv \mathcal{M}^\alpha / \rho, \tag{6.7}$$

where we have used (2.20). If follows from this the constitutive equations

$$\theta = \frac{\partial \tilde{\varepsilon}}{\partial \eta}, \qquad T_\alpha = \rho_0 \left(\frac{\partial \tilde{\varepsilon}}{\partial f^\alpha} \right)_\perp, \qquad \mu^\alpha = -\left(\frac{\partial \tilde{\varepsilon}}{\partial \mathcal{B}_\alpha} \right)_\perp. \tag{6.8}$$

We uncouple the matter-field interactions described by (6.6) by considering the following very special representation for *isotropic* bodies:

$$\varepsilon = \varepsilon_1(f, N, \eta) - \frac{1}{2} \frac{\lambda}{\rho} \mathcal{B}^2, \qquad \lambda = \text{const.}, \tag{6.9}$$

so that eqs. (6.8) yield

$$\theta = \frac{\partial \varepsilon_1}{\partial \eta}, \qquad t^{\beta\alpha} = \rho \left(\frac{\partial \varepsilon_1}{\partial f_\alpha} \right)_\perp f^\beta, \qquad \mathcal{M}^\alpha = \lambda \mathcal{B}^\alpha. \tag{6.10}$$

Setting $\mu^{-1} = 1 - \lambda$, so that $\mathcal{H}^\alpha = \mathcal{B}^\alpha - \mathcal{M}^\alpha = \mu^{-1} \mathcal{B}^\alpha$, an easy calculation allows us to find the expression of $T^{\alpha\beta}_{(\text{tot})}$ as (for $\mu \neq 1$, astrophysical case)

$$T^{\alpha\beta}_{(\text{tot})} = \rho(1+\varepsilon_1) u^\alpha u^\beta - t^{\beta\alpha} + \mu \mathcal{H}^2 (\tfrac{1}{2} g^{\alpha\beta} + u^\alpha u^\beta) - \mu \mathcal{H}^\alpha \mathcal{H}^\beta. \tag{6.11}$$

The last two terms form the electromagnetic energy-momentum tensor used by Lichnerowicz (up to the signature of the metric). Substituting from (6.11) into $(6.3)_1$ and projecting the result on the direction of **u** and orthogonally to it, we obtain after a somewhat lengthy calculation:

$$\rho D_u \left(\varepsilon_1 + \frac{1}{2} \frac{\mu}{\rho} \mathcal{H}^2 \right) = \left(t^{\alpha\beta} + \mu \mathcal{H}^\alpha \mathcal{H}^\beta - \frac{1}{2} \mu \mathcal{H}^2 P^{\alpha\beta} \right) \overset{\perp}{\nabla}_\beta u_\alpha, \tag{6.12}$$

and the Euler-Cauchy equations of the motion in the form [compare eq. (2.26)]

$$\rho \tilde{F}^{\alpha}_{\cdot\beta} a^\beta = P^{\alpha}_{\cdot\gamma} \overset{\perp}{\nabla}_\beta t^{\alpha\beta} + \mu [\mathcal{H}^\alpha \, \nabla_\beta \mathcal{H}^\beta + \mathcal{H}^\beta \, (\nabla_\beta \mathcal{H}^\alpha)_\perp - \overset{\perp}{\nabla}{}^\alpha (\mathcal{H}^2/2)] \tag{6.13}$$

where $\tilde{F}_{\alpha\beta}$ is the "magnetic" tensorial index of the continuum [compare eq. (2.27)] defined by

$$\tilde{F}_{\alpha\beta} = \left(1 + \varepsilon_1 + \frac{\mu}{\rho} \mathcal{H}^2 \right) P_{\alpha\beta} - \rho^{-1} t_{\alpha\beta} = \tilde{F}_{\beta\alpha} = (\tilde{F}_{\alpha\beta})_\perp. \tag{6.14}$$

Equations (6.1), (6.2), (6.12), (6.13), together with $(6.10)_{1-2}$, Einstein's equations (2.21) and the kinematical compatibility condition

$$(\partial_\tau f^\alpha)_\perp = (\partial_X u^\alpha)_\perp, \tag{6.15}$$

constitute the complete set of equations for the present theory. If ε_1 depends on **f** only through ρ, then the latter reduces to the theory of relativistic magnetohydrodynamics as given, for instance, by Lichnerowicz [15]. Other models have been developed by Bressan [21].

Some other useful equations can be extracted from the above system. For instance, by projecting (6.1) along and orthogonally to $\mathbf{u}$, we obtain

$$u^\alpha u^\beta \nabla_\alpha \mathcal{H}_\beta + \nabla_\alpha \mathcal{H}^\alpha = 0, \tag{6.16}$$

$$\mathcal{H}^\beta \nabla_\alpha u^\alpha + (D_u \mathcal{H}^\beta)_\perp - \mathcal{H}^\alpha (\nabla_\alpha u^\beta)_\perp = 0. \tag{6.17}$$

Taking the inner product of (6.17) with $\mathcal{H}_\beta$ yields

$$\mathcal{H}^2 \nabla_\alpha u^\alpha + D_u(\mathcal{H}^2/2) + u^\beta \mathcal{H}^\alpha \overset{\perp}{\nabla}_\alpha \mathcal{H}_\beta = 0. \tag{6.18}$$

Substituting then from (6.18) in (6.13), taking the inner product of the resulting equation with $\mathcal{H}$, and accounting for (6.16), we arrive at

$$\rho(1+\varepsilon_1)\nabla_\beta \mathcal{H}^\beta + \mathcal{H}_\alpha \nabla_\beta t^{\alpha\beta} = 0, \tag{6.19}$$

which can be used instead of (6.12).

6.2. *Infinitesimal discontinuities in isotropic solids*[6])

The system of equations we have to consider is the following one:

$$D_u\rho + \rho \nabla_\alpha u^\alpha = 0, \qquad D_u f^\alpha = D_f u^\alpha, \qquad D_u \eta = 0; \tag{6.20}$$

$$u^\alpha u^\beta \nabla_\alpha \mathcal{H}_\beta + \nabla_\alpha \mathcal{H}^\alpha = 0; \tag{6.21}$$

$$\mathcal{H}^\beta \nabla_\alpha u^\alpha + (D_u \mathcal{H}^\beta)_\perp - \mathcal{H}^\alpha (\nabla_\alpha u^\beta)_\perp = 0; \tag{6.22}$$

$$\rho(1+\varepsilon_1)\nabla_\beta \mathcal{H}^\beta + \mathcal{H}_\alpha \nabla_\beta t^{\alpha\beta} = 0; \tag{6.23}$$

$$\rho \tilde{F}^{\alpha}_{\cdot\beta} D_u u^\beta - P^{\alpha}_{\cdot\gamma} \overset{\perp}{\nabla}_\beta t^{\gamma\beta} - \mu[\mathcal{H}^\alpha \nabla_\beta \mathcal{H}^\beta + \mathcal{H}^\beta (\nabla_\beta \mathcal{H}^\alpha)_\perp - \overset{\perp}{\nabla}{}^\alpha (\mathcal{H}^2/2)] = 0; \tag{6.24}$$

$$t^{\alpha\beta} = \rho \frac{\partial \phi}{\partial f_\alpha} f^\beta, \qquad \varepsilon_1 = \phi(f, N, \eta) \quad \text{in relativistic magnetoelasticity,} \tag{6.25}$$

$$t^{\alpha\beta} = -pP^{\alpha\beta}, \qquad \varepsilon_1 = \phi(\rho, \eta), \qquad p = \rho^2 \frac{\partial \phi}{\partial \rho} \quad \text{in relativistic MHD.} \tag{6.26}$$

With $\mathcal{U} \neq 0$, eqs. (6.20) immediately yield

$$\delta\rho = -\rho \mathcal{U}^{-1}\,\delta u, \qquad \delta f = f\mathcal{U}^{-1}\,\delta u, \qquad \delta f^\alpha_\perp = f\mathcal{U}^{-1}\,\delta u^\alpha_\perp, \qquad \delta\eta = 0. \tag{6.27}$$

Consider the case of principal infinitesimal discontinuities propagating in an initial *longitudinal* state of deformation, i.e., $f^\alpha = f\lambda^\alpha$, $S_{\alpha\beta} f^\beta = 0$, (6.25) yields

$$t^{\alpha\beta} = \rho(\phi_f \lambda^\alpha + 2\phi_N S^{\alpha}_{\cdot\mu} f^\mu) f^\beta = \rho\phi_f \lambda^\alpha f^\beta \tag{6.28}$$

and, with $\delta f = \lambda_\alpha\,\delta f^\alpha$,

$$\delta t^{\alpha\beta} = \delta\rho(\phi_f \lambda^\alpha f^\beta) + \rho\phi_{ff} \lambda^\alpha f^\beta\,\delta f + \rho\phi_f \lambda^\alpha\,(\delta f^\beta)_\perp \tag{6.29}$$

on account of $(6.27)_4$. On substituting from $(6.27)_{1-3}$ and taking account of initial conditions, it comes

$$\delta t^{\alpha\beta} = \mathcal{U}^{-1}(\rho\phi_{ff} f^2 \lambda^\alpha \lambda^\beta\,\delta u + \rho\phi_f f \lambda^\alpha\,\delta u^\beta_\perp). \tag{6.30}$$

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[6]) Only the main steps in the derivation are reproduced.

Setting $\mathscr{H}_{\|} \equiv \mathscr{H}_{\alpha}\lambda^{\alpha}$, from (6.21)–(6.23) we obtain

$$L\mathscr{U}(u_{\beta}\,\delta\mathscr{H}^{\beta}) + (l_{\alpha}\,\delta\mathscr{H}^{\alpha}) = 0, \tag{6.31}$$

$$\mathscr{U}\,\delta\mathscr{H}^{\beta} + \mathscr{H}^{\beta}\,\delta u - \mathscr{H}_{\|}\,\delta u^{\beta} = 0, \tag{6.32}$$

$$\mathscr{U}(1+\varepsilon_1)(l_{\beta}\,\delta\mathscr{H}^{\beta}) + L(\phi_{ff}f^2\mathscr{H}_{\|})\delta u = 0 \tag{6.33}$$

on account of (6.30). Finally, (6.24) yields

$$\rho L\mathscr{U}\tilde{F}^{\alpha}_{\cdot\beta}\,\delta u^{\beta} - \rho L\mathscr{U}^{-1}\phi_{ff}f^2\lambda^{\alpha}\,\delta u - \mu\mathscr{H}^{\alpha}(l_{\beta}\,\delta\mathscr{H}^{\beta}) - \mu L\mathscr{H}_{\|}\,\delta\mathscr{H}^{\alpha} + \mu L\lambda^{\alpha}\,\delta(\mathscr{H}^2/2) = 0. \tag{6.34}$$

But from (6.31) and (6.32)

$$l_{\beta}\,\delta\mathscr{H}^{\beta} = L\mathscr{U}\mathscr{H}_{\beta}\,\delta u^{\beta}, \tag{6.35}$$

$$\delta\mathscr{H}^{\alpha} = \mathscr{U}^{-1}(\mathscr{H}_{\|}\,\delta u^{\alpha} - \mathscr{H}^{\alpha}\,\delta u), \tag{6.36}$$

$$\delta(\mathscr{H}^2/2) = \mathscr{U}^{-1}(\mathscr{H}_{\|}\mathscr{H}_{\beta}\,\delta u^{\beta} - \mathscr{H}^2\,\delta u). \tag{6.37}$$

On substituting from eqs. (6.35)–(6.37) into (6.34), we arrive at a linear system of three equations for δu^{α}. On using $(3.4)_1$ and a similar decomposition for $\mathscr{H}^{\alpha}$, i.e., $\mathscr{H}^{\alpha} = \mathscr{H}^{\alpha}_{\perp} + \mathscr{H}_{\|}\lambda^{\alpha}$, this system reads

$$[\rho\mathscr{U}^2(\tilde{F}^{\alpha}_{\cdot\mu} - (\mu/\rho)\mathscr{H}^{\alpha}\mathscr{H}_{\perp\mu}) - \mu\mathscr{H}^2_{\|}S^{\alpha}_{\cdot\mu} + \mu\mathscr{H}_{\|}\lambda^{\alpha}\mathscr{H}_{\perp\mu}]\,\delta u^{\mu}_{\perp} + [\rho\mathscr{U}^2(\tilde{F}^{\alpha}_{\cdot\beta}\lambda^{\beta} - (\mu/\rho)\mathscr{H}^{\alpha}\mathscr{H}_{\|}) - (\rho\phi_{ff}f^2 + \mu\mathscr{H}^2_{\perp})\lambda^{\alpha} + \mu\mathscr{H}_{\|}\mathscr{H}^{\alpha}_{\perp}]\,\delta u = 0.$$

Setting

$$A^2_{\perp} = \mu\mathscr{H}^2_{\perp}/\rho, \qquad A^2_{\|} = \mu\mathscr{H}^2_{\|}/\rho, \qquad F_{\|} = 1 + \varepsilon - \phi_f f + A^2_{\perp}, \qquad \tilde{c}^2_{\|} = \phi_{ff}f^2 + A^2_{\perp},$$

and projecting the above equation in the direction of $\boldsymbol{\lambda}$ and onto π_{λ} yields the equations ($|\mathscr{U}| \neq 1$)

$$(1-\mathscr{U}^2)(\mu\mathscr{H}_{\|}/\rho)\mathscr{H}_{\perp\alpha}\,\delta u^{\alpha}_{\perp} + (\mathscr{U}^2F_{\|} - \tilde{c}^2_{\|})\,\delta u = 0,$$

$$\{\mathscr{U}^2[S^{\alpha}_{\cdot\gamma}\tilde{F}^{\gamma}_{\cdot\beta}S^{\beta}_{\cdot\mu} - (\mu/\rho)\mathscr{H}^{\alpha}_{\perp}\mathscr{H}_{\perp\mu}] - A^2_{\|}S^{\alpha}_{\cdot\mu}\}\,\delta u^{\mu}_{\perp} + (1-\mathscr{U}^2)(\mu\mathscr{H}_{\|}/\rho)\mathscr{H}^{\alpha}_{\perp}\,\delta u = 0. \tag{6.38}$$

Consider the special case for which $\mathscr{H}_{\perp 3} = 0$. Then the compatibility condition for solving (6.38) splits in two parts and yields

$$\mathscr{U}^2 = \mathscr{U}^2_1 \equiv A^2_{\|}/F_m, \qquad F_m = 1 + \varepsilon + (\mu\mathscr{H}^2/\rho), \tag{6.39}$$

and

$$(\mathscr{U}^2 - \mathscr{C}^2_{\|})(\mathscr{U}^2 - \mathscr{C}^2_{\perp}) = (1-\mathscr{U}^2)^2\mathscr{A}^2_{\|}\mathscr{A}^2_{\perp}, \tag{6.40}$$

where we have defined various relativistic speeds and Alfvén numbers by

$$\mathscr{C}^2_{\|} \equiv \tilde{c}^2_{\|}/F_{\|}, \qquad \mathscr{C}^2_{\perp} \equiv A^2_{\|}/F_{\perp}, \qquad F_{\perp} \equiv 1 + \varepsilon + A^2_{\|}, \qquad \mathscr{A}^2_{\|} = A^2_{\|}/F_{\|}, \qquad \mathscr{A}^2_{\perp} = A^2_{\perp}/F_{\perp}.$$

On setting

$$\mathscr{C}^2 \equiv \tfrac{1}{2}(\mathscr{C}^2_{\|} + \mathscr{C}^2_{\perp} - 2\mathscr{A}^2_{\|}\mathscr{A}^2_{\perp}), \qquad \alpha^2 \equiv 1 - \mathscr{A}^2_{\|}\mathscr{A}^2_{\perp},$$

$$\mathscr{D} \equiv \mathscr{C}^4 - \mathscr{C}^2_{\|}\mathscr{C}^2_{\perp}\alpha^2\left[1 - 2\left(\frac{\mathscr{A}_{\|}}{\mathscr{C}_{\|}}\right)^2\left(\frac{\mathscr{A}_{\perp}}{\mathscr{C}_{\perp}}\right)^2\right] > 0,$$

The roots of (6.40) are obtained as

$$\mathcal{U}_S^2 = \alpha^{-2}(\mathcal{C}^2 - \sqrt{\mathcal{D}}), \qquad \mathcal{U}_F^2 = \alpha^{-2}(\mathcal{C}^2 + \sqrt{\mathcal{D}}). \tag{6.41}$$

A long proof allows one to show that, in general, $\mathcal{U}_S^2 \leqq \mathcal{U}_I^2 < \mathcal{U}_F^2$ (cf. the non-relativistic case in [22] and relativistic MHD in [12]). (6.39) corresponds to a purely transverse (so-called *intermediate*) mode. $(6.41)_1$ and $(6.41)_2$ correspond to so-called *slow* and *fast* magnetoelastic modes. These two modes in general are neither purely transverse no purely longitudinal in so far as elastic oscillations are concerned. All modes, however, offer purely transverse magnetic oscillations. Two special cases of obvious interest are:

(i) *Purely longitudinal initial magnetic field*: Then (6.39) and (6.41) yield

$$\mathcal{U}_S^2 = \mathcal{U}_I^2 = A_{\|}^2/(1+\varepsilon+A_{\|}^2), \qquad \mathcal{U}_F^2 = (\phi_{ff} f^2)/(1+\varepsilon-\phi_f f);$$

(ii) *Purely transverse initial magnetic field*: Then we have

$$\mathcal{U}_S^2 = \mathcal{U}_I^2 = 0, \qquad \mathcal{U}_F^2 = (\phi_{ff} f^2 + A_{\perp}^2)/(1+\varepsilon-\phi_f f + A_{\perp}^2).$$

The vanishing double root corresponds to *stationary* modes (in a co-moving frame). The simplicity of the above-obtained results follows from the simple initial mechanical state. The case of general three-dimensional initial states of strains requires a generalization of the purely elastic case studied in Ref. [6].

6.3. *Notions on shock waves*

Here we give only some notions on how the shock-wave problem can be envisaged in relativistic magnetoelasticity. Magnetoelastic shocks can occur for certain initial conditions since the solution of Paragraph 6.2 can break out in a finite interval of time, the corresponding system of field equations being quasi-linear hyperbolic and each weak-discontinuity magnitude being governed by an equation of the type (4.14) along its ray.

According to eq. (3.8) quantities of the form $l_\alpha T^{\alpha\beta}$ are conserved across a shock W if the conservation law $\nabla_\alpha T^{\alpha\beta} = 0$ holds true in $\mathcal{B} - W$. We can apply this property to the general equations (2.22), (6.1) and $(6.3)_1$, in which $T^{\alpha\beta}_{(\mathrm{tot})}$ is given by (6.4). Let the symbolism $\propto \mathrm{INV}(W, j)$ indicate that the quantity to which it applies is conserved through W and has j independent scalar components in a local chart of V^4. Then, S^- representing the state ahead of the shock,

$$m(S^-) = \rho(l^\alpha u_\alpha) \propto \mathrm{INV}(W, 1), \tag{6.42}$$

$$V^\alpha(S^-) = (\mathcal{H}^\sigma l_\sigma) u^\alpha - \mathcal{H}^\alpha (u^\sigma l_\sigma) \propto \mathrm{INV}(W, 4), \tag{6.43}$$

$$W^\alpha(S^-) = m(S^-)\left(1+\varepsilon_1 + \frac{\mu}{\rho}\mathcal{H}^2\right) u^\alpha - t^{\alpha\beta} l_\beta + \frac{1}{2}\mu \mathcal{H}^2 l^\alpha - \mu \mathcal{H}^\alpha(\mathcal{H}^\sigma l_\sigma) \propto \mathrm{INV}(W, 4). \tag{6.44}$$

But (6.43) shows that $V^\alpha l_\alpha = 0$, so that V^α is tangential and in fact is INV $(W, 3)$. Define

$$H(S^-) \equiv m^{-2}(S^-) V^\alpha V_\alpha = \frac{\mathcal{H}^2}{\rho^2} - \frac{(\mathcal{H}^\sigma l_\sigma)^2}{m^2}. \tag{6.45}$$

Let v_α be the tangential component of u_α. Then (6.43) yields

$$V^\beta(S^-) = (\mathscr{H}^\sigma l_\sigma)u^\beta - \frac{m(S^-)}{\rho}\mathscr{H}^\beta. \tag{6.46}$$

Call κ^β the tangential component of $\mathscr{H}^\beta$, so that (6.46) renders

$$\kappa^\beta = \frac{\rho}{m(S^-)}[(\mathscr{H}^\sigma l_\sigma)v^\beta - V^\beta(S^-)]. \tag{6.47}$$

Defining a magnetic pressure by $p_m = \frac{1}{2}\mu\mathscr{H}^2$ and using (6.45) allows us to transform (6.44) to

$$W^\alpha(S^-) = [m(S^-)(1+\varepsilon_1)u^\alpha - t^{\alpha\beta}l_\beta] + p_m l^\alpha + \mu m(S^-)H(S^-)(\rho u^\alpha) + \frac{\rho\mu}{m(S^-)}V^\alpha(S^-)(\mathscr{H}^\sigma l_\sigma). \tag{6.48}$$

This can be decomposed as

$$W^\alpha(S^-) = X^\alpha(S^-) + \tilde{e}(S^-)l^\alpha,\ X^\alpha(S^-) \propto \mathrm{INV}\,(W, 3),\ \tilde{e}(S^-) \propto \mathrm{INV}\,(W, 1). \tag{6.49}$$

We find that

$$e(S^-) \equiv \tilde{e}(S^-) - \mu m^2(S^-)H(S^-) = p_m + (l^\sigma l_\sigma)^{-1}[m(S^-)(1+\varepsilon_1)(u^\sigma l_\sigma) - t^{\alpha\beta}l_\alpha l_\beta] \propto \mathrm{INV}\,(W, 1). \tag{6.50}$$

Call $\mathscr{T}^\alpha$ the tangential component of $[m(S^-)(1+\varepsilon_1)u^\alpha - t^{\alpha\beta}l_\beta]$. Then we have

$$X^\alpha(S^-) = \mathscr{T}^\alpha + \mu m(S^-)H(S^-)(\rho v^\alpha) + \frac{\rho\mu}{m(S^-)}(\mathscr{H}^\sigma l_\sigma)V^\alpha(S^-) \propto \mathrm{INV}\ (W, 3). \tag{6.51}$$

Other invariants can be found as follows. For instance,

$$\mathscr{X}(S^-) \equiv -X^\alpha(S^-)V_\alpha(S^-) = -W^\alpha(S^-)V_\alpha(S^-) \propto \mathrm{INV}\,(W, 1). \tag{6.52}$$

But it follows from the definition of v^α that

$$v^\alpha v_\alpha = -\left[1 + \frac{m^2(S^-)}{\rho^2(l^\sigma l_\sigma)}\right],\ \rho v^\alpha V_\alpha = -\rho(\mathscr{H}^\sigma l_\sigma). \tag{6.53}$$

On account of $(6.53)_2$ and (6.51) we therefore have

$$\mathscr{X}(S^-) = -\mathscr{T}^\alpha V_\alpha(S^-) \propto \mathrm{INV}\,(W, 1).$$

Finally, consider

$$\mathscr{H}(S^-) \equiv m^{-2}(S^-)X^\alpha(S^-)X_\alpha(S^-) \propto \mathrm{INV}\,(W, 1). \tag{6.55}$$

On setting (these are *not* invariants)

$$\chi \equiv \mathscr{H}^2 + \frac{m^2(S^-)}{(l^\sigma l_\sigma)}H(S^-), \tag{6.56}$$

$$A_\alpha \equiv (\rho/m(S^-))(\mathscr{H}^\sigma l_\sigma)V_\alpha + m(S^-)H(S^-)(\rho v_\alpha)$$
$$= (m(S^-)/\rho)\mathscr{H}^2 v_\alpha - (\mathscr{H}^\sigma l_\sigma)\kappa_\alpha, \qquad A^\alpha l_\alpha \equiv 0, \tag{6.57}$$

it is possible to show after a lengthy, but simple, calculation that

$$\mathscr{H}(S^-) = -\mu^2 H(S^-)\mathscr{X} + m^{-2}(S^-)\mathscr{T}^\alpha(\mathscr{T}_\alpha + 2\mu A_\alpha) \propto \mathrm{INV}\,(W, 1). \tag{6.58}$$

In spite of the formalism eq. (6.58) is none other than the *Hugoniot jump condition* for magnetoelastic perfect conductors in relativity. It is also valid in relativistic magnetohydrodynamics since no mechanical constitutive assumptions have been made.

In the case of one-dimensional motions in isotropic magnetoelasticity (6.28) holds good; we have thus

$$\mathscr{T}^\alpha = m(S^-)[(1+\varepsilon_1)v^\alpha + \phi_f fL(l^\sigma l_\sigma)^{-1}l^\alpha] - 2\rho L\phi_N f S^{\alpha}_{\cdot\mu} f^\mu. \tag{6.59}$$

This, with (6.45), (6.56), (6.57) and (6.53) allows us to find the complete expression of the Hugoniot condition $\mathscr{H}(S^-) \propto \mathrm{INV}\,(W, 1)$. This will not be done here. We simply remark that the thermodynamic study of magnetoelastic shocks in relativity must be based on the discussion of the properties of the Hugoniot curve associated with the invariant (6.58). This invariant has not the usual form of a Hugoniot invariant. Rather, on account of (6.55), it is most like the square of the invariant considered in classical magnetoelasticity. Indeed, if we had worked along the same lines as in Paragraph 5.1 a long calculation, that we do not reproduce here, would have yielded the jump condition

$$m[\![\bar\varepsilon]\!] + \langle \bar T^\alpha\rangle[\![u_\alpha]\!] = -m\mu(\langle\mathscr{H}^2\rangle - \langle\mathscr{H}\rangle^2)[\![\tau]\!], \tag{6.60}$$

where $\tau = \rho^{-1}$, $\bar\varepsilon = \varepsilon_1 + (\mu\mathscr{H}^2/\rho)$, $\bar T^\alpha = m(1+\bar\varepsilon)u^\alpha - t^{\alpha\beta}l_\beta$. Equation (6.60) has the same structure as in classical cases[7]). It can be shown on the basis either of (6.58) or of (6.60) that some results of Paragraph 5.1 are directly generalized. For instance, instead of (5.11) one obtains

$$[\![\eta]\!] = 0(|[\![\mathbf{f}]\!]^3|, |[\![\mathscr{H}]\!]|^3) \tag{6.61}$$

for *weak magnetoelastic shocks*, whose speed does not differ much from the speed (evaluated ahead of the shock) of weak discontinuities found in Paragraph 6.2.

6.4. *The case of relativistic magnetohydrodynamics*

For perfect relativistic *hydrodynamics* where (4.13) holds true, setting $\bar\tau \equiv F/\rho$, it is immediately deduced from (6.58) that

$$F^2 - (l^\sigma l_\sigma)^{-1}m^2(S^-)\bar\tau^2 \propto \mathrm{INV}\,(W, 1). \tag{6.62}$$

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[7]) Indeed, at the nonrelativistic limit, e.g., for *classical magnetoelasticity*, (6.60) yields

$$m[\![\varepsilon]\!] + \langle\mathbf{T}\rangle\cdot[\![\mathbf{v}]\!] = -\frac{m\mu}{4}[\![\mathscr{H}_\perp]\!]^2[\![\tau]\!]$$

since, then $\langle\mathscr{H}^2\rangle - \langle\mathscr{H}\rangle^2 \equiv \frac{1}{4}[\![\mathscr{H}_\perp]\!]^2$ (Compare [22]). For *classical MHD*, one obtains (compare [17])

$$[\![\varepsilon + \langle p\rangle\tau]\!] = -\frac{\mu}{4}[\![\mathscr{H}_\perp]\!]^2[\![\tau]\!].$$

This is Lichnerowicz's result [12]. For *relativistic MHD*, on setting $\alpha \equiv \bar{\tau} + \mu H(S^-)$, the so-called Alfvén variable, (6.58) yields

$$F^2 - (l^\sigma l_\sigma)^{-1} m^2(S^-)\bar{\tau}^2 + 2\mu\bar{\tau}\chi - \mu^2 H(S^-)\chi \propto \text{INV}\,(W, 1). \tag{6.63}$$

This can be written in terms of jumps and mean values as

$$[\![F^2]\!] - 2\langle\bar{\tau}\rangle[\![p]\!] + 2\mu\langle\bar{\tau}\rangle[\![\alpha]\!] + \mu\langle\alpha\rangle[\![\chi]\!] = 0 \tag{6.64}$$

across W. This is Lichnerowicz's form. We refer the reader to this author [18] for an exhaustive study of the corresponding Hugoniot curve.

7. Final remarks: Initial states and existence theorems

Attempts have been made to study nonlinear wave propagation in other models of relativistic continua. For instance, Cissoko [23] considers the case of relativistic *anisotropic* MHD. The same author, starting from a simplified version of [24], envisages the case of relativistic conducting "ferrofluids" (nonlinear magnetic constitutive equations). Coll [25] devotes some attention to detonation waves in relativistic MHD. Viscosity, however, is seldom accounted for, except in the case of weak-signal detection (Cf. the work of Gambini [27] based on [2] for viscoelastic solids) and discontinuities in Madore [28].

One important feature of nonlinear wave propagation is the fact that propagation occurs through an initial state of finite deformation and/or pressure and finite bias electromagnetic fields. This initial state may be such as to produce remarkable results. For instance, it has been proven by the author on the sole hypothesis that the body be isotropic [6] that the characteristic speeds of longitudinal and transverse elastic waves propagating through an initial state of high pressure p_0 (case of neutron stars) are related by the universal relationship

$$\mathcal{U}_\parallel^2 = \tfrac{4}{3}\mathcal{U}_\perp^2 + a^2(\mathfrak{M}_0), \tag{7.1}$$

where a^2, which is defined by (4.13) with p_0 replacing p, is the sound speed of a fictitious relativistic perfect fluid which would have a law of compression corresponding to the initial state $\mathfrak{M}_0$.

The mathematical question arises, therefore, as to the existence of such initial states. This problem has been solved for relativistic hydrodynamics [29] and relativistic MHD (Cf. [15] and the mathematical discussion of Friedrichs [30]). For more involved descriptions such as those of relativistic elasticity and magnetoelasticity, the problem remains open.

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