Quantum Theory in Real Hilbert Space

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Abstract. Relativistic Quantum Theory is brought to a form, where all operators, including time reversal, are linear: Hilbert space is real. Instead of the imaginary number $i = \sqrt{-1}$, an operator $\breve{J}$ ($\breve{J}^2 = -1$) is introduced, which commutes with all observables and with the orthogonal operators representing ortho-chronous Lorentz transformations, and anti-commutes with the orthogonal representation of pseudo-chronous Lorentz transformations. It is shown, that $\breve{J}$ is necessary in order to have an uncertainly principle (§ 2). Furthermore it follows that momentum-energy and angular momentum-centre of energy are pseudo-chronous quantities. Therefore, the Hamiltonian operator does not change sign under time reversal (§ 5). Lorentz transformations are considered as passive (= coordinate frame-) transformations (§ 7).

In the annexes the following topics are discussed: A possible generalisation of quantum theory involving non linear operators (A-1); The dictionary between conventional theory in complex Hilbert space and the proposed formalism in real Hilbert space (A-2) and (A-3); The dictionary between a quantum theory in quaternion Hilbert space and our real theory (A-4). Also an error, frequently found in literature, concerning the representation of the Lorentz group is pointed out (A-5).

Introduction and Conclusion

This article presents the essential of the lectures on *Relativistic Quantum Theory* (*QT*) *of Fields*, given at the universities of Geneva and Lausanne during the past 20 years. The problem was to show students, *why the imaginary unit enters quantum theory*. We start therefore from a theory built entirely upon *real numbers* and are lead to introduce an operator J (with $J^2 = -1$), in order to have an *uncertainty principle* (UP) between the *mean square errors* $\langle \Delta F^2 \rangle$ and $\langle \Delta G^2 \rangle$ of two observables F and G. Observables are *symmetric tensors* (or *symmetric linear operators*) in real Hilbert space (RHS) $F_{ab} = F_{ba}$, or

$$F^T = F,\ G^T = G, \ldots \qquad (0.1)^{**})$$

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**) A^T is the *transposed operator:* $A^T_{ab} = A_{ba}$, which plays the analogous role as the *hermitian conjugate* $\hat{A}^\dagger$ ($\hat{A}^\dagger_{pq} = \hat{A}_{qp}^*$) in *Complex Hilbert Space* (CHS), see Annex (A-2).

The criterium for the impossibility of measuring F and G simultaneously, is a non vanishing commutator

$$FG - GF = [F, G] = - [F, G]^T \neq 0. \tag{0.2}$$

The expectation value $\langle [F, G] \rangle$ vanishes, because $[F, G]$ is an *antisymmetric tensor*. Therefore, only the *positive definit observable*

$$P = - [F, G]^2 = P^T \tag{0.3}$$

can occur in

$$\langle \Delta F^2 \rangle \langle \Delta G^2 \rangle \geqslant \lambda^2 \langle P \rangle. \tag{0.4}$$

λ is a real number. *Unless otherwise mentioned* (Annexes (A-2), (A-3) and (A-4)) *all numbers occurring in this paper are real.*

We show, that this uncertainty principle leads to a contradiction, unless $\lambda^2 = 0$, in which case (0.4) is a triviality. We show, in § 2, that the only other possibility consists in introducing an *antisymmetric operator* $J_{(FG)}$ which has an inverse $J^{-1}_{(FG)}$ (and may therefore, without loss of generality, be normalised to -1) and which commutes with F and G.

$$J^T_{(FG)} = - J_{(FG)}, \quad J^2_{(FG)} = - J^T_{(FG)} J_{(FG)} = - 1 \tag{0.5}$$

$$[J_{(FG)}, F] = [J_{(FG)}, G] = 0. \tag{0.6}$$

We may now form the symmetrical operator

$$C_{(FG)} = J_{(FG)} [\mathrm{F}, G] = C^T_{(FG)} \tag{0.7}$$

and expect a uncertainty principle of the form

$$\langle \Delta F^2 \rangle \langle \Delta G^2 \rangle \geqslant \lambda^2 \langle C_{(FG)} \rangle^2. \tag{0.8}$$

Let H be a third observable. $C_{(FG)}$ being an observable, $J_{(CH)}$ has to commute with $C_{(FG)}$ and H. Thus, the simplification to assume but one universal

$$J_{(FG)} = J \tag{0.9}$$

commuting with all observables, seems natural.

Finhelstein, Jauch and Speiser[3]) have shown that only three possibilities: RHS, CHS (Complex Hilbert Space) and QHS (Quaternion Hilbert Space) are possible in Quantum theory (QT). Thus three anticommuting J's (J_1, J_2, J_3) may exist. We have analysed QHS in terms of RHS in the annex (A-4).

We begin (§ 1) by an analysis of probability, which leads us necessarily to RHS. The linearity of the operators is a further assumption, which may eventually be omitted, because classical statistical mecanics does not necessarily require the Jakobean cyclic identity, but may give

$$\overrightarrow{FGH}\,\{F, \{G, H\}\} \neq 0 \tag{0.10)*}$$

for the generalised Poisson brackets (see annex (A-1) and [2])).

Therefore, the corresponding identity for linear operators

$$\overrightarrow{FGH}\, J\,[F, J\,[G, H]] = 0 \tag{0.10J}$$

may not hold. In § 2 we discuss the uncertainty principle (UP). In § 3 we introduce the representation of the linear group $\{L\}$ (which leaves the metric tensor $g^{\alpha\beta} = g^{(\alpha\beta)}$**) invariant) by orthogonal operators O in RHS:

$$O^T = O^{-1}, \qquad L \to e^{J\,\lambda}\, O. \tag{0.11}$$

In § 4 we show, that the metric $g^{\alpha\beta}$ of the differential manifold $x = \{x^\alpha\}$, $\alpha\,\beta \ldots = 1\,2\,\ldots\,n$, has necessarily the *thermodynamic signature* (STUECKELBERG and WANDERS[4])[5])) if the existence of *fundamental state* $\Psi^{(0)}$, *the vacuum*, is postulated:

$$\text{signat}\,(g^{\alpha\beta}) = \pm\,(1\,1\ldots 1-1). \tag{0.12}$$

This gives a preference to one coordinate $x^n = t$, the time. Thus, $\{L\}$ is the *full Lorentz group in n-dimensional space* (including *time reversal* $L = T$). Furthermore it is shown, that $J = \breve{J}$ is a *pseudochronous operator*

$$'\breve{J} = O^{-1}\,\breve{J}\,O = \text{sig}\,(L'^{n}_{\ n})\,\breve{J} \tag{0.13)***}$$

if $$'x'^\alpha = L'^\alpha_{\ \alpha}\,(x^\alpha + L^\alpha) : {}'x = Lx;\ \det\,(L'^\alpha_{\ \alpha}) \neq 0 \tag{0.14)****}$$

satisfies $$g'^{\alpha\,'\beta} = L'^\alpha_{\ \alpha}\;L'^\beta_{\ \beta}\;g^{\alpha\,\beta}. \tag{0.15}$$

Multilocal ortho-chronous observables transform according to

$$\left.\begin{aligned} 'F'^{\alpha\,'\beta\ldots}\,('x\,'y\ldots) &= L'^\alpha_{\ \alpha}\,L'^\beta_{\ \beta\ldots}\,F^{\alpha\,\beta\ldots}\,(L^{-1}\,'x\,L^{-1}\,'y\ldots) \\ &= O^{-1}\,F'^{\alpha\,'\beta\ldots}\,('x\,'y\ldots)\,O \end{aligned}\right\} \tag{0.16}$$

while *pseudo-chronous quantities* transform according to

$$'\breve{F}'^{\alpha\ldots}\,('x\ldots) = \text{sig}\,(L'^{n}_{\ n})\,L'^\alpha_{\ \alpha}\ldots\,\breve{F}^\alpha(L^{-1}\,'x\ldots) = O^{-1}\,\breve{F}'^{\alpha\ldots}\,('x\ldots)\,O \tag{0.17$\breve{F}$}$$

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*) $\overrightarrow{ABC}$ stands for the cyclic sum.

**) $F^{(\alpha\beta\gamma\ldots)}$ is a totally symmetric tensor, while $F^{[\alpha\beta\gamma\ldots]}$ is a totally antisymmetric tensor in α-space.

***) sig (λ) is the sign function sig $(\lambda) = \pm 1$ for $\lambda \gtrless 0$.

****) Frame transformations are written with the primes to the left: $'x'^\alpha \leftarrow x^\alpha$; $'\Psi'_a \leftarrow \Psi_a$.

We shall later*) also use *pseudo-chorous quantities*, which transform according to ($ik \ldots = 1\,2 \ldots n-1$)

$$\left.\begin{aligned}\overset{\frown}{F}{}'^{\alpha\cdots}\,('x\ldots) &= \mathrm{sig}\left(\det\,(L'^{i}{}_{i})\right) L'^{\alpha}{}_{\alpha}\ldots\,\overset{\frown}{F}{}^{\alpha\cdots}\,(L^{-1}\,'x\ldots)\\ &= O^{-1}\,\overset{\frown}{F}{}'^{\alpha\cdots}\,('x\ldots)\,O\end{aligned}\right\}\quad (0.17\overset{\frown}{F})$$

and finally *pseudo-quantities* $\overset{\circ}{F} = \overset{\circ}{F}$:

$$\left.\begin{aligned}\overset{\circ}{F}{}'^{\alpha\cdots}\,('x\ldots) &= \mathrm{sig}\left(\det\,(L'^{\alpha}{}_{\alpha})\right) L'^{\alpha}{}_{\alpha}\ldots\,\overset{\circ}{F}{}^{\alpha\cdots}\,(L^{-1}\,'x\ldots)\\ &= O^{-1}\,\overset{\circ}{F}{}'^{\alpha\cdots}\,('x\ldots)\,O.\end{aligned}\right\}\quad (0.17\overset{\circ}{F})$$

Let us remark, that we consider (§§ 3, 4 and 5) L always as *passive transformations*. As a matter of fact, we show (§ 7), that this *passive point of view* is perfectly reasonable in QT, because a statistical analysis of observations at two epochs t' and t'', is independent of whether t'' is later or earlier than t' in the *thermodynamic time scale* ($\breve{S}(t'') > \breve{S}(t')$; $\breve{S}(t) =$ entropy > 0, at epoch t, cf. [4]) and [6])[7])).

In § 5 we analyse the infinitesimal group $L(\delta\,\lambda^{\cdot}\,\delta\omega^{[\cdot\cdot]})$

$$'x'^{\alpha} = x'^{\alpha} + \delta\,\lambda'^{\alpha} + \delta\,\omega'^{\alpha}{}_{\alpha}\,x^{\alpha} = x'^{\alpha} + \delta\,\lambda'^{\alpha} + \frac{1}{2}\,\delta\,\omega^{[\mu\nu]}\,\Sigma_{\mu\nu}{}'^{\alpha}{}_{\alpha}\,x^{\alpha} \qquad (0.18)$$

$$\Sigma_{\mu\nu}{}'^{\alpha}{}_{\alpha} = \Sigma_{[\mu\nu]}{}'^{\alpha}{}_{\alpha} = \delta'^{\alpha}_{\mu}\,g_{\nu\alpha} - \delta'^{\alpha}_{\nu}\,g_{\mu\alpha} \qquad (0.19)$$

generating the continuous group $\{L_{(\mathrm{cont})}\}$. The generators of the corresponding Lie group $\{O\}$, with $n + (1/2)\,n\,(n-1)$, parameters λ^{μ} and $\omega^{\mu\nu} = \omega^{[\mu\nu]}$ are $-\breve{J}\,\breve{\Pi}_{\mu}$ and $\breve{J}\,\breve{M}_{\mu\nu} = \breve{J}\,\breve{M}_{[\mu\nu]}$. The pseudo-chronous observables are the *pseudo-chronous momentum-energy vector* $\breve{\Pi}_{\mu}$ and the *pseudo-chronous angular momentum-centre of energy tensor* $\breve{M}_{\mu\nu}$. We arrive at the relation

$$(\partial^{x}_{\mu} + \partial^{y}_{\mu} + \cdots)\,F^{\alpha\cdots}\,(xy\ldots) = -\,\breve{J}\,[\breve{\Pi}_{\mu},\,F^{\alpha\cdots}\,(xy\ldots)] \qquad (0.20)$$

$$\left.\begin{aligned}&\big(([x_{\mu},\partial^{x}_{\nu}] + [y_{\mu},\partial^{y}_{\nu}] + \cdots)\,\delta^{\alpha}_{\alpha'}\,\delta^{\beta}_{\beta'}\ldots + \Sigma_{\mu\nu}{}^{\alpha}{}_{\alpha'}\,\delta^{\beta}_{\beta'}\ldots\\ &+ \delta^{\alpha}_{\alpha'}\,\Sigma_{\mu\nu}{}^{\beta}{}_{\beta'}\ldots + \cdots\big)\,F^{\alpha'\beta'\cdots}\,(xy\ldots) = -\,\breve{J}\,[\breve{M}_{\mu\nu},\,F^{\alpha\beta\cdots}\,(xy\ldots)].\end{aligned}\right\}\quad (0.21)$$

From the structure relation of the generators of $\{L_{(\mathrm{cont})}\}$ ($-\,\partial_{\mu}$ and $N_{\mu\nu}{}^{\alpha}{}_{\alpha'} = [x_{\mu},\,\partial_{\nu}]\,\delta^{\alpha}_{\alpha'} + \Sigma_{\mu\nu}{}^{\alpha}{}_{\alpha'}$), the commutation relations

$$\breve{J}\,[\breve{\Pi}_{\mu},\,\breve{\Pi}_{\nu}] = 0 \qquad (0.22)$$

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*) In a following article on *real representations of the spinor group* $\{\pm\,\delta^{A}_{B},\ \pm\,\gamma^{\alpha A}{}_{B},\ \pm\,\gamma^{[\alpha_1\alpha_2]\,A}{}_{B},\ \ldots\ \pm\,\gamma^{[\alpha_1\ldots\alpha_n]\,A}{}_{B}\}$.

$$\breve{J}[\breve{M}_{\mu\nu}, \breve{M}_{\sigma\tau}] = - g_{\mu\sigma} \breve{M}_{\nu\tau} - g_{\nu\tau} \breve{M}_{\mu\sigma} + g_{\mu\tau} \breve{M}_{\nu\sigma} + g_{\nu\sigma} \hat{M}_{\mu\tau} \tag{0.23}$$

$$\breve{J}[\breve{\Pi}_{\mu}, \breve{M}_{\sigma\tau}] = g_{\mu\sigma} \breve{\Pi}_{\tau} - g_{\mu\tau} \breve{\Pi}_{\sigma} \tag{0.24}$$

follow*).

In § 6, we show, that $\breve{\Pi}_{\mu}$ and $\breve{M}_{\mu\nu}$ can be expressed in terms of an ortho-chronous observable, the *momentum energy tensor*

$$\Theta^{\alpha\beta}(x) = \Theta^{(\alpha\beta)}(x), \quad \partial_{\alpha} \Theta^{\alpha\beta}(x) = 0 \tag{0.25}$$

as integrals over a surface $'\tau(x) = 0$, whose surface element $d\,\breve{\sigma}_{\alpha}(x)$ is a *time-like pseudo-chronous vector* (signat $(g^{\alpha\beta}) = (1,1 \ldots 1 - 1)$)

$$d\breve{\sigma}_{\alpha}(x)\, d\breve{\sigma}^{\alpha}(x) < 0; \quad d\breve{\sigma}_{n}(x) > 0 \tag{0.26}$$

$$\breve{\Pi}^{\mu} = \int_{'\tau(x)=0} d\breve{\sigma}_{\alpha}\, \Theta^{\alpha\mu}(x); \quad \breve{M}^{\mu\nu} = \int_{'\tau(x)=0} d\breve{\sigma}_{\alpha}\, (x^{\mu} \Theta^{\alpha\nu} - x^{\nu} \Theta^{\alpha\mu})(x). \tag{0.27}$$

In the annexes, we consider *classical statistical mecanics* with $\overset{\curvearrowright}{_{FGH}}\{F, \{G, H\}\} \neq 0$ (A-1), *hermitian CHS* (A-2), *unitary and antiunitary transformations U* and *V* in CHS (A-3), quaternion Hilbert space (QHS) in (A-4), and an *error frequently found in literature* due to a wrong definition of the representation O (in RHS) or U (in CHS) (A-5).

§ 1. Analysis of Probability

Let F and G be two observables, whose spectra, assumed discrete, are

$$F: \{F^{(i)}\} = \{F^{(1)} < F^{(2)} < \ldots < F^{(i)} < \ldots < F^{(\omega_F)}\} \tag{1.1F}$$

$$G: \{G^{(k)}\} = \{G^{(1)} < G^{(2)} < \ldots < G^{(k)} < \ldots < G^{(\omega_G)}\} \tag{1.1G}$$

and let $W^{(i)}$, resp. $W^{(k)}$ be the probabilities that F takes the value $F^{(i)}$ (resp. G the value $G^{(k)}$)

$$W^{(i)}, W^{(k)} > 0 \qquad \sum_i W^{(i)} = \sum_k W^{(k)} = 1. \tag{1.2}$$

Then we may, without loss of generality, write $W^{(i)}$ as a sum of squares

$$W^{(i)} = \sum_{\alpha=1}^{\alpha=\omega} \Psi^2_{i\alpha}; \quad W^{(k)} = \sum_{\beta=1}^{\beta='\omega_k} {'\Psi}^2_{k\beta}, \tag{1.3}$$

which introduces an ω_i- (resp. $'\omega_k$-) fold degeneracy of the spectral term $F^{(i)}$ (resp. $G^{(k)}$). Now let us introduce two indices a and $'a$:

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*) Due to a wrong sign in the representation $L \to e^{\breve{J}\lambda}O$, these commutations relation are frequently wrong in several books on QT of fields (see Annex (A-5)).

$$\left.\begin{aligned} a\,b\ldots &= 1\,2\ldots \textstyle\sum \omega_i = 1\,2\ldots\omega_R \\ 'a\,'b\ldots &= '1\,'2\ldots \textstyle\sum {'\omega_k} = '1\,'2\ldots{'\omega_R} \end{aligned}\right\} \tag{1.4}$$

and write the degeneracies in the form

$$F^{(a)} = F^{(i\alpha)}, \quad F^{(i\alpha)} = F^{(i)} \qquad G^{('a)} = G^{(k\beta)},\ G^{(k\beta)} = G^{(k)}. \tag{1.5}$$

Now we may chose the arbitrary large numbers ω_R and $'\omega_R$ equal, and represent $\Psi_a = \Psi_{i\alpha}$, and $'\Psi_{'a} = {'\Psi_{k\beta}}$ as components of *the same abstract vector* $\boldsymbol{\Psi}$ (*state vector*), referred to *two different orthogonal coordinate frames* in an *Euclidien space* of $\omega_R = {'\omega_R}$ dimensions. In general, this number ω_R will be infinite. Therefore, we call this space the *Real Hilbert Space* (RHS). The two sets of components are related to each other by an orthogonal matrix $O = \{O_{'a\,a}\}$.

Using the summation convention, we write:

$$'\Psi_{'a} = O_{'a\,a}\,\Psi_a; \qquad '\Psi = O\,\Psi; \qquad O^T = O^{-1} \tag{1.6}$$

the expectation values are now

$$\langle F\rangle_\Psi = \textstyle\sum_i W^{(i)}\,F^{(i)} = \Psi_a\,F_{ab}\,\Psi_b \equiv (\Psi, F\,\Psi) \tag{1.7}$$

$$F_{ab} = F^{(a)}\,\delta_{ab} \tag{1.8}$$

$$\langle G\rangle_\Psi = \textstyle\sum_k W^{(k)}\,G^{(k)} = {'\Psi_{'a}}\,{'G_{'a'b}}\,{'\Psi_{'b}} = \Psi_a\,G_{ab}\,\Psi_b \equiv (\Psi, G\,\Psi) \tag{1.9}$$

$$'G_{'a'b} = G^{('a)}\,\delta_{'a'b}; \qquad G_{ab} = O^T_{a\,'a}\,{'G_{'a'b}}\,O_{'bb} \tag{1.10}$$

where

$$(\Phi, \Psi) = (\Psi, \Phi) = \Phi_a\,\Psi_a = {'\Psi_{'a}}\,{'\Phi_{'a}} \tag{1.11}$$

is the scalar product between *vectors in RHS*. F, G and all observables are *symetrical tensors in RHS:*

$$F^T = F,\ G^T = G,\quad H^T = H, \ldots \tag{1.12}$$

In the a-frame, F is diagonal (1.8) and in the $'a$-frame $'G$ is diagonal (1.10). The transposed operator of an operator A, $A = \{A_{ab}\}$ is defined by

$$(\Phi, A\,\Psi) = (A^T\,\Phi, \Psi); \quad A^T_{ab} = A_{ba}. \tag{1.13}$$

Now, $\boldsymbol{F}$ and $\boldsymbol{G}$ are *two tensor ellipsoids in RHS or a-space:* The length of their principal axes are given by the spectra (1.1F) and (1.1G). The length of the axes are thus independent of the orientation of the a-space vector $\boldsymbol{\Psi}$. However the *relative orientation of the* two ellipsoids $\boldsymbol{F}$ and $\boldsymbol{G}$ i. e. the *relative orientation of the a-frame and 'a-frame* in abstract RHS is not necessarily independent of $\boldsymbol{\Psi}$. Thus, $O = \{O_{'aa}\}$ may depend on $\boldsymbol{\Psi}$. This introduces the possibility of assuming F and G to be more general operators than linear ones[2]) (see (0.10) (0.10J) and Annex (A-1)).

§ 2. The Uncertainty Principle

In order to express the uncertainty principle (UP), we introduce in (0.4) the error operators

$$\varDelta F = F - 1\,\langle F\rangle_\Psi; \quad \varDelta G = G - 1\,\langle G\rangle_\Psi \tag{2.1}$$

from which we form the *mean square errors* $\langle\varDelta F^2\rangle_\Psi$ and $\langle\varDelta G^2\rangle_\Psi$ in (0.4). There are two possibilities:

$$\langle\varDelta F^2\rangle_\Psi\,\langle\varDelta G^2\rangle_\Psi \geqslant \begin{cases} \lambda^2\,\langle P\rangle_\Psi & (2.2P) \\ \lambda^2\,\langle C\rangle_\Psi^2 & (2.2C) \end{cases}$$

where P is a positive observable of dimension $[F]^2\,[G]^2$ and C is an observable of dimension $[F]\,[G]$. λ is a number to be determined.

Let us demonstrate, that the first choice (0.4) or (2.2P) leads to a contradiction: We express (2.2P) or (0.3), (0.4) in the a-frame, where F is diagonal. Then, if $[F, G] \neq 0$, G has nondiagonal elements in this frame. Suppose further that F has the value $F^{(a')}$ i.e. $\Psi_a = \Psi'_a = \pm\,\delta_{aa'}$. Then we have

$$[F,\,G]_{ab} = (F^{(a)} - F^{(b)})\,G_{ab}; \quad G_{ab} \neq 0 \tag{2.3}$$

and (on account of $G^T = G$)

$$P_{ab} = -\,[F,\,G]_{ab}^{\,2} = \sum\nolimits_c (F^{(a)} - F^{(c)})\,(F^{(b)} - F^{(c)})\,G_{ac}\,G_{bc}. \tag{2.4}$$

Therefore the expectation value is

$$\langle P\rangle_{\Psi'} = \sum\nolimits_c (F^{(a')} - F^{(c)})^2\,(G_{a'c})^2 = (\text{finite }')^2 > 0. \tag{2.5}$$

Now $\langle\varDelta F^2\rangle_{\Psi'} = 0$. Let the spectre of G (1.1 G) be bounded. Then we have $\langle\varDelta G^2\rangle_{\Psi'} \leqslant (G^{(1)} - G^{(\omega_G)})^2 = (\text{finite})^2$ and (2.2P) (or (0.4)) reads

$$0 \cdot (\text{finite})^2 \geqslant \lambda^2\,(\text{finite }')^2 \tag{2.6}$$

which has only the trivial solution $\lambda = 0$, corresponding to the trivial statement

$$\langle\varDelta F^2\rangle_\Psi\,\langle\varDelta G^2\rangle_\Psi \geqslant 0 \tag{2.7}$$

The only other possibility, (2.2C) is to introduce an observable C, linear in F and linear in G. This implies the existence of an antisymmetric tensor in a-space $J_{(FG)} = -\,J^T_{(FG)}$ commuting with F and G:

$$C = J_{(FG)}\,[F,\,G] = C^T \tag{2.8}$$

In order to deduce the UP, we form, with an arbitrary number ξ,

$$\left.\begin{aligned} |(\varDelta F + \xi\,J\varDelta G)\,\Psi\,|^2 &= (\Psi, (\varDelta F - \xi\,\varDelta G J)\,(\varDelta F + \xi\,J\varDelta G)\,\Psi) \\ &= \langle\varDelta F^2\rangle_\Psi - \langle J^2\varDelta\,G^2\rangle_\Psi\,\xi^2 + \langle J\,[F,\,G]\rangle_\Psi\,\xi \\ &= f(\xi) \geqslant f_{\min}\,(\xi') \geqslant 0. \end{aligned}\right\} \tag{2.9)*}$$

*) We have written J for $J_{(FG)}$.

In order to make appear $\langle \Delta G^2 \rangle_\Psi$ in (2.9), it is necessary that $J^2_{(FG)}$ is a number $\neq 0$. Being antisymmetric, this number must be negative. As ξ is an arbitrary number, we lose no generality in normalising $J^2_{(FG)} = -1$. Thus, we arrive at the conditions (0.5) and (0.6).

The minimum $f_{\min}(\xi')$ of $f(\xi)$ is easily found to be at

$$\xi' = -1/2 \langle J_{(FG)} [F, G] \rangle_\Psi \langle \Delta G^2 \rangle_\Psi^{-1} \tag{2.10}$$

which we insert in the last inequality (2.9). Multiplying with $\langle \Delta G^2 \rangle_\Psi$, we find the inequality (0.8) with $\lambda^2 = 1/4$. Assuming but one universal J (see text following (0.8)), we arrive at the UP:

$$\langle \Delta F^2 \rangle_\Psi \langle \Delta G^2 \rangle_\Psi \geqslant \frac{1}{4} \langle J[F, G] \rangle_\Psi^2 . \tag{2.11}$$

§ 3. The linear inhomogenous group $\{L\}$ in x-space

$\{L\}$ is defined by its general element L (0.14) and the condition (0.15), stipulating the invariance of the metric tensor $g^{\alpha\beta}$. A classical observable transforms according to (0.16) or (0.17). It will be useful to combine the ,indices'

$$\{\alpha\, \beta \ldots x\, y \ldots\} \equiv X \tag{3.1}$$

and define

$$\left.\begin{aligned} {}'F^{'X} &\equiv {}'F^{'\alpha\,'\beta\ldots}('x\,'y\ldots) = L^{'X}{}_X\, F^X \\ &= L^{'\alpha}{}_\alpha\, L^{'\beta}{}_\beta \ldots \int d^n x\, \delta('x - Lx) \int d^n y\, \delta('y - Ly) \ldots F^{\alpha\beta\ldots}(x\, y\ldots) \\ &= L^{'\alpha}{}_\alpha\, L^{'\beta}{}_\beta \ldots F^{\alpha\beta\ldots}(L^{-1}\,'x\; L^{-1}\,'y\ldots). \end{aligned}\right\} \tag{3.2}$$

Let us now consider how the expectation value

$$\langle F^{\alpha\ldots}(x\ldots)\rangle_\Psi = (\Psi, F^{\alpha\ldots}(x\ldots)\,\Psi) = \Psi_a\, F^X_{ab}\, \Psi_b \tag{3.3}$$

transformes under $\{L\}$. There are *two possibilities: Either* we leave Ψ_a *unchanged* and write

$$\langle 'F^{'\alpha\ldots}('x\ldots)\rangle_\Psi = L^{'\alpha}{}_\alpha \ldots (\Psi, F^{\alpha\ldots}(L^{-1}\,'x\ldots)\,\Psi) \tag{3.4}$$

which expresses the fact that $'F^{'X}$ is the *transformed operator* (3.2) (0.16) (0.17). *Or*, we may express the transformed expectation value in terms of the initial operator $F^{'X}$ with the index $'X = ('\alpha\ldots\,'x\ldots)$ and in terms of a *transformed vector* $'\Psi_{'a}$

$$'\Psi_{'a} = O_{'a\,a}\,\Psi_a = \Psi_a\, O^T_{a\,'a} \tag{3.5}$$

in the form

$$\langle 'F'^{\alpha\ldots}('x\ldots)\rangle_{\Psi} = \langle F'^{\alpha\ldots}('x\ldots)\rangle_{'\Psi} = '\Psi_{'a}\, F'^{X}_{'a'b}\, '\Psi_{'b}. \qquad (3.6)$$

Equating (3.4) and (3.6) we obtain an *identity*

$$\Psi_a\, L'^{\alpha}{}_{\alpha}\ldots F^{\alpha\ldots}_{ab}(L^{-1}\,'x\ldots)\,\Psi_b \equiv \Psi_a\, O^T_{a'a}\, F'^{\alpha\ldots}_{'a'b}('x\ldots)\, O_{'b\,b}\,\Psi_b \qquad (3.7)$$

between two *quadratic forms* in Ψ_a. Or, these forms are equal, for an arbitrary Ψ, if and only if*)

$$L'^{\alpha}{}_{\alpha}\ldots F^{\alpha\ldots}_{ab}(L^{-1}\,'x\ldots) = O^T_{a'a}\, F'^{\alpha\ldots}_{'a'b}('x\ldots)\, O_{'b\,b}. \qquad (3.8)$$

We may write this identity (multiplying by $O\ldots O^{-1}$)

$$F'^{X}_{'a'b} = L'^{X}{}_{X}\, O_{'a\,a}\, F^{X}_{ab}\, O^T_{b'b} \qquad (3.9)$$

Thus F^X_{ab} *considered as a ,vector' in* $X = \{\alpha\ldots x\ldots\}$*-space and as a symmetrix tensor in a-space, is left invariant, if it is transformed with respect to its three indices X and a, b.* This is in perfect analogy to metric tensor $g^{\alpha\beta}$ in (0.15) and to the *α-vector mixed bispinor* $\gamma^{\alpha\,A}{}_{B}$ (cf.[1]) and an article, to appear in this journal, on Spinor Calculus in RHS).

Now it is easily seen, that the $\{O\}$ group is a *ray representation* of the $\{L\}$-group: Write $L_{(1)}$ and $O^{(1)}$ in (3.9) and consider a second transformation $L_{(2)}$ and $O^{(2)}$ leading from the frames $'X\,'a$ to the frames $''X\,''a$:

$$F''^{X}_{''a''b} = L''^{X}_{(2)\,'X}\, O^{(2)}_{''a'a}\, O^{(2)}_{''b'b}\, F'^{X}_{'a'b} \qquad (3.10)$$

and substitute (3.9) in (3.10): From

$$F''^{X}_{''a''b} = (L_{(2)}\,L_{(1)})''^{X}{}_{X}\,(O^{(2)}\,O^{(1)})_{''a\,a}\,(O^{(2)}\,O^{(1)})_{''b\,b}\,F^{X}_{a\,b} \qquad (3.11)$$

it is seen that $L \to O$ and $L \to (-O)$ is a two valued representation. However, since J commutes with any observable F, G, we may write

$$L \to e^{\lambda J}\;O,\;\; L^{-1} \to O^{-1}\,e^{-\lambda J} \qquad (3.12)$$

where the number λ is an arbitrary phase. We see, that O does not necessarily commute with J.

To illustrate the identity

$$\langle 'F'^{X}\rangle_{\Psi} = \langle F'^{X}\rangle_{'\Psi}$$

we have drawn the Fig. 1 and 2:

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*) In CHS, where $\langle \widehat{\Psi}, \widehat{A}\,\widehat{\Psi}\rangle$ is a *complex number* (cf. Annex (A-2)) the unitary transformation $U^\dagger = U^{-1}$ replaces $O^T = O^{-1}$. Thus we have two real identities, and the condition

$$L'^{\alpha}{}_{\alpha}\,\widehat{A}^{\alpha}_{p\,q}\,(L^{-1}\,'x) = U^{-1}_{p'p}\,\hat{A}'^{\alpha}_{p'q}\,('x)\,U_{'q\,q} \qquad (3.8\,\hat{A})$$

is valable for *all operators* $\widehat{A}$, whether hermitian or not.

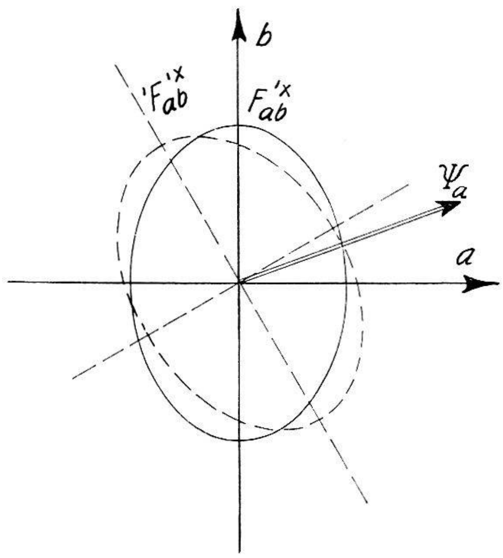

Fig. 1
a-frame

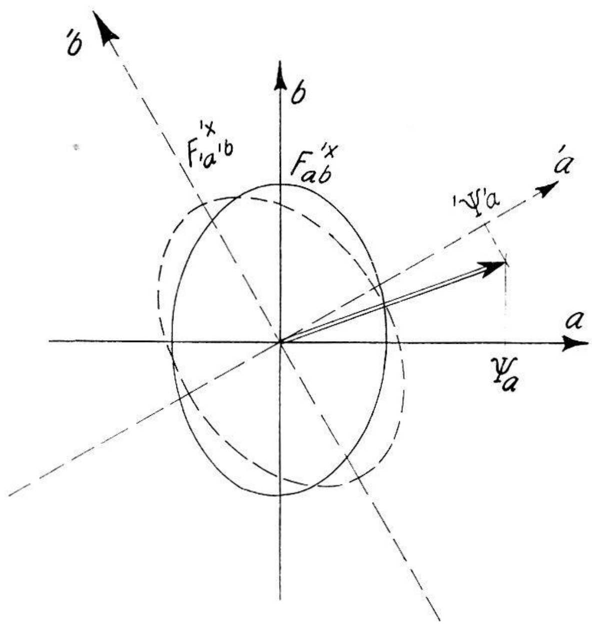

Fig. 2
a-frame and $'a$-frame

Either (Fig. 1), we form $\Psi_a\,'F^{'X}_{ab}\,\Psi_b$ in the *a-frame* from the $('X \leftarrow X)$-*transformed tensor*

$$'F^{'X}{}_{ab} = L^{'X}{}_X\, F^X_{ab}$$

(which has, on account to the $\{L\}$-invariance, the same length of the principal axes as the untransformed tensor $F^{'X}_{ab}$) with respect to Ψ_a.

Or (Fig. 2), we form $'\Psi_{'a}\, F^{'X}_{'a'b}\, \Psi_{'b}$ in the *'a-frame* from the tensor $F^{'X}_{'a'b}$ (which has, in the $'a$-frame, the same components as $F^{'X}_{ab}$ has in the a-frame) with respect to $'\Psi_{'a}$.

From the two figures follwes immediately:

$$'\Psi_{'a} = O_{'a\,a}\,\Psi_{a};\quad F'^{X}_{'a\,'b} = O_{'a\,a}\,O_{'b\,b}\,'F'^{X}_{ab} \tag{3.13}$$

or

$$'F'^{X}_{ab} = L'^{X}{}_{X}\,F^{X}_{ab} = O^{-1}_{a\,'a}\,F'^{X}_{'a\,'b}\,O_{'b\,b}; \tag{3.14}$$

$$'F'^{\alpha\cdots}('x\ldots) = L'^{\alpha}{}_{\alpha}\ldots F^{\alpha\cdots}(L^{-1}\,'x\ldots) = O^{-1}\,F'^{\alpha\cdots}('x\ldots)\,O$$

§ 4. The Thermodynamic Signature of $g^{\alpha\beta}$ and the Pseudo-Chronous Character of $J = \breve{J}$

We need the hypotheses that a *particular state $\Psi^{(0)}$ of the cosmos exists, the vacuum,* which is *homogeneous and isotropic*. Let us consider, for the simplicity, a *local scalar observable* $F(x)$, and write the UP for two events x and y in the form

$$\langle J\,[F(x),\, F(y)]\rangle_{\Psi^{(0)}} \equiv \breve{f}(x\,y) = -\,\breve{f}(y\,x). \tag{4.1}$$

Homogeneity requires:

$$\breve{f}(x\,y) = \breve{f}(x-y). \tag{4.2}$$

Isotropy would further require:

$$\breve{f}(x\,y) = \breve{f}\big((x-y)^2\big), \tag{4.3}$$

$$(x-y)^2 = (x-y)_{\alpha}\,(x-y)^{\alpha}. \tag{4.4}$$

However, (4.3) is in contradiction with the antisymmetry (4.1) of the commutator. There is only one way to turn this difficulty: We have to give to the differentiable manifold $\{x^{\alpha}\}$ *one privileged axis*, $x^n = t$, the time. By this we understand that the metric has the *thermodynamic signature:*

$$\text{signat}\,(g^{\alpha\beta}) = \pm\,(1\,1\ldots 1\,-1). \tag{4.5}$$

We have shown in an earlier paper[4]), that this signature is necessary for a phenomenological relativistic thermodynamics. Thus we may define a function

$$\left.\begin{aligned} \breve{f}(x\,y) &= \text{sig}\,(x^n - y^n)\,f\big((x-x)^2\big) \\ f\big((x-y)^2\big) &= 0 \quad \text{for } x-y = \text{spatial} \end{aligned}\right\} \tag{4.6}$$

which is homogeneous and ,quasi-isotropic'. Now it is easily seen, that $\breve{f}$ is a pseudo-chronous bilocal scalar

$$'\breve{f}('x\,'y) = \text{sig}\,(L'^{n}{}_{n})\,\breve{f}\,(L^{-1}\,'x\;L^{-1}\,'y) \tag{4.7}$$

because, for $L'^{n}{}_{n} > 0$, we have

$$'\breve{f}\,('x\,'y) = \breve{f}\,(L^{-1}\,'x\;L^{-1}\,'y) = \breve{f}\,(x\,y);\quad L'^{n}{}_{n} > 0 \tag{4.8}$$

while, for $L'^n_{\;n} < 0$, the relation ($x = L^{-1}\,'x$) is

$$\left.\begin{aligned} '\breve{f}\,('x\,'y) &= -\,\mathrm{sig}\,(x^n - y^n)\,f\,((x-y)^2) \\ &= -\,\breve{f}\,(x\,y); \quad L'^n_{\;n} < 0. \end{aligned}\right\} \tag{4.9}$$

Thus we may write, using the homogeneity and the pseudo-chronous isotropy of $\breve{f}$:

$$'\breve{f}\,('x\,'y) = \mathrm{sig}\,(L'^n_{\;n})\,\breve{f}\,(x\,y). \tag{4.10}$$

Now, consider the transformed value of the observable $J[F(x),\,F(y)]$, which is, according to (3.5) and (3.6), the expectation value with respect to $\Psi^{(0)}$ of

$$\left.\begin{aligned} &O^{-1}\,(J\,[F('x),\,F('y)])\,O \\ &= O^{-1}\,J\,O\,['F('x),\,'F('y)] \end{aligned}\right\} \tag{4.11}$$

in

$$\langle O^{-1}\,J\,O\,['F\,('x),\,'F\,('y)]\rangle_{\Psi^{(0)}} = '\breve{f}\,('x\,'y) \tag{4.12}$$

or $'F('x) = F\,(L^{-1}\,'x) = F(x)$, according to (0.16). Thus, making use of the relation (4.10), we obtain

$$\langle(O^{-1}\,J\,O)\,[F(x),\,F(y)]\rangle_{\Psi^{(0)}} = \mathrm{sig}\,(L'^n_{\;n})\,\breve{f}(x\,y). \tag{4.13}$$

Comparing this relation to (4.1), we find (0.13):

$$J \equiv \breve{J}; \quad O^{-1}\,\breve{J}\,O = \mathrm{sig}\,(L'^n_{\;n})\,\breve{J} \equiv '\breve{J}. \tag{4.14}$$

(4.14) *defines* the transformed operator $'\breve{J}$:

The transformation law $\breve{J} \to '\breve{J}$ is now analogous to the law (0.17) for an x-independant operator. Note however, that $'\breve{J}$ is but a *definition*, because we have established the identities expressed in the second equation (0.16) (and (0.17)) by comparing (3.4) and (3.6) in (3.7) only for observables, i. e. for symmetric a-space tensors $F, G\ldots$ and not for antisymmetric a-space tensors like $\breve{J}$.

§ 5. Infinitesimal Lorentz transformations

After having introduced the pseudo-euclidian signature with one privileged axis $x^n = t$ in § 4, our group $\{L\}$ is the *full Lorentz-group in n dimensions*. Writing down the infinitesimal transformation $L = L\,(\delta\,\lambda^{\cdot}\,\delta\,\omega^{[\cdot\,\cdot]})$ ((0.18) (0.19)) we find

$$\left.\begin{aligned} &L'^{\alpha}_{\;\alpha}\,F^{\alpha}(L^{-1}\,'x) = (\delta'^{\alpha}_{\;\alpha} + \tfrac{1}{2}\,\delta\omega^{\mu\nu}\,\Sigma_{\mu\nu}{}'^{\alpha}_{\;\alpha})\,F^{\alpha}('x^{\cdot} - \delta\lambda^{\cdot} - \delta\omega^{\cdot}{}_{\nu}\,'x^{\nu}) \\ &= F'^{\alpha}\,('x) + \delta\lambda^{\mu}\,(-\,'\partial_{\mu})\,F'^{\alpha}\,('x) + \tfrac{1}{2}\,\delta\omega^{\mu\nu}\,'N_{\mu\nu}{}'^{\alpha}_{\;\alpha}\,F^{\alpha}('x) \end{aligned}\right\} \tag{5.1}$$

where $-'\partial_\mu$ and

$$'N_{\mu\nu}{}'^{\alpha}{}_{\alpha} = ['x_\mu, '\partial_\nu]\,\delta'^{\alpha}{}_{\alpha} + \Sigma_{\mu\nu}{}'^{\alpha}{}_{\alpha} \tag{5.2}$$

are the *generators* of the $n + (1/2)\, n\, (n-1)$*-parameter Lie-group* $\{L_{(\mathrm{cont})} = L\,(\lambda^{\cdot}\, \omega^{[\cdot\cdot]})\}$, which is the continuous subgroup of $\{L\}$. The generators satisfy the *Lie structure relations:*

$$[-\partial_\mu, -\partial_\nu] = 0 \tag{5.3}$$

$$[N_{\mu\nu}, N_{\sigma\tau}] = -g_{\mu\sigma} N_{\nu\tau} - g_{\nu\tau} N_{\mu\sigma} + g_{\mu\tau} N_{\nu\sigma} + N_{\nu\sigma} g_{\mu\tau} \tag{5.4}$$

$$[-\partial_\mu, N_{\sigma\tau}] = g_{\mu\sigma}(-\partial_\tau) - g_{\mu\tau}(-\partial_\sigma). \tag{5.5}$$

Now the corresponding orthogonal operator $O(\delta\lambda^{\cdot}\, \delta\omega^{[\cdot\cdot]})$ can be written as

$$O(\delta\lambda^{\cdot}\, \delta\,\omega^{[\cdot\cdot]}) = 1 + \delta\lambda^\mu\,(-\breve{J}\,\breve{\Pi}_\mu) + \frac{1}{2}\,\delta\omega^{\mu\nu}\,(\breve{J}\,\breve{M}_{\mu\nu}). \tag{5.6}$$

The *symmetric operators* $\breve{\Pi}_\mu$ and $\breve{M}_{[\mu\nu]}$ are *pseudo-chronous observables* and commute with $\breve{J}$, because $O_{(\mathrm{cont})} \to L_{(\mathrm{cont})}$, contains neither time- nor space-reflections. In particular, the generators of the group $\{O_{(\mathrm{cont})}\}$: $-\breve{J}\,\breve{\Pi}_\mu$ and $\breve{J}\,\breve{M}_{\mu\nu}$ must satisfy the Lie structure relations (5.3)–(5.5) of $L_{(\mathrm{cont})}$.

Multiplying by $\breve{J}^{-1}$, these are (0.22)–(0.24).

The identity (3.9), which relates L and O is, for the infinitesimal element

$$\left.\begin{aligned} F'^{\alpha}('x) &= F'^{\alpha}('x) + \delta\lambda^\mu\,(-'\partial_\mu\, F'^{\alpha}('x) - \breve{J}[\breve{\Pi}_\mu, F'^{\alpha}('x)]) \\ &\quad + \frac{1}{2}\,\delta\omega^{\mu\nu}\,('N_{\mu\nu}{}'^{\alpha}{}_{\alpha}\, F^{\alpha}('x) + \breve{J}[\breve{M}_{\mu\nu}, F'^{\alpha}('x)]) \end{aligned}\right\} \tag{5.10}$$

and leads thus to (0.20) and (0.21).

The sign of $\breve{\Pi}_\mu$ is chosen in order to give, for signat $(g^{\alpha\beta}) = (1\,1 \ldots 1 -1)$, the relation $(\partial_n = \partial_t, \breve{\Pi}^n = -\breve{\Pi}_n \equiv \breve{H})$

$$\partial_t\, F(\vec{x}, t) = \breve{J}\,[\breve{H}, F(\vec{x}\, t)], \tag{5.11}$$

where the Heisenberg operator $F(\vec{x}\, t)$ and the Schrodinger operator $\overline{F}(\vec{x})$ are related by

$$\langle F(\vec{x}\, t)\rangle_\Psi = \langle \overline{F}(\vec{x})\rangle_{\overline{\Psi}(t)};\quad \overline{\Psi}(t) = e^{-\breve{J}\breve{H}t}\,\Psi \tag{5.12}$$

or

$$F(x) = F(\vec{x}\, t) = e^{\breve{J}\breve{H}t}\, \overline{F}(\vec{x})\, e^{-\breve{J}\breve{H}t}. \tag{5.13}$$

We write $\breve{H}$ with the pseudo-chronous sign $\breve{}$, because it is the $n-th$ component of $\breve{\Pi}^\alpha$. For time reflection, $('x^i = x^i, 'x^n = -x^n)$ we have therefore:

$$'\breve{H} = \breve{H},\ '\vec{\breve{\Pi}} = -\vec{\breve{\Pi}}. \tag{5.14}$$

The *energy operator* $\breve{H} = \breve{\Pi}^n$ does not change its sign, while the *momentum operator* $\breve{\vec{\Pi}}$ changes sign, because velocities change sign.

In order to show that $\breve{M}^{ik}$ is the *angular momentum operator*, we consider the transformation for an infinitesimal displacement of the origine:

$$'x'^{\mu} = x'^{\mu} + \delta\lambda'^{\mu} \tag{5.15}$$

$$'\breve{M}'^{\mu\,'\nu} = O^{-1}\,\breve{M}'^{\mu\,'\nu}\,O = \breve{M}'^{\mu\,'\nu} + \delta\lambda^{\sigma}\,J[\breve{\Pi}_{\sigma}, \breve{M}'^{\mu\,'\nu}] \tag{5.16}$$

or, using (0.24)

$$'\breve{M}'^{\mu\,'\nu} = \breve{M}'^{\mu\,'\nu} + (\delta\lambda'^{\mu}\,\breve{\Pi}'^{\nu} - \delta\lambda'^{\nu}\,\breve{\Pi}'^{\mu}). \tag{5.17}$$

This shows that the arm-length of the moment with respect to the primed frame ($'\alpha$-frame) is larger by the amount $\delta\lambda'^{\mu}$ than the arm-length with respect to the α-frame.

§ 6. The momentum-energy density operator

$\breve{\Pi}^{\mu}$ and $\breve{M}^{[\mu\nu]}$ can be expressed as integrals over an arbitrary time-like surface $'\tau(x) = 0$, whose covariant n-component $d\breve{\sigma}_n(x)$ of the surface-element $d\breve{\sigma}_{\alpha}(x)$ is positive in every α-frame, if we choose the signature $+ (1\,1 \ldots 1 - 1)$. This means that $d\breve{\sigma}_{\alpha}(x)$ is a *pseudo-chronous time-like vector* (0.26). Then it follows from Gauss' theorem, that the pseudo-chronous quantities $\breve{\Pi}^{\mu}$ and $\breve{M}^{\mu\nu}$ are independant of the surface $'\tau(x) = 0$ chosen, if (0.25) holds.

To verify the transformation law, let us transform (6.2) according to

$$'\breve{\Pi}^{\mu} = \int_{'\tau(x)=0} d\,\breve{\sigma}_{\alpha}\,(x)\;O^{-1}\,\breve{\Theta}^{\alpha\mu}\,(x)\;O = L^{\mu}{}_{\sigma} \int_{'\tau(x)=0} d\,\breve{\sigma}_{\alpha}\,(x)\;L^{\alpha}{}_{\beta}\,\Theta^{\beta\sigma}\,(L^{-1}\,x) \tag{6.5}$$

and write $y = L^{-1}x$. Then, from the pseudo-chronous character of $d\breve{\sigma}_{\alpha}$ follows

$$d\,\breve{\sigma}_{\alpha}\,(Ly)\;L^{\alpha}{}_{\beta} = \mathrm{sig}\,(L^{n}{}_{n})\;d\,\breve{\sigma}_{\beta}\,(y) \tag{6.6}$$

where $d\breve{\sigma}_n\,(Ly)$ is orientated parallel to $x^n = (Ly)^n$, while $d\breve{\sigma}_n(y)$ is orientated parallel to $y^n = (L^{-1}x)^n$.

Thus we have finally, writing $'\mu$ for μ and μ for σ

$$'\breve{\Pi}'^{\mu} = \mathrm{sig}\,(L'^{n}{}_{n})\;L'^{\mu}{}_{\mu} \int_{'\tau(Ly)=\tau(y)=0} d\,\breve{\sigma}_{\beta}\,(y)\;\Theta^{\beta\mu}\,(y). \tag{6.7}$$

The integral being independent of the particular surface $\tau(x) = 0$ or $'\tau(x) = 0$ chosen, we may write:

$$'\breve{\Pi}'^{\mu} = \mathrm{sig}\,(L'^{n}{}_{n})\;L'^{\mu}{}_{\mu}\,\breve{\Pi}^{\mu}. \tag{6.8}$$

Analogously, we transform

$$'\breve{M}^{\mu\nu} = O^{-1}\,\breve{M}^{\mu\nu}\,O = \int\limits_{\tau(x)=0} d\breve{\sigma}_\alpha(x)\,L^\alpha{}_\beta\left(x^\mu L^\nu{}_\tau\,\Theta^{\beta\tau}(L^{-1}x) - x^\nu L^\mu{}_\sigma\,\Theta^{\beta\sigma}(L^{-1}x)\right) \tag{6.9}$$

and substitute $x = Ly$: $x^\mu = L^\mu{}_\sigma\,(y^\sigma + L^\sigma)$. Using again (6.6), we obtain

$$\left.\begin{aligned} '\breve{M}'^{\mu\,'\nu} &= \mathrm{sig}\,(L'^n{}_n)\,L'^\mu{}_\mu\,L'^\nu{}_\nu \int\limits_{\tau(\breve{y})=0} d\,\breve{\sigma}_\beta\,(y)\,\big((y^\mu + L^\mu)\,\Theta^{\beta\nu}(y) \\ &- (y^\nu + L^\nu)\,\Theta^{\beta\mu}(y)\big) = \mathrm{sig}\,(L'^n{}_n)\,L'^\mu{}_\mu\,L'^\nu{}_\nu\,(\breve{M}^{\mu\nu} + L^\mu\,\breve{\Pi}^\nu - L^\nu\,\breve{\Pi}^\mu) \end{aligned}\right\} \tag{6.10}$$

Thus we have verified the pseudo-chronous character of momentum-energy $\breve{\Pi}^\mu$ and of angular momentum-centre of energy $\breve{M}^{[\mu\nu]}$, expressed as surface integrals of an ortho-chronous momentum-energy density $\Theta^{(\alpha\beta)}(x)$ over a surface $\tau(x) = 0$, with a pseudo-chronous time-like surface element $d\breve{\sigma}_\alpha(x)$.

§ 7. Physical Meaning of the Passive Time Reversal

To our *passive interpretation* of time reversal, it has been objected, that only the *active interpretation* has a physical sense, because an observation at an epoch t' changes the earlier state (in the thermodynamic sense) of the system Ψ into a later state Ψ' corresponding to the measure of an observable $F = F^{(i')}$. However, we may consider an observer which makes only correlation experiments:

This observer makes a great number, say N, of experiments, at two epochs $t'_{(1)}$ and $t''_{(1)}$, $t'_{(2)}$ and $t''_{(2)}$ etc. ..., separated always by Δt. Let us suppose first that t measures the thermodynamic time order, and that

$$t'' - t' = t''_{(1)} - t'_{(1)} = t''_{(2)} - t'_{(2)} = \ldots = t''_{(N)} - t'_{(N)} = \Delta\,t > 0 \tag{7.1}$$

Every time, an observer observes $F^{(i')}$ at the earlier epoch t', he will observe $G = \{G^{(k)}\}$ at the later epoch and thus be able to make a statistics

$$F^{(i')} \to G^{(1)}, G^{(2)}, \ldots, G^{(k)}, \ldots G^{(\omega_G)} \tag{7.2}$$

giving transition probabilities

$$W(\overleftarrow{k, i'}) \geqslant 0;\quad \sum\nolimits_k W(\overleftarrow{k, i'}) = 1. \tag{7.3}$$

However, he is free to evaluate his statistics the other way round: Every time he registers $G^{(k'')}$ at the later epoch t'', he makes statistics of the corresponding measures of $F = F^{(i)}$ at the earlier epoch t'. Thus, he obtains transition probabilities

$$W(\overrightarrow{k'', i}) \geqslant 0,\ \sum\nolimits_i W(\overrightarrow{k'', i}) = 1. \tag{7.4}$$

The coefficients (7.3) and (7.4) are of course equal

$$W(\overrightarrow{k, i}) = W(\overleftarrow{k, i}) = W(k, i). \tag{7.4b}$$

The arrows $\leftarrow$ (evolution in the thermodynamic sense) and $\rightarrow$ (evolution in the opposite sense) are thus superfluous. This means, that quantum-mechanical ‚predictions' can be made for the future as well as for the past. If the system ist not degenerate, we may write $F^{(a)}$ and $G^{('a)}$ for $F^{(i)}$ and $G^{(k)}$ (see (1.5)) and our correlation coefficients are

$$W('a, a) = (O_{'a\,a})^2 = (O^T_{a\,'a})^2. \tag{7.5}$$

They correspond to the doubly normalised transition amplitudes:

$$W(k, i) \geqslant 0; \quad \sum_k W(k, i) = 1; \quad \sum_i W(k, i) = 1 \tag{7.6}$$

used by INAGAKI, PIRON and WANDERS[9]) to prove the Boltzmann H theorem for the most general case (STUECKELBERG [8])), while the usual proves assume, instead of (7.6), detailed balancing

$$W(k, i) = W(i, k) \tag{7.7}$$

which is known to be insufficient[8])[9]).

Therefore, passive time reversal can be verified experimentally.

§ 8. Acknowledgments

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Annex 1. Remark on Generalised Poisson Brackets

In order to establish the *Boltzmann H-theoreme* in classical statistical mechanics, we have to start from a covariant theory of motion in *phase space* $x = \{x^\alpha\}$ ($\alpha\,\beta \ldots = 1\,2 \ldots \omega$), which satisfies the *theorem of Liouville*. The *conservation of energy* $H = H\,(x)$ leads to an equation of motion for $x^\alpha = z^\alpha(t)$

$$\dot{z}^\alpha(t) = \partial_t\, z^\alpha(t) = (\partial_\beta\, H\,(z(t))\, j^{\beta\alpha}\,(z(t)), \tag{A-1.1}$$

where

$$j^{\alpha\beta}(x) = j^{[\alpha\beta]}(x) \tag{A-2.2}$$

is an *antisymmetrical tensor in phase space*. The *scalar density* of the Gibbs ensemble is $\mathfrak{w}(x, t) \geqslant 0$. It satisfies the equation of continuity

$$\int d^\omega x\ \mathfrak{w}(x, t) = 1; \quad \partial_t \mathfrak{w}(z, t) + \partial_\alpha\big(\dot{z}^\alpha \mathfrak{w}(z, t)\big) = 0 \tag{A-2.2}$$

and transforms according to

$$'\mathfrak{w}('x, t) = \left|\det\left(A'^{\mu}{}_{\mu}(x)\right)\right| \mathfrak{w}(x, t) \geqslant 0. \qquad \text{(A-2.3)}$$

$$'x'^{\alpha} = \psi'^{\alpha}(x); \quad A'^{\alpha}{}_{\alpha}(x) = \partial_{\alpha}\,\psi'^{\alpha}(x)$$

Let $d\Omega(x) = \mathfrak{g}(x)\ |\ d\varphi^{[12\ldots\omega]}\ | = \mathfrak{g}\ d^{\omega}x$ be the invariant scalar volume element, where $d\varphi^{[\alpha_1\alpha_2\ldots\alpha_\omega]} = \det\,(\overset{(\beta)}{dx^{\alpha}})$ is the antisymmetric tensor of the parallelipiped, formed from ω non coplanar vectors $d\,\overset{(\beta)}{x^{\alpha}}$. Then we may introduce the *scalar of the density* $w(x, t)$

$$w(x, t) = \frac{dW(x, t)}{d\Omega(x)} = \frac{\mathfrak{w}}{\mathfrak{g}}\,(x, t) > 0 \qquad \text{(A-2.4)}$$

where $\mathfrak{g}(x)$ is the ,density of volume'. To form such a density we have only the antisymmetric contravariant tensor $j^{[\mu\nu]}$ at our disposal, which is the *fundamental tensor in phase space,* analogous to the metric tensor $g^{(\alpha\beta)}$ in Riemann space.

Therefore we put $\qquad \mathfrak{g} = \left|\det\,(j^{\mu\nu})^{-1/2}\right| > 0 \qquad$ (A-2.5)

because it has the right transformation property. It is $\neq 0$ if, and only if, $\omega = 2\,f$ is an *even number.* In terms of $w(x, t)$, the continuity equation (A-2.2) takes the form

$$\partial_t w + D_{\alpha}(z^{\alpha} w) = 0; \quad D_{\alpha} = \partial_{\alpha} + G_{\alpha}; \quad G_{\alpha} = \partial_{\alpha}\log\mathfrak{g}, \qquad \text{(A-2.6)}$$

where D_{α} is the *covariant divergence operator.* The *theorem of Liouville* states, that the *scalar of the density* w remains constant, if we follow an orbit $x^{\alpha} = z^{\alpha}(t)$:

$$\frac{d}{dt}\,w\big(z(t), t\big) \equiv \dot{w}(z, t) = (\partial_t w + \dot{z}^{\alpha}\,\partial_{\alpha} w)\,(z, t) = 0. \qquad \text{(A-2.7)}$$

This implies (see (A-2.6):

$$D_{\alpha}\dot{z}^{\alpha} = D_{\alpha}\big((\partial_{\beta}H)\,j^{\beta\alpha}\big) = (\partial_{\alpha}\partial_{\beta}H)\,j^{[\alpha\beta]} + (\partial_{\beta}H)\,D_{\alpha}\,j^{[\alpha\beta]} = 0 \quad \text{(A-2.8)}$$

and is a covariant condition*) for the fundamental tensor:

$$D_{\alpha}\,j^{[\alpha\beta]} = q^{\beta} = 0. \qquad \text{(A-2.9.}j\text{)*)}$$

We may express it in terms of the density $\mathfrak{j}^{\alpha\beta} = \mathfrak{g} j^{\alpha\beta}$

$$\partial_{\alpha}\,\mathfrak{j}^{[\alpha\beta]} = \mathfrak{q}^{\beta} = 0. \qquad \text{(A-2.9}\mathfrak{j}\text{)*)}$$

(A-2.9) is formally analogous to the *second set of Maxwell's equations,* if no electric charges $\mathfrak{q}^{\beta}$ are present.

) $D_{\alpha}\,F^{[\alpha\beta\gamma\ldots]} = G^{[\beta\gamma\ldots]}$ or $\partial_{\alpha}\mathfrak{F}^{[\alpha\beta\ldots]} = \mathfrak{G}^{[\beta\gamma\ldots]}$ (A-2.9)
and $\partial_{[\alpha}F_{\beta\gamma\ldots]} = G_{[\alpha\beta\gamma\ldots]}$ are covariant relations. (A-2.13*)

An observable $F(x)$ varies with time, according to

$$\dot{F}(z(t)) = \dot{z}^\alpha \partial_\alpha F = (\partial_\beta H)\, j^{[\beta\alpha]} \partial_\alpha F \equiv \{H, F\} = -\{F, H\} \quad \text{(A-2.10)}$$

where $\{H, F\}$ defines a *generalised Poisson bracket*.

It immediately follows, from

$$\{F, \{G, H\}\} = (\partial_\alpha F)\, j^{[\alpha\beta]}\, \partial_\beta((\partial_\gamma G)\, j^{[\gamma\delta]}\, \partial_\delta H)$$

$$= (\partial_\alpha F)\, j^{[\alpha\beta]}(\partial_\beta \partial_\gamma G)\, j^{[\gamma\delta]}\, \partial_\delta H + (\partial_\alpha F)\, j^{[\alpha\beta]}\, (\partial_\gamma G)\, j^{[\gamma\delta]}\, \partial_\beta \partial_\delta H$$

$$+ (\partial_\alpha F)\, (\partial_\beta G)\, (\partial_\gamma H)\, j^{[\alpha\varrho]}\, \partial_\rho j^{[\beta\gamma]}$$

that the cyclic sum is

$$\underset{FGH}{\looparrowright} \{F, \{G, H\}\} = (\partial_\alpha F)\, (\partial_\beta G)\, (\partial_\gamma H) \underset{\alpha\beta\gamma}{\looparrowright} j^{[\alpha\varrho]}\, \partial_\rho\, j^{[\beta\gamma]}. \quad \text{(A-2.11)}$$

Thus, the Jakobi identity is not necessarily satisfied.

If we require this identity, we must have

$$\underset{\alpha\beta\gamma}{\looparrowright} j^{\alpha\varrho} \partial_\rho\, j^{\beta\gamma} \equiv q^{[\alpha\beta\gamma]} = 0, \quad \text{(A-2.12)}$$

to which we may give the form of the *first set of Maxwell's equations* in the absence of magnetic charge $q_{[\alpha\beta\gamma]}$

$$\underset{\alpha\beta\gamma}{\looparrowright} \partial_\alpha j_{[\beta\gamma]} \equiv q_{[\alpha\beta\gamma]} = 0. \quad \text{(A-2.13)*)}$$

if we introduce the inverse tensor $j_{\alpha\beta}$:

$$j_{\alpha\rho} j^{\beta\varrho} = \delta^\beta_\alpha; \quad j_{\alpha\beta} = \min\, (j^{\alpha\beta})/\det\, (j^{\mu\nu}). \quad \text{(A-2.14)}$$

From this definition follows

$$j_{\alpha\rho}\, j_{\beta\sigma}\, j^{\varrho\sigma} = j_{\alpha\rho}\, \delta^\varrho_\beta = j_{\alpha\beta}. \quad \text{(A-2.15)}$$

We may thus raise and lower indices, with these antisymmetric fundamental tensors $j^{[\alpha\beta]}$ and $j_{[\alpha\beta]}$

$$F_{\alpha\beta\ldots} = j_{\alpha\alpha'}\, j_{\beta\beta'\ldots}\, F^{\alpha'\beta'}\ldots \qquad F^{\alpha\beta} = j^{\alpha\alpha'}\, j^{\beta\beta'} \ldots F_{\alpha'\beta'\ldots}. \quad \text{(A-2.16)}$$

In particular, it follows from

$$\partial_\rho\, \delta^\gamma_{\gamma'} = (\partial_\rho\, j^{\beta\gamma})\, j_{\beta\gamma'} + j^{\beta\gamma}\, \partial_\rho\, j_{\beta\gamma'} = 0$$

that

$$\partial_\rho\, j^{\beta\gamma} = j^{\beta\beta'}\, j^{\gamma\gamma'}\, \partial_\rho\, j_{\beta'\gamma'}. \quad \text{(A-2.17)}$$

Introducing this expression into (A-2.12), we arrive at the first set of ,Maxwell's equations' (A-2.13), with the conditions that ,magnetic charges' $q_{[\alpha\beta\gamma]}$ vanish.

This totally antisymmetric tensor of, magnetic charge' $q^{[\alpha\beta\gamma]}$ in phase space with a fundamental tensor $j^{[\alpha\beta]}$ plays an analogous role to the Riemann-Christoffel Tensor $R_{([\alpha\beta][\gamma\delta])}$ in a space with a metric $g_{(\alpha\beta)}$: If $q^{[\alpha\beta\gamma]}$ vanishes, a coordinate frame $'x^{'\alpha}$ exists, where $j^{[\alpha\beta]}$ has the form

$$\{'j^{'\alpha'\beta}\} = \left\{\begin{array}{c|c|c} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & 0 & 0 \\ \hline 0 & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & 0 \\ \hline 0 & 0 & \ddots \end{array}\right\}; \ \det ('j^{'\alpha'\beta}) = 1. \qquad \text{(A-2.19)}$$

This is analogous to the existence of an euclidian or pseudo-euclidian frame $'g_{'\alpha'\beta} = \pm\, \delta_{'\alpha'\beta}$ in the flat space, if $R_{([\alpha\beta][\gamma\delta])} = 0$. The proof of this theorem is given for instance by WHITTAKER[10]) (see also PAULI[12]):

It states, that the Pfaff differential expression formed from a_α in

$$j_{\alpha\beta} = \partial_\alpha\, a_\beta - \partial_\beta\, a_\alpha \qquad \text{(A-2.20)}$$

consisting of ω terms

$$a_\alpha(x)\, dx^\alpha = \sum_{i=1}^{i=f} p_i(x)\, d\, q_i(x) + \begin{cases} 0;\ 2f \leqslant n \\ dq_{l+1};\ 2f+1 \leqslant n \end{cases} \qquad \text{(A-2.21)}$$

can always be expressed in the form of the right-hand side.

Now put (as $\omega = 2f$)

$$p_i(x) = 'x^{2i-1},\ q_i(x) = 'x^{2i},\ i = 1\,2 \ldots f. \qquad \text{(A-2.22)}$$

Then we have

$$'a_{2i}('x) = 'x^{2i-1};\quad 'a_{2i-1}('x) = 0 \qquad \text{(A-2.22)}$$

and the only non zero component of $'j^{'\alpha'\beta}$

$$'\partial_{2i-1}\, 'a_{2i}('x) = 'j_{2i-1,\,2i} = +1, \qquad 'j^{2i,\,2i-1} = \min ('j_{2i,\,2i-1}) = -1. \qquad \text{(A-2.23)}$$

Introducing

$$'H('x) = H(x) = 'H(p, q) \qquad \text{(A-2.24)}$$

we have

$$\left.\begin{aligned} '\dot{z}^{2i-1} &= \dot{p}_i = '\partial_{2i}\, 'H('z)\, 'j^{2i,\,2i-1} = -\,'\partial_{2i}\, 'H('z) = -\frac{\partial}{\partial q_i}\, 'H(p, q) \\ \dot{z}^{2i} &= \dot{q}_i = '\partial_{2i-1}\, 'H('z)\, j^{2i-1,\,2i} = '\partial_{2i-1}\, 'H('z) = +\frac{\partial}{\partial p_i}\, 'H(pq). \end{aligned}\right\} \qquad \text{(A-2.25)}$$

Therefore we see, that the Jakobi identity is *by no means necessary to establish the H-theorem:* Only the ,second Maxwell set' (absence of "electric charges' $\mathfrak{q}^\alpha$) has to be satisfied. The presence of 'magnetic charges' $q_{[\alpha\beta\gamma]}$ in the "first Maxwell set' does *not invalidate* Liouville's *theorem* (from which the Boltzmann H-theorem follows). To this generalised statistical mechanics corresponds aQT, in which the observables are no more linear operators, because of:

$$\underset{FGH}{\curvearrowright} J\,[F, J\,[G, H]] \neq 0. \qquad \text{(A-2.26)}$$

Annex 2. Complex Hilbert Space (CHS)

If we restrict ourselves to the ortho-chronous subgroup $\{L_{(\mathrm{ochr})}\}$ with $L'^{n}_{(\mathrm{ochr})n} > 0$, all operators $\{F, G, \ldots, \breve{J}, O_{(\mathrm{ochr})}\}$ commute with $\breve{J}$:

$$[\breve{J}, O_{(\mathrm{ochr})}] = [\breve{J}, F] = \ldots = 0; \quad \breve{J}^T = -\breve{J}; \quad \breve{J}^2 = -1. \qquad \text{(A-2.1)}$$

We may now establish a relation (dictionary) between our QT in RHS and the conventional QT in CHS. To do this, we consider the ω_R-dimensional RHS as a product space between a 2-dimensional RHS and an $\omega_C = 1/2\,\omega_R$-dimensional RHS. We write

$$\Phi = (\Phi_{(r)}\,\Phi_{(i)}); \; \Psi = \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(i)} \end{pmatrix} \qquad \text{(A-2.2)}$$

and the arbitrary operator A ($[\breve{J}, A] = 0$) as the Kronecher product ($\times$)

$$A = 1 \times A_{(r)} + j \times A_{(i)}; \quad \breve{J} = j \times 1 \qquad \text{(A-2.3)}$$

$$A_{(r)} = \{A_{(r)\,pq}\} \quad \text{and} \quad A_{(i)} = \{A_{(i)\,pq}\}$$

are $\omega_C = 1/2\,\omega_R$-dimensional matrices ($pq \ldots = 1\,2 \ldots \omega_C$) and

$$j = \begin{pmatrix} 0 & -\lambda \\ +\lambda & 0 \end{pmatrix}; \quad j^2 = -1; \quad \lambda^2 = 1. \qquad \text{(A-2.4)}$$

Now let us consider the ω_C-dimensional complex Matrix ($i = +\sqrt{-1}$)

$$\widehat{A} = A_{(r)} + i\,A_{(i)}; \; \widehat{A}^{\dagger} = A^T_{(r)} - i\,A^T_{(i)} \qquad \text{(A-2.5)*)}$$

and the two ω_C-dimensional complex vectors

$$\widehat{\Phi} = \Phi_{(r)} + i\,\Phi_{(i)}; \quad \widehat{\Psi} = \Psi_{(r)} + i\,\Psi_{(i)} \qquad \text{(A-2.6)}$$

where $\Phi_{(r)}, \Phi_{(i)}, \Psi_{(r)}, \Psi_{(i)}, A_{(r)}$ and $A_{(i)}$ are the real ω_C-dimensional vectors and matrices. Now, we define the usual complex matrix element in CHS by

$$\left.\begin{aligned}
&\langle \widehat{\Phi}, \widehat{A}\,\widehat{\Psi} \rangle = \big((\Phi_{(r)} - i\,\Phi_{(i)}), (A_{(r)} + i\,A_{(i)})\,(\Psi_{(r)} + i\,\Psi_{(i)})\big) \\
&= \big((\Phi_{(r)}, A_{(r)}\Psi_{(r)}) + (\Phi_{(i)}, A_{(r)}\Psi_{(i)}) - (\Phi_{(r)}, A_{(i)}\Psi_{(i)}) \\
&+ (\Phi_{(i)}, A_{(i)}\Psi_{(r)})\big) + i\big((\Phi_{(r)}, A_{(i)}\Psi_{(r)}) + (\Phi_{(i)}, A_{(i)}\Psi_{(i)}) + \\
&+ (\Phi_{(r)}, A_{(r)}\Psi_{(i)}) - (\Phi_{(i)}, A_{(r)}\Psi_{(r)})\big) = (\Phi_{(r)}\,\Phi_{(i)}) \\
&\begin{pmatrix} A_{(r)} & -A_{(i)} \\ A_{(i)} & A_{(r)} \end{pmatrix} \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(i)} \end{pmatrix} - i(\Phi_{(r)}\,\Phi_{(i)}) \begin{pmatrix} -A_{(i)} & -A_{(r)} \\ A_{(r)} & -A_{(i)} \end{pmatrix} \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(i)} \end{pmatrix} \\
&= \langle \widehat{A}\,\widehat{\Psi}, \widehat{\Phi} \rangle^{*}.
\end{aligned}\right\} \qquad \text{(A-2.6)*)}$$

---

*) An * denotes the conjugate complex number. An $\dagger$ signifies the Hermitian conjugate operator:

$$\widehat{A}^{\dagger}_{pq} = \widehat{A}^{*}_{qp}; \; \widehat{A}^{\dagger} = (\widehat{A}^{*})^T = (\widehat{A}^T)^{*}. \qquad \text{(A-2.9)}$$

This expression is equal to

$$\langle \widehat{\Psi}, \widehat{A}\, \widehat{\Phi} \rangle = (\Phi, A\, \Psi) - i(\Phi, \breve{J}\, A\, \Psi) \qquad \text{(A-2.7)}$$

if we choose $\lambda = +1$ in (A-2.3), i. e.

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{(A-2.8)}$$

(A-2.7) is the dictionary we proposed to establish.

It is worth noting, that the definition of j in (A-2.3) is univoque: (A-2.8).

Annex 3. Unitary (U) and Anti-Unitary Operators (V) in CHS

We write our dictionary between CHS and RHS (A-2.7) in the form

$$\langle \widehat{\phi}, \widehat{A}^{\alpha}(x)\, \widehat{\Psi} \rangle = (\Phi, (1 - i\, \breve{J})\, A^{\alpha}(x)\, \Psi) \qquad \text{(A-3.1)}$$

where the left hand side is the scalar product in complex Hilbert space (= CHS) and the right hand side is the scalar product in RHS (i. e. all symbols, except i, stand for real quantities (vectors, operators)). Let us consider the transformed quantity (matrix element of $A^{\alpha}(x)$ between the states $\boldsymbol{\Phi} \leftarrow \boldsymbol{\Psi}$)

$$\left.\begin{aligned} &\langle \widehat{\Phi}, '\widehat{A}'^{\alpha}('x)\, \widehat{\Psi} \rangle = L'^{\alpha}{}_{\alpha}\, (\Phi, (1 - i \,\mathrm{sig}\, (L'^{n}{}_{n})\, \breve{J})\, A^{\alpha}(L^{-1}\, 'x)\, \Psi) \\ &\equiv \langle '\widehat{\Phi}, \widehat{A}'^{\alpha}('x)\, '\widehat{\Psi} \rangle = ('\Phi, (1 - i\, \breve{J})\, A'^{\alpha}('x)\, '\Psi) \end{aligned}\right\} \qquad \text{(A-3.2)}$$

a) *Orthochronous transformations*

$(L'^{n}_{(\mathrm{ochr})n} > 0)$: We have

$$'\Psi_{'a} = O_{'a a}\, \Psi_{a};\ \begin{pmatrix} '\Psi_{(r)} \\ '\Psi_{(i)} \end{pmatrix} = (1 \times O_{(r)} + j \times O_{(i)}) \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(i)} \end{pmatrix} \qquad \text{(A-3.4)}$$

$$'\widehat{\Psi}_{'p} = U_{'p p}\, \widehat{\Psi}_{p};\ '\widehat{\Psi} = (O_{(r)} + i\, O_{(i)})\, \widehat{\Psi} \equiv U \widehat{\Psi} \qquad \text{(A-3.5)}$$

$$'\widehat{\Phi}^{*}_{'p} = \widehat{\Phi}^{*}_{p}\, U^{*T}_{p'p} = \widehat{\Phi}^{*}_{p}\, U^{\dagger}_{p'p};\ '\widehat{\Phi}^{*} = (O_{(r)} - i\, O_{(i)})\, \widehat{\Phi}^{*} \equiv U^{*}\, \widehat{\Phi}^{*} \equiv \widehat{\Phi}^{*}\, U^{\dagger} \qquad \text{(A-3.5*)}$$

From the orthogonality of O

$$\left.\begin{aligned} &O^{T}\, O = (1 \times O^{T}_{(r)} - j \times O^{T}_{(i)})\, (1 \times O_{(r)} + j \times O^{T}_{(i)}) = \\ &1 \times (O^{T}_{(r)}\, O_{(r)} + O^{T}_{(i)}\, O_{(i)}) + j \times (O^{T}_{(r)}\, O_{(i)} - O^{T}_{(i)}\, O_{(r)}) = 1 \times 1 = 1 \end{aligned}\right\} \qquad \text{(A-3.6)}$$

follows

$$O^{T}_{(r)}\, O_{(r)} + O^{T}_{(i)}\, O_{(i)} = 1;\quad O^{T}_{(r)}\, O_{(i)} - O^{T}_{(i)}\, O_{(r)} = 0. \qquad \text{(A-3.8)}$$

Thus, according to the definition of U and $U^{\dagger}$ in (A-3.5), and (A-3.5*),

$$U^{\dagger}\, U = 1 \qquad \text{(A-3.9)}$$

$U = \{U_{pq}\}$ is an *unitary matrix* in $\omega_C = 1/2\,\omega_R$- dimensional CHS ($pq \ldots = 1\,2 \ldots \omega_C$; $a\,b \ldots = 1\,2 \ldots \omega_R$). Because of sig $(L'^n{}_n) = 1$ and of (A-3.1), we may write the second member of (A-3.2) in the form $L'^\alpha{}_\alpha \langle \widehat{\phi}, \widehat{A}^\alpha(L^{-1\prime}x)\, \widehat{\Psi}\rangle$. Thus we obtain finally (see (A-2.5)) the identity

$$\widehat{A}'^\alpha('x) = L'^\alpha{}_\alpha\, U\, \widehat{A}^\alpha(L^{-1}\,'x)\, U^\dagger \tag{A-3.10}$$

i. e. *the tensor* $\widehat{A}_{pq}(x) = \widehat{A}^X_{pq}$ *is invariant if it is transformed with respect to all its indices*:

$$\widehat{A}'^X_{'p'q} = L'^X_{(\mathrm{ochr})X}\, U_{'pp}\, U^*_{'qq}\, \widehat{A}^X_{pq} \tag{A-3.11}$$

and U is an unitary representation of $L_{(\mathrm{ochr})}$

$$L_{(\mathrm{ochr})} \rightarrow e^{i\lambda}\, U. \tag{A-3.12}$$

b) *Pseudochronous Transformations* $(L'^n_{(\mathrm{pchr})n} < 0)$

We *try* (A-3.2), posing for the transformed matrix element (A-3.2)

$$\left.\begin{aligned}&\langle \widehat{\Phi}, '\widehat{A}'^\alpha('x)\, \widehat{\Psi}\rangle = \langle '\widehat{\Phi}, \widehat{A}'^\alpha('x)\, '\widehat{\Psi}\rangle = \\ &L'^\alpha{}_\alpha(\Phi, (1 + i\,\breve{J})\, A^\alpha(L^{-1}\,'x)\, \Psi) = L'^\alpha{}_\alpha \langle \widehat{\Phi}, \widehat{A}^\alpha(L^{-1}\,'x)\, \widehat{\Psi}\rangle^*.\end{aligned}\right\} \tag{A-3.13}$$

It is, in virtue of the definition (A-3.1), *a linear function of the untransformed conjugate complex element.* We shall see however, that the identity in $\widehat{A}'^\alpha_{'p'q}('x)$ leads now to a contradiction. In order to show that, write, introducing a *non-linear operator* V

$$'\widehat{\Psi} = (V\,\widehat{\Psi}),\ '\widehat{\Phi} = (V\,\widehat{\Phi})$$

the second and fourth member of (A-3.13) in the Form:

$$\langle (V\,\widehat{\Phi}), V(V^{-1}\,(\widehat{A}'^\alpha('x)\,(V\,\widehat{\Psi})))\rangle = L'^\alpha{}_\alpha \langle \widehat{A}^\alpha(L^{-1}\,'x)\,\widehat{\Psi}, \widehat{\Phi}\rangle. \tag{A-3.14}$$

Thus, we must have a relation

$$(V\,\widehat{\Psi})_{'p} = U_{'pp}\,\widehat{\Psi}^*{}_p;\quad (V^{-1}\,\widehat{\Psi})_p = U^T_{p'p}\,\widehat{\Psi}^*_{'p} \tag{A-3.15}$$

$$(V\,\widehat{\Phi})^*_{'p} = U^*_{'pp}\,\widehat{\Phi}_p = \widehat{\Phi}_p\, U^\dagger_{p'p};\ (V^{-1}\widehat{\Phi})^*_p = U^\dagger_{p'p}\,\widehat{\Phi}_{'p}. \tag{A-3.15*}$$

where U is an unitary matrix, in order to have

$$\left.\begin{aligned}\langle V(\widehat{\Phi}), V(\widehat{\Psi})\rangle = \langle \widehat{\Phi}, \widehat{\Psi}\rangle^* = \langle \widehat{\Psi}, \widehat{\Phi}\rangle;\\ U^\dagger\, U = U^T\, U^* = 1.\end{aligned}\right\} \tag{A-3.16}$$

On account of (A-3.16), the identity (A-3.14) (second = forth member in (A-3.13)) has the form:

$$\big(V^{-1}\,(\widehat{A}'^\alpha('x)\,(V\,\widehat{\Psi}))\big)^*_p\,\widehat{\Phi}_p = (L'^\alpha{}_\alpha\, A^\alpha(L^{-1}\,'x)\,\widehat{\Psi})^*_p\,\widehat{\Phi}_p, \tag{A-3.17}$$

which must hold for all $\widehat{\Phi}_q$. Thus we have

$$(V^{-1}(\widehat{A}'^{\alpha}('x)\,(V\widehat{\Psi})))_p = (L'^{\alpha}{}_{\alpha}\,\widehat{A}^{\alpha}(L^{-1}\,'x)\,\widehat{\Psi})_p \qquad \text{(A-3.18)}$$

from this identity in $\widehat{\Psi}_p$ (see (A-3.15)) follows

$$U^T_{p'p}\,\widehat{A}^*_{'p'q}('x)\,U^*_{'qq} = L^{\alpha}{}_{'\alpha}\,\widehat{A}'^{\alpha}_{pq}(L^{-1}\,'x) \qquad \text{(A-3.19)}$$

or

$$A^{*'X}_{'p'q} = L'^X{}_X\,U^*_{'pp}\,U_{'qq}\,\widehat{A}^X_{pq}. \qquad \text{(A-3.20)}$$

This identy can evidently not be satisfied in general, because no linear transformation $(L'^{\alpha}{}_{\alpha}\,U^* \ldots U^T)$ exists, which transforms *all tensors* $\widehat{A}^X_{pq}$ in hermitian CHS into their conjugate complex. Thus we conclude, that pseudo-chronous transformations are not given by (A-3.13), but we must have

$$\langle\widehat{\Phi},\,'\widehat{A}'^{\alpha}('x)\,\widehat{\Psi}\rangle = \langle'\widehat{\Phi},\widehat{A}^{*'\alpha}('x)\,'\widehat{\Psi}\rangle = L'^{\alpha}{}_{\alpha}\,\langle\widehat{\Phi},\widehat{A}^{\alpha}(L^{-1}\,'x)\,\Psi\rangle^*. \qquad \text{(A-3.13*)}$$

In this case, $\widehat{A}'^{\alpha}\,('x)$ (in the third member of (A-3.2) and in the first members of equations (A-3.14), (A-3.17)–(A-3.20)) has to be replaced by $\widehat{A}^{*'\alpha}('x)$.

(A-3.20) is now analogous to (A-3.11)

$$\widehat{A}'^X_{'p'q} = L'^X_{(\mathrm{pchr})X}\,U^*_{'pp}\,U_{'qq}\,\widehat{A}^X_{pq}. \qquad \text{(A-3.20*)}$$

This rather complicated formalism for time reversal in CHS shows clearly the advantage of RHS, where all transformations $L \to e^{\lambda\breve{J}}O$ are linear.

Annex 4. Quaternion Hilbert Space (QHS)

QHS has been introduced by FINKELSTEIN, JAUCH and SPEISER[3]). They pose, with $i_{\alpha}^2 = -1$; $i_1\,i_2 = -\,i_2\,i_1 = i_3$ cycl.:

$$\widehat{\Phi} = \Phi_{(r)} + i_1\,\Phi_{(1)} + i_2\,\Phi_{(2)} + i_3\,\Phi_{(3)} = \{\widehat{\Phi}_p\} \qquad \text{(A-4.1)}$$

$$\widehat{\Psi} = \Psi_{(r)} + i_1\,\Psi_{(1)} + i_2\,\Psi_{(2)} + i_3\,\Psi_{(3)} = \{\widehat{\Psi}_q\} \qquad \text{(A-4.2)}$$

$$\widehat{A} = A_{(r)} + i_1\,A_{(1)} + i_2\,A_{(2)} + i_3\,A_{(3)} = \{\widehat{A}_{pq}\}. \qquad \text{(A-4.3)}$$

The QHS has $\omega_Q = 1/4\,\omega_R$-dimensions: $p\,q, \ldots = 1\,2 \ldots \omega_Q$.. The scalar product is defined by

$$\langle\widehat{\Phi},\,\widehat{A}\,\widehat{\Psi}\rangle = \widehat{\Phi}^*_p\,\widehat{A}_{pq}\,\widehat{\Psi}_q \qquad \text{(A-4.4)}$$

where $\widehat{\Phi}^*_p$ is the conjugated quaternion of $\widehat{\Phi}_p$ $(i_{\alpha} \to -\,i_{\alpha})$. All numbers, $\widehat{\Phi}_p$, $\widehat{A}_{pq}$, $\widehat{\psi}_q$ and $\langle\widehat{\Phi},\,\widehat{A}\,\widehat{\Psi}\rangle$ are now quaternions ($\Phi_{(r)} \ldots A_{(r)} \ldots$ are real vectors and tensors). A straight forward calculation gives

$$\langle\widehat{\Phi},\,\widehat{A}\,\widehat{\Psi}\rangle_{(r)} = (\Phi,\,A\,\Psi) \qquad \text{(A-4.5)}$$

with

$$\Phi = (\Phi_{(r)}\,\Phi_{(1)}\,\Phi_{(2)}\,\Phi_{(3)})\,;\ \Psi = \begin{pmatrix}\Psi_{(r)}\\ \Psi_{(1)}\\ \Psi_{(2)}\\ \Psi_{(3)}\end{pmatrix} \tag{A-4.6}$$

$$A = 1 \times A_{(r)} + j_1 \times A_{(1)} + j_2 \times A_{(2)} + j_{(3)} \times A_{(3)}. \tag{A-4.7}$$

The j_α *'s* are four by four matrices and satisfy the same commutation laws as the quaternions i_α *'s* ($j^2 = -1$; $j_1 j_2 = - j_2 j_1 = j_3$ cycl). They are KRONECKER products of the *'pseudo-quaternions'*:

$$\left.\begin{aligned} &1 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \quad j = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right), \quad k = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right), \quad l = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\\ &\quad - j^2 = k^2 = l^2 = 1; \quad jk = -kj = l,\ kl = -lk = -j\\ &\quad lj = -jl = k \end{aligned}\right\} \tag{A-4.8}$$

$$j_1 = \left(\begin{smallmatrix}j&0\\0&j\end{smallmatrix}\right) = 1 \times j; \quad j_2 = \left(\begin{smallmatrix}0&-k\\k&0\end{smallmatrix}\right) = j \times k; \quad j_3 = \left(\begin{smallmatrix}0&-l\\l&0\end{smallmatrix}\right) = j \times l. \tag{A-4.9}$$

However, in order to form the vector components of the quaternion, we need three further four by four matrices

$$\left.\begin{aligned} &k_1 = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right) = k \times 1; \ k_2 = \left(\begin{smallmatrix}k&0\\0&k\end{smallmatrix}\right) = 1 \times k; \quad k_3 = \left(\begin{smallmatrix}k&0\\0&-k\end{smallmatrix}\right) = k \times k\\ &\quad k_1 k_2 = k_2 k_1 = k_3 \text{ cycl.}; \quad k_\alpha^2 = 1\\ &\quad [k_\alpha, j_\alpha] = 0; \quad (k_\alpha, j_\beta) = 0; \quad \alpha \neq \beta \end{aligned}\right\} \tag{A-4.10)*}$$

in order to write

$$\langle \widehat{\Phi}, \widehat{A}\,\widehat{\Psi}\rangle_\alpha = -(\Phi, (k_\alpha j_\alpha \times 1)\, A\, \Psi). \tag{A-4.11}$$

Introducing, analogous to (A-2.3)

$$J_\alpha = j_\alpha \times 1; \quad K_\alpha = k_\alpha \times 1, \tag{A-4.12}$$

we find

$$\left.\begin{aligned} \langle \widehat{\Phi}, \widehat{A}\,\widehat{\Psi}\rangle &= (\Phi, A\Psi) - i_1(\Phi, K_1 J_1 \Psi) - i_2(\Phi, K_2 J_2 \Psi)\\ &\quad - i_3(\Phi, K_3 J_3 \Psi). \end{aligned}\right\} \tag{A-4.13}$$

Thus, QT in QHS is not equivalent to QT in RHS, because we need – in addition to the three anticommuting operators

$$J_\alpha^2 = -1; \quad J_1 J_2 = - J_2 J_1 = J_3 \text{ cycl.} \tag{A-4.14}$$

– three further operators

$$\left.\begin{aligned} &K_\alpha^2 = 1; \quad K_1 K_2 = K_2 K_1 = K_3 \text{ cycl.}\\ &[K_\alpha, J_\alpha] = 0; \quad (K_\alpha, J_\beta) = 0, \quad \alpha \neq \beta. \end{aligned}\right\} \tag{A-4.15)*}$$

*) We write $[A, B] = AB - BA$ for the commutator and $(A, B) = AB + BA$ for the anticommutator. This is in strict analogy to our notation for antisymmetric tensors $j^{[\alpha\beta]} = -j^{[\alpha\beta]}$ and for symmetric tensors $g^{(\alpha\beta)} = g^{(\beta\alpha)}$.

Annex 5. An Error in the Representation Theory Frequently Found in Literature

Instead of our identity, following from (3.4) and (3.5)

$$(\Psi, {}'F'^{\alpha}('x)\,\Psi) = ({}'\Psi, F'^{\alpha}('x)\,{}'\Psi) \qquad \text{(A-5.1)}$$

many authors start from the identity

$$({}'\overline{\Psi}, {}'\overline{F}'^{\alpha}('x)\,{}'\overline{\Psi}) = (\overline{\Psi}, \overline{F}'^{\alpha}('x)\,\overline{\Psi}), \qquad \text{(A-5.2)*)}$$

which is analogous to the relation (5.2) between the Schroedinger and the Heisenberg representation. We refer particularily to the otherwise excellent book by JAUCH and ROHRLICH[11]) (to be referred to as $J\,R$). They define

$${}'\overline{\Psi} = \overline{O}\,\overline{\Psi},\ \overline{O}^T = \overline{O}^{-1} \qquad \text{(A-5.3)}$$

and, as we have done,

$${}'\overline{F}'^{\alpha}('x) = L'^{\alpha}{}_{\alpha}\,\overline{F}^{\alpha}(L^{-1}\,'x). \qquad \text{(A-5.4)}$$

From the identity in ${}'\overline{\Psi}_{'a}$ they find, of course

$${}'\overline{F}'^{\alpha}('x) = \overline{O}\,\overline{F}'^{\alpha}('x)\,\overline{O}^{-1}. \qquad \text{(A-5.5)}$$

Now, they pretend that $\{e^{\breve{J}\lambda}\overline{O}\}$ is a ray representation of $\{L\}$. The identity which follows from (A-5.4) and (A-5.5) is, explicitly written:

$$L'^{\alpha}{}_{\alpha}\,\overline{F}^{\alpha}_{'a'b}(L^{-1}\,'x) = \overline{O}_{'aa}\,\overline{F}'^{\alpha}_{ab}('x)\,\overline{O}^{-1}_{b'b} \qquad \text{(A-5.6)}$$

(Eq. (1–43) and (1–42) p. 11 of JR). According to JR, $\overline{O}$ (and not $\overline{O}^{-1} = \overline{O}^T \equiv O$, as we found in (3.8)) is a representation of L. They give no proof of their statement.

We shall give an argument, which may have lead them to this contradictory statement: Write $L_{(1)}$ in (A-5.4) and $\overline{O}^{(1)}$ in (A-5.5). $L_{(1)}$ transforms from the $X = \{x\,\alpha\}$-frame to the $'X$-frame. Let $L_{(1)}$ be followed by $L_{(2)}$ transforming $''X \leftarrow {}'X$:

$$''\overline{F}''^{\alpha}(''x) = L_{(2)}{}''^{\alpha}{}_{'\alpha}\,\overline{F}'^{\alpha}(L^{-1}_{(2)}\,''x) \qquad \text{(A-5.4}^{(2)}\text{)}$$

$$''\overline{\Psi}_{''a} = \overline{O}^{(2)}_{''a'a}\,\overline{\Psi}_{'a} \qquad \text{(A-5.3}^{(2)}\text{)}$$

$$''\overline{F}''^{\alpha}(''x) = \overline{O}^{(2)}\,\overline{F}''^{\alpha}(''x)\,\overline{O}^{(2)-1}. \qquad \text{(A-5.5}^{(2)}\text{)}$$

Now, substitute (5.4$^{(1)}$) into (5.4$^{(2)}$) i. e.

$$''\overline{F}''^{\alpha}(''x) = (L_{(2)}\,L_{(1)})''^{\alpha}{}_{\alpha}\,\overline{F}^{\alpha}((L_{(2)}\,L_{(1)})^{-1}\,''x) \qquad \text{(A-5.6)}$$

and (A-5.5$^{(1)}$) into (A-5.5$^{(2)}$)

$$''\overline{F}''^{\alpha}(''x) = (\overline{O}^{(2)}\,\overline{O}^{(1)})\,\overline{F}''^{\alpha}(''x)\,(\overline{O}^{(2)}\,\overline{O}^{(1)})^{-1}. \qquad \text{(A-5.7)}$$

---

*) We denote vectors and operators satisfying (A-5.2) by a bar, in order to distinguish them from the vectors and operators in the text and in (A-5.1).

If we eliminate $''\overline{F}''^{\alpha}(''x)$ from the last two equations, we find writing $X = \{\alpha\, x\}$

$$(L_{(2)}\, L_{(1)})^{''X}{}_{X}\, \overline{F}^{X}_{''a''b} = (\overline{O}^{(2)}\, \overline{O}^{(1)})_{''a\,a}\, \overline{F}^{''X}_{ab}\, (\overline{O}^{(2)}\, \overline{O}^{(1)})^{-1}_{b''b}. \qquad \text{(A-5.8)}$$

Thus, it seems to follow from (A-5.8) that $\{e^{\breve{J}\lambda}\overline{O}\}$ is a representation of $\{L\}$. This is of course contradictory to the theory presented in the text (§ 3). C. Piron and H. Ruegg shall publish a note in this journal, which shows how the two contradictory points of view can be understood.

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