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# Marsden-Ratiu Reduction and $\mathbf{W}_3^2$ Algebra

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*Abstract* The $W_3^2$ algebra is deduced by the Marsden-Ratiu reduction in the bi-Hamiltonian framework proposed by Magri et al and compared with the usual derivations via the Drinfeld-Sokolov formalism. It is observed that the choice of A in the first Poisson tensor must be different for $W_3^2$ algebra.

## 1. Introduction

It has been known since a long time that the KdV equation $U_t = U_{xxx} + 6UU_x$ can be written as a Hamiltonian system with respect to two different Poisson structures(1). This property leads to a sequence of commuting Hamiltonians which can be constructed through recursion. The second hamiltonian structure in this hierarchy coincides with the canonical Lie-Poisson structure on the dual of Virasoro algebra(2). On the other hand, in a fundamental paper, Drinfeld-Sokolov(3)presented a procedure to associate generalised KdV-type equations with any Kac-Moody algebra, which also enjoy the property of being bi-Hamiltonian. The Drinfeld-Sokolov reduction is essentially algebraic, a fundamental role being played by the idea of gauge invariance. On the other hand in the formulation of Magri et al(4), a different explanation of the Hamiltonian reduction and the generation of Virasoro algebra was given using a geometrical reduction process, viz. the Marsden-Ratiu procedure. In the present paper, we utilise the idea of Marsden-Ratiu reduction and the theory of bi-Hamiltonian manifold to deduce classical $W_3^2$algebra, which is associated with the generalised DS hierarchies. We also study the co-adjoint invariance of the structure of $W_3^2$.

This paper is organized as follows. In section (2) we briefly review the Marsden-Ratiu reduction[5]scheme and the associated bi-Hamiltonian manifold and then apply it to derive the $W_3^2$. In this context we have observed that some generalization of the formalism of ref (6) is needed for the $W_3^2$case. In section (3) the co-adjoint invariance is discussed.

# 2. Formulation

Recall that, according to classical mechanics an integrable system is a dynamical system on a symplectic manifold $M$ which admits a complete set of constants of motion in involution. These constants are usually constructed by means of a group of symmetry $G$ acting symplectically on the phase space. As a first step towards developing the idea of bi-Hamiltonian manifold, we replace $G$ by a "Poisson-action of the algebra of observables on $M$ defined by the second Poisson structure. Manifolds endowed with a pair of "compatible Poisson brackets $P_0$ and $P_1$, are called bi-Hamiltonian manifolds, such that one of them selects the Hamiltonians and the other selects the vector fields[7].

The Marsden-Ratiu reduction scheme considers a submanifold $S$ of $M$, a foliation $E$ of $S$ and the quotient space $N = S/E$. The foliation $E$ is defined by the intersection with $S$ of a distribution $D$ in $M$, defined only at the points of $S$. The submanifold $S$ is a symplectic leaf of the first Poisson tensor $P_0$. The distribution $D$ is the image of the kernel of $P_0$ with respect to $P_1$. We then have the following general result:

The quotient space $N = S/E$ is a bi-Hamiltonian manifold. On $N$ there exists a unique Poisson $\{,\}_N^\lambda$ such that

$$\{f,g\}_N^\lambda \circ \pi = \{F,G\}_M^\lambda \circ i$$

for any pair of functions $F$ and $G$ which extend the functions $f$ and $g$ of $N$ into $M$, and are constant on $D$. Here $\pi$ stands for the projection $\pi : S \mapsto N$ and $i$ denotes the inclusion. This means that the function $F$ satisfies the conditions,

$$\begin{aligned} F \circ i &= f \circ \pi \\ \{F, K\}_1 &= 0 \end{aligned}$$

for any function $K$ whose differential at the point of $S$, belongs to the kernel of $P_0$. To proceed let us consider $g = sl(3, C)$, and set

$$\begin{aligned} S =& V_{11}e_{11} + V_{22}e_{22} + V_{33}e_{33} + V_1 e_{12} + V_{-1}e_{21} + \\ & V_3 e_{13} + V_{-3}e_{31} + V_2 e_{23} + V_{-2}e_{32} \end{aligned} \tag{1}$$

a map from the circle $S^1$ into the Lie algebra $sl(3, c)$. The entries of this matrix are periodic functions of the coordinate $x$ on the circle. Let us consider this matrix as a point on the manifold $M$. We then have

$$\begin{aligned}\dot{S} =& \dot{V}_{11}e_{11} + \dot{V}_{22}e_{22} + \dot{V}_{33}e_{33} + \dot{V}_1 e_{12} + \\ & \dot{V}_{-1}e_{21} + \dot{V}_3 e_{13} + \dot{V}_{-3}e_{31} + \dot{V}_2 e_{23} + \dot{V}_{-2}e_{32},\end{aligned} \tag{2}$$

a tangent vector to $M$ at the point $S$. Let

$$V = \alpha_1 e_{11} + \alpha_2 e_{22} + \alpha_3 e_{33} + \beta_1 e_{12} + \beta_2 e_{21} + \delta_1 e_{13} + \delta_2 e_{31} + \gamma_1 e_{23} + \gamma_2 e_{23} \tag{3}$$

denote a covector at the point $S$. They are arbitrary loops from $S^1$ into $g$. To be consistent with the $sl(3,c)$ algebra , we must have

$$\sum V_{ii} = 0; \sum \alpha_i = 0, i = 1,2,3 \tag{4}$$

The space $M$ is essentially an infinite dimensional Lie algebra with a canonical co-cycle

$$\omega(\dot{S}_1, \dot{S}_2) = \int_{S^1} \mathrm{Tr}\left(\dot{S}_1 \frac{d\dot{S}_2}{dx}\right) dx \tag{5}$$

the linear map $\Omega : g \mapsto g^*$ associated with this co-cycle is

$$\Omega(V) = \frac{dV}{dx} \tag{6}$$

According to the general construction of bi-Hamiltonian manifolds, the space $M$ is endowed with two Poisson tensors $P_0$ and $P_1$ defined by

$$P_0(V) = [A, V] \tag{7a}$$

$$P_1(V) = V_x + [V, S] \tag{7b}$$

Here $V_x$ denotes the derivative of the loop $V$ with respect to the co-ordinate $x$ on $S^1$, and $A$ is a constant matrix. The crucial point is the choice of $A$. Specific Lie algebraic method is given in reference (6) only for the Drinfeld-Sokolov type reductions. There it was stipulated that $A$ should belong to the centre of the Borel subalgebra. But in the case of $W_3^2$ we are to modify this prescription. We have observed that if we consider $A$ to be a constant strictly lower triangular matrix belonging to $sl(3,c)$ algebra , then we can arrive at $W_3^2$. But the ansatz given in ref. (6) leads only to $W_3$. So we set

$$A = e_{21} + e_{31} + e_{32} \tag{8}$$

The Poisson tensor $P_0$ leads to

$$\begin{aligned}\dot{V}_{11} &= -\beta_1 - \delta_1 \\ \dot{V}_{22} &= \beta_1 - \gamma_1 \\ \dot{V}_{33} &= \delta_1 + \gamma_1 \\ \dot{V}_{-1} &= \alpha_1 - \alpha_2 - \gamma_1 \\ \dot{V}_{-2} &= \beta_1 + \alpha_2 - \alpha_3 \\ \dot{V}_{-3} &= \alpha_1 + \beta_2 - \gamma_2 - \alpha_3 \\ \dot{V}_1 &= -\delta_1 \\ \dot{V}_2 &= \delta_1 \\ \dot{V}_3 &= 0\end{aligned} \tag{9}$$

Similarly from the second Poisson tensor $P_1$ we get

$$\begin{aligned}
\dot{V}_{11} &= \alpha_{1x} + \beta_1 V_{-1} + \delta_1 V_{-3} - \beta_2 V_1 - \delta_2 V_3 \\
\dot{V}_{22} &= \alpha_{2x} + \beta_2 V_1 + \gamma_1 V_{-2} - \beta_1 V_{-1} - \gamma_2 V_2 \\
\dot{V}_{33} &= \alpha_{3x} + \delta_2 V_3 + \gamma_2 V_2 - \delta_1 V_{-3} - \gamma_1 V_{-2} \\
\dot{V}_{-1} &= \beta_{2x} + \beta_2(V_{11} - V_{22}) + (\alpha_2 - \alpha_1)V_{-1} + \gamma_1 V_{-3} - \delta_2 V_2 \\
\dot{V}_{-2} &= \gamma_{2x} + \gamma_2(V_{22} - V_{33}) + (\alpha_3 - \alpha_2)V_{-2} - \beta_1 V_{-3} + \delta_2 V_1 \\
\dot{V}_{-3} &= \delta_{2x} + d_2(V_{11} - V_{33}) + (\alpha_3 - \alpha_1)V_{-3} + \gamma_2 V_{-1} - \beta_2 V_{-2} \\
\dot{V}_1 &= \beta_{1x} + \beta_1(V_{22} - V_{11}) + (a_1 - \alpha_2)V_1 + \delta_1 V_{-2} - V_3\gamma_2 \\
\dot{V}_2 &= \gamma_{1x} + \gamma_1(V_{33} - V_{22}) + (a_2 - \alpha_3)V_2 + \delta_1 V_{-1} - \beta_2 V_3 \\
\dot{V}_3 &= \delta_{1x} + \delta_1(V_{33} - V_{11}) + (a_1 - \alpha_3)V_3 + \beta_1 V_2 - \gamma_1 V_1
\end{aligned} \tag{10}$$

Let us note that the vector field defined by the first bi-vector $P_0$ are tangent to the affine hyperplanes $V_3 = V_{30}$ (where $V_{30}$ is a given periodic function); so the symplectic leaves of $P_0$ are affine hyperplanes.

Since $\dot{V}_3 = 0$, from the Poisson tensor $P_0$, let us choose $V_3 = 1$, so that

$$S = V_{11}e_{11} + V_{22}e_{22} + V_{33}e_{33} + V_1 e_{12} + V_{-1}e_{21} + e_{13} + V_{-3}e_{31} + V_2 e_{23} + V_{-2}e_{32} \tag{11}$$

The kernel of $P_0$ is formed by the covectors with

$$\begin{gathered}
\delta_1 = \beta_1 = \gamma_1 = 0 \\
\alpha_1 = \alpha_2 = \alpha_3 = 0 \\
\text{along with } \beta_2 = \gamma_2 \text{ and } V_1 + V_2 = 0
\end{gathered} \tag{12}$$

Now the flows given by the second Poisson tensor suggest that the distribution $D$ is spanned by the following vector fields,

$$\begin{aligned}
\dot{V}_{11} &= -\beta_2 V_1 - \delta_2 \\
\dot{V}_{22} &= \beta_2 V_1 - \gamma_2 V_2 \\
\dot{V}_{33} &= \delta_2 + \gamma_2 V_2 \\
\dot{V}_{-1} &= \beta_{2x} + \beta_2(V_{11} - V_{22}) - \delta_2 V_2 \\
\dot{V}_{-2} &= \gamma_{2x} + \gamma_2(V_{22} - V_{33}) + \delta_2 V_1 \\
\dot{V}_{-3} &= \delta_{2x} + \delta_2(V_{11} - V_{33}) + \gamma_2 V_{-1} - \beta_2 V_{-2} \\
\dot{V}_1 &= -\gamma_2 \\
\dot{V}_2 &= \beta_2
\end{aligned} \tag{13}$$

So from these equations we obtain the elements of the matrix $V$,

$$\begin{aligned}
\beta_2 &= \dot{V}_2 \\
\gamma_2 &= -\dot{V}_1 \\
\delta_2 &= V_{33} + V_1 V_2
\end{aligned} \tag{14}$$

By using equation (13) in (14), we obtain

$$(V_{22} - V_2 V_1)^{\cdot} = 0$$

So we get an invariant functional of $S$, viz

$$U_0 = V_{22} - V_2 V_1 \tag{15}$$

Similarly we obtain, after a laborious computation , the other three invariants, viz.

$$\begin{aligned} U_1 &= V_2(V_{22} - V_{11}) + V_{-1} - V_2^2 V_1 - V_{2x} \\ U_2 &= V_1(V_{11} + 2V_{22}) + V_{-2} - V_1^2 V_2 + V_{1x} \\ U_3 &= -V_{11}V_{33} + \frac{1}{4}(V_{22} + 6V_1V_2)V_{22} - \frac{3}{4}V_1^2 V_2^2 \\ &\quad + V_1 V_{-1} + V_2 V_{-2} + V_{-3} + V_{11x} + \frac{1}{2}V_{22x} - \frac{1}{2}V_2 V_{1x} - \frac{1}{2}V_1 V_{2x} \end{aligned} \tag{16}$$

These invariants closely resemble those found in ref. (9) in the discussion of the twisted version of the $W_3^2$ algebra. Geometrically speaking, $U_0$, $U_1$, $U_2$, $U_3$ are the final variables of the quotient space $N = S/E$ which is the space of functions on $S^1$ and equations (15) and (16) give the projection $\pi : S \mapsto N$. These four invariants turn out to be the generators of the $W_3^2$ algebra because their Poisson brackets yield,

$$\begin{aligned} \{U_0(x), U_0(y)\} &= -\frac{2}{3}\delta'(x-y) \\ \{U_0(x), U_1(y)\} &= U_1(x)\delta(x-y) \\ \{U_0(x), U_2(y)\} &= -U_2(x)\delta(x-y) \\ \{U_1(x), U_2(y)\} &= -\delta'(x-y) + 3U_0(x)\delta(x-y) + \{U_3(x) + \frac{3}{2}U_0'(x) - 3U_0^2(x)\}\delta(x-y) \\ \{U_3(x), U_0(y)\} &= -U_0(x)\delta'(x-y) \\ \{U_3(x), U_1(y)\} &= -\frac{3}{2}U_1(x)\delta'(x-y) - \frac{1}{2}U_1'(x)\delta(x-y) \\ \{U_3(x), U_2(y)\} &= -\frac{3}{2}U_2(x)\delta'(x-y) - \frac{1}{2}U_2'(x)\delta(x-y) \\ \{U_3(x), U_3(y)\} &= \frac{1}{2}\delta'''(x-y) - 2U_3(x)\delta'(x-y) - U_3'(x)\delta(x-y) \end{aligned} \tag{17}$$

The Poisson brackets (17) correspond to the reduction of the second Poisson tensor $P_1$. To obtain these Poisson brackets we use the fact that the fundamental Poisson brackets between the different $V_i$'s are isomorphic to the Lie commutation relations with a central extension , and are given by

$$\{V_a(z), V_b(z')\} = f_{abc}V_c(z)\delta(z-z') - k(T^a, T^b)\delta'(z-z') \tag{18}$$

where

$$S(z) = V_a(z)T^a \tag{19}$$

and $T^a$ denotes the generators of the Lie algebra $sl(3)$ with commutation relations

$$[T^a, T^b] = f_{abc}T^c \tag{20}$$

This fundamental Poisson bracket is, in turn, derived from the basic definition,

$$\{V_a(z), V_b(z)\} = ([dV_a, \partial + S], dV_b) \tag{21}$$

where $S$ is the symplectic leaf containing the different $V_i$'s as its entries.

As a simple exercise, we calculate $\{V_{-1}(x), V_{-2}(y)\}$. We obtain

$$dV_{-1} = \delta V_{-1}(x)/\delta S(z) = e_{12}\delta(x-z)$$

and

$$dV_{-2} = \delta V_{-2}(z)/\delta S(y) = e_{23}\delta(z-y) \tag{22}$$

After using the expression for $S$ given in (11), we get $\{V_{-1}(x), V_{-2}(y)\} = -V_{-3}(x)\delta(x-y)$. Exactly the same result is obtained on using (18). Finally, we calculate one Poisson bracket from the set (17) explicitly. We have

$$\begin{aligned}\{U_0(x), U_0(y)\} =&\{V_{22}(x) - V_2(x)V_1(x), V_2(y) - V_2(y)V_1(y)\}\\ =&\{V_{22}(x), V_{22}(y)\} - \{V_{22}(x), V_2(y)\}V_1(y) -\\ &V_2(y)\{V_{22}(x), V_1(y)\} - V_{-2}(x)\{V_1(x), V_{22}(y) -\\ &\{V_2(x), V_{22}(y)\}V_1(x) + V_2(x)V_1(y)\{V_1(x), V_2(y)\} +\\ &V_1(x)V_1(y)\{V_2(x), V_2(y)\} + V_2(y)V_1(x)\{V_2(x), V_1(y)\} +\\ &V_2(x)V_2(y)\{V_1(x), V_1(y)\}\\ =&\{V_{22}(x), V_{22}(y)\}\end{aligned} \tag{23}$$

after cancelling several terms in pairs using the antisymmetry of the Poisson brackets, whence

$$\begin{aligned}\{U_0(x), U_0(y)\} &= -k\delta'(x-y)\\ &= -\frac{2}{3}\delta'(x-y), \text{ choosing } k = \frac{2}{3}\end{aligned} \tag{24}$$

The above discussion shows how the Poisson brackets (17) are obtained and thus the classical $W_3^2$ algebra is derived. Thus through a rather new choice of the constant matrix $A$ of the first Poisson tensor $P_0$ we have deduced the classical $W_3^2$ algebra. Our choice of the symplectic leaf is further justified by the discussion in ref. (10). For comparison we can mention in short the case of $W_3$ algebra. Here the symplectic leaf is considered to be

$$S = V_{11}(e_{11} - e_{33}) + V_1e_{12} + V_{-1}e21 + V_3e_{13} + V_{-3}e_{31} + V_2e_{23} + V_{-2}e_{32} \tag{25}$$

where $V_1 = V_2 = 1$ and $V_3 = 0$ is the required condition. Further

$$A = e_{31} \tag{26}$$

The covector $V$ is found to be

$$V = \frac{\alpha}{2}(e_{11} - e_{33}) + \beta_1 e_{12} + \beta_2 e_{21} + \delta_1 e_{13} + \delta_2 e_{31} + \gamma_1 e_{23} + \gamma_2 e_{32} \tag{27}$$

Proceeding as before we get two invariants, viz.

$$\begin{aligned} U_1 &= V_{11}^2 + V_{-1} + V_{-2} + 2V_{11x} \\ U_0 &= V_{11}(V_{-1} - V_{-2}) + V_{-3} + V_{11}V_{11x} + V_{11xx} + V_{-1x}, \end{aligned} \tag{28}$$

instead of four, as in the case of $W_3^2$ algebra. The algebra generated by $U_1$ and $U_0$ is found to be the $W_3$ algebra of Zamolodchikov. Finally we may mention again that the difference actually comes from the fact that in case of $W_3$, "A" belongs to the centre of the strictly lower triangular matrices, while in case of $W_3^2$ it is itself a strictly lower triangular matrix.

# 3. Co-adjoint Invariance

After our derivation of $W_3^2$ from the bi-Hamiltonian framework we can compare our results with those obtained in the gauge transformation frame-work. This method actually generates the $W$-algebra via the co-adjoint action invariance of certain functionals. Such an approach was used in ref. (8) to deduce the Lie-Poisson structure on the dual of the Virasoro algebra, the underlying algebra being the $sl(3,c)$ Kac-Moody algebra on $S^1$. We now briefly comment on the results in case of $sl(3,c)$ leading to $W_3^2$. It is now well-known that if $G$ is an affine Lie group and $g$ its Lie algebra then the dual space $g^*$ of $g$ is defined as the space of linear functionals of $g$. The coadjoint action is given by the formulae,

$$\mathrm{ad}^*_{(Y,\mu)}(v,k) = ([Y,v] + kY, 0) \tag{29}$$

$$\mathrm{Ad}^*_{(\phi,\mu)}(v,k) = (\phi v \phi^{-1} + k\phi'\phi^{-1}, k) \tag{30}$$

where $(v(x), k)$ belongs to the dual space. In the case of $sl(3,c)$ algebra, the phase space points are specified as ,

$$v(x) = V_{11}e_{11} + V_{22}e_{22} + V_{33}e_{33} + V_1 e_{12} + V_{-1}e_{21} + V_3 e_{13} + V_{-3}e_{31} + V_2 e_{23} + V_{-2}e_{32} \tag{31}$$

We put the constraint $V_3 = 1$. The maximal co-adjoint action which does not change this constraint is given by (30) with $\phi$ given as

$$\phi = e_{11} + e_{22} + e_{33} + Ae_{21} + Be_{31} + Ce_{32}, \text{ that is, } Ad^*_{(\phi,\mu)}(v,k) = (\bar{v}, k). \tag{32}$$

Simple algebra gives

$$A = \bar{V}_2 - V_2;\ B = V_{11} - \bar{V}_{11} - \bar{V}_1(\bar{V}_2 - V_2);\ C = V_1 - \bar{V}_1$$

and we also obtain that

$$
\begin{aligned}
&V_{22} - V_2 V_1 = \bar{V}_{22} - \bar{V}_2 \bar{V}_1 \\
&V_2(V_{22} - V_{11}) - V_2^2 V_1 + V_{-1} - V_{2x} = \bar{V}_2(\bar{V}_{22} - \bar{V}_{11}) - \bar{V}_2^2 \bar{V}_1 + \bar{V}_{-1} - \bar{V}_{2x}
\end{aligned}
\tag{33}
$$

and so on. The upshot is that we get back the four quantities $U_0$ $U_1$,$U_2$,and $U_3$ as the invariants of the co-adjoint action whereas the bi-Hamiltonian approach suggests that they are invariants of the flow. This can be seen to be related to the fact that we actually construct the dynamics via the co-adjoint action.

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