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Stimulated emission of particles by 1+1 dimensional black holes[1]

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Abstract. The stimulated emission of massless bosons by a relativistic and the CGHS black hole are studied for real and complex scalar fields. The radiations induced by one-particle and thermal states are considered and their thermal properties investigated near the horizon. These exhibit both thermal and *non-thermal* properties for the two black-hole models.

1 Introduction

The quantum theory of fields in curved space-times is the study of the propagation of quantum fields in classical gravitational fields [1]. It is a theory of quantum relativity, in the sense that fields and quantum states look different in distinct non-inertial frames [2]. The relativity of the particle concept in curved space-time implies that particles may be created spontaneously (i.e. from the vacuum) for geometrical reasons. This phenomenon was first analyzed by Parker in cosmology [3] and by Hawking in the gravitational fields of 1+3 dimensional black holes [4]. It has been shown in particular that the spontaneous emission of particles is thermal at late times near the event horizon of black holes [5]. This discovery subsequently led to the understanding of a profound connection between gravity and thermodynamics [6].

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Wald [7] was the first to consider the stimulated emission of particles by 1+3 dimensional black holes. In the presence of an incoming radiation the emitted radiation must be perturbed somehow or other, so one may wonder if the emission of particles is still thermal near the event horizon of black holes for a non-vacuum incoming state. In the case of the emission of bosons stimulated by one-particle states, Wald showed that the mean number of particles is indeed still thermal *asymptotically* close to the event horizon, and that the energy of the incoming state must be very large to modify the emission of particles close to the event horizon. In this case, he also observed that the mean number of particles emitted is equal to or greater than that for the spontaneous emission. This phenomenon was referred to as *amplification* of the emitted radiation by Audresch and Müller [8], who confirmed the results obtained by Wald on the basis of a more detailed analysis. In the case of fermions, the emitted radiation is generally *attenuated* by an incoming state [9].

In this paper I consider the creation of bosons by two 1+1 dimensional black holes: the relativistic and CGHS black holes. These were introduced in refs [10] and [11] respectively, where some of their semi-classical properties were studied. In particular, for the CGHS black-hole model, it was shown that the spontaneous emission of particles is thermal at late times near the event horizon. For the relativistic black-hole model, this emission was shown to be thermal everywhere immediately after the formation of the black hole.

This paper is specifically concerned with the study of the emission of massless bosons stimulated by one-particle and thermal states. Both real and complex scalar fields are considered. I show that the emitted radiation induced by non-vacuum incoming states exhibits both thermal and *non-thermal* properties near the horizon. The most remarkable results are the following:

1) for both black-hole models and for some one-particle incoming states, the two-point function may be non-thermal near the horizon;

2) for both black-hole models and for one-particle and thermal incoming states, the energy-momentum tensor of the emitted radiation is always thermal close to the horizon;

3) for the relativistic black-hole model, the mean number of particles stimulated by an incoming thermal state is *not thermal for any test function.*

The cited results of Wald are also confirmed for the two 1+1 dimensional black-hole models, but I do not use S matrix formalism, following Gallay and Wanders [12], since the problem is not implementable in these cases [10, 13].

The second section of this paper is devoted to a review of the relativistic and CGHS black-hole models. For later use, the results obtained in ref. [10] on the dynamics of the scalar field in curved space-times are summarized in the third section and those for the spontaneous emission of bosons in the fourth. The emission of bosons stimulated by one-particle or thermal states is considered in the fifth section. The mean values of the two-point function, energy-momentum tensor, number of created particles and current are computed in these states and compared with their corresponding thermal mean values. In the last section the thermal properties of the stimulated radiation are discussed for both black-hole models.

2 Black-hole models in 1+1 dimensions

The following equation defines a relativistic classical theory of gravity in 1+1 dimensional space-times [14]:

$$R(x) \;=\; 8\pi G\, T(x), \tag{2.1}$$

where $T(x) = T^{\mu}{}_{\mu}(x)$ is the trace of the energy-momentum tensor and G is Newton's constant. The relativistic black-hole model [10] is defined assuming that $T(x)$ is given by

$$T(x) \;=\; \frac{M}{8\pi G}\, \delta(x^+ - x_o^+), \tag{2.2}$$

where $x^{\pm} = (x^0 \pm x^1)\,/\sqrt{2}$ and the constant M is strictly positive. Equation (2.2) describes a pulse of classical matter. A solution of eqs (2.1) and (2.2) is given by

$$ds^2 \;=\; \begin{cases} dx^+\, dx^-, & \text{if } \; x^+ < x_o^+, \\ \dfrac{dx^+\, dx^-}{M\,(\Delta - x^-)}, & \text{if } \; x^+ > x_o^+, \end{cases} \tag{2.3}$$

where Δ is an arbitrary constant reflecting the invariance of the trace (2.2) under translations of x^-. The solution (2.3) is not continuous at $x^+ = x_o^+$. Another set of conformal coordinates $(y^+, y^-) \in \mathbb{R}^2$ is defined by the transformation

$$\begin{cases} x^+(y^+) \;=\; y^+, \\ x^-(y^-) \;=\; \Delta - e^{-My^-}, \end{cases} \tag{2.4}$$

and in these new coordinates the metric (2.3) is given by

$$ds^2 = \begin{cases} M\, e^{-My^-}\, dy^+ dy^-, & \text{if } \; y^+ < y_o^+, \\ dy^+ dy^-, & \text{if } \; y^+ > y_o^+, \end{cases} \tag{2.5}$$

where $y_o^+ = x_o^+$.

The CGHS black-hole model [11] is based on the the action

$$S \;=\; \frac{1}{2\pi} \int d^2x\, \sqrt{-g}\, \left\{ e^{-2\phi} \left[R + 4\,(\nabla\phi)^2 + 4\,\lambda^2 \right] - \frac{1}{2}\,(\nabla f)^2 \right\}, \tag{2.6}$$

where g is the metric, ϕ the dilatonic field, λ^2 the cosmological constant and f a classical matter field. This action is related to that for non-critical strings with $c = 1$ and defines the

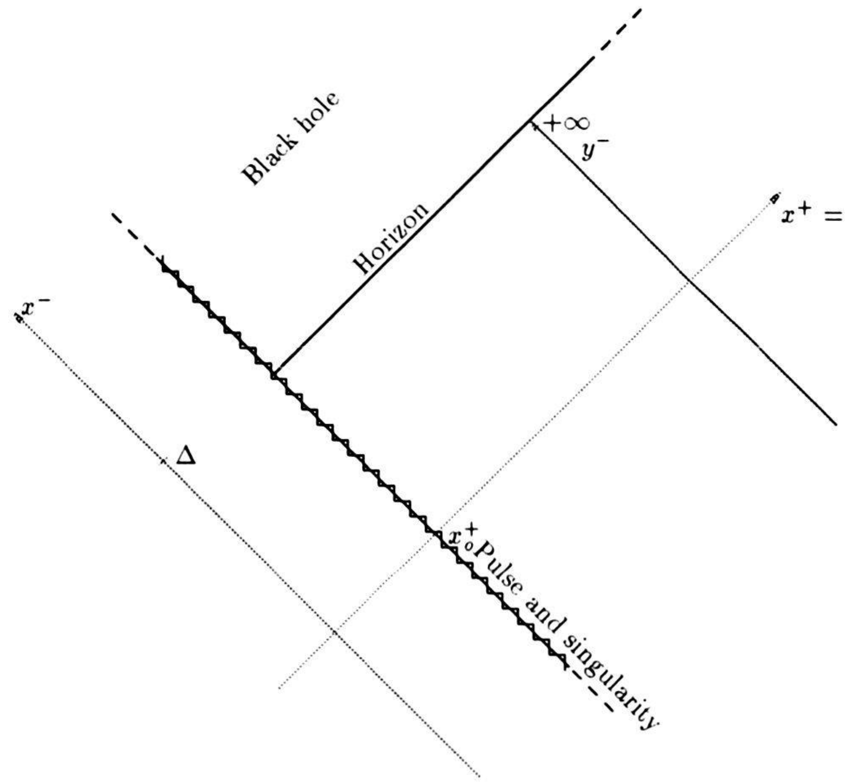

Figure 1: Space-time structure of the relativistic black-hole model.

dilatonic gravity. If $T^f_{\mu\nu}(x)$ is the energy-momentum tensor of the matter field f, the CGHS black-hole model is defined by assuming that

$$\begin{aligned} T^f_{++}(x) &= \tfrac{1}{2}\,(\partial_+ f)^2 &&= M\,\delta(x^+ - x_o^+), \\ T^f_{--}(x) &= \tfrac{1}{2}\,(\partial_- f)^2 &&= 0, \\ T^f_{+-}(x) &= \tfrac{1}{2}\,\partial_+ f\,\partial_- f &&= 0, \end{aligned} \tag{2.7}$$

where $M > 0$. A continuous solution of the field equations, deduced from the action (2.6) under the constraints (2.7), is given by

$$ds^2 = \begin{cases} dx^+ dx^-, & \text{if } x^+ < x_o^+, \\ \dfrac{dx^+ dx^-}{1 + (M/\lambda)\,e^{\lambda x^-}\,(\,e^{-\lambda x^+} - e^{-\lambda x_o^+}\,)}, & \text{if } x^+ > x_o^+. \end{cases} \tag{2.8}$$

From this result the scalar curvature is computed:

$$R(x) = \begin{cases} 0, & \text{if } x^+ < x_o^+, \\ \dfrac{4\lambda^2}{1 - e^{\lambda(x^+ - x_o^+)} + (\lambda/M)\,e^{\lambda(x^+ - x^-)}}, & \text{if } x^+ > x_o^+. \end{cases} \tag{2.9}$$

The curvature (2.9) is not continuous at $x^+ = x_o^+$ and it is singular on the curve given by

$$x_S^-(x^+) = x_H^- - \frac{1}{\lambda}\log\left[1 - e^{-\lambda(x^+ - x_o^+)}\right], \qquad x^+ \in \mathbb{R}, \tag{2.10}$$

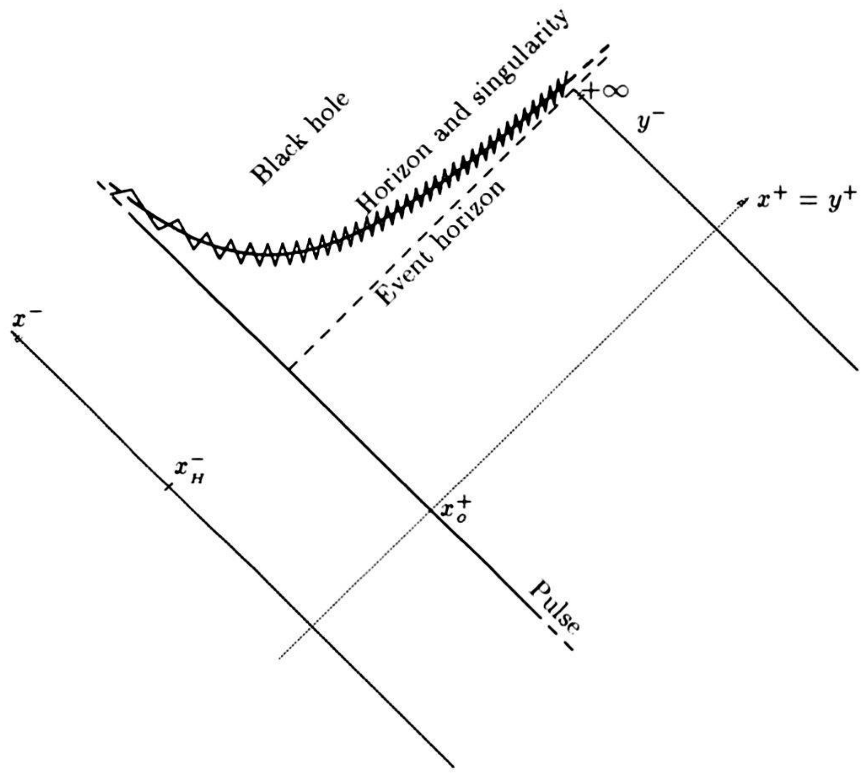

Figure 2: Space-time structure of the CGHS black-hole model.

where we have defined

$$x_H^- \;=\; x_o^+ + \frac{1}{\lambda}\log\frac{\lambda}{M}. \tag{2.11}$$

Another set of conformal coordinates $(y^+, y^-) \in \mathbb{R}^2$ is defined by the transformation

$$\begin{cases} x^+(y^+) &= \; y^+, \\ x^-(y^-) &= \; x_H^- - \frac{1}{\lambda}\log\left[1 + \frac{\lambda}{M}e^{-\lambda(y^- - y_o^+)}\right], \end{cases} \tag{2.12}$$

where $y_o^+ = x_o^+$. In these new coordinates the metric (2.8) is given by

$$ds^2 = \begin{cases} \dfrac{dy^+dy^-}{1 + (M/\lambda)\; e^{\lambda(y^- - y_o^+)}}, & \text{if } y^+ < y_o^+, \\ \dfrac{dy^+dy^-}{1 + (M/\lambda)\; e^{\lambda(y^- - y^+)}}, & \text{if } y^+ > y_o^+. \end{cases} \tag{2.13}$$

For the two space-time models described, the coordinates x and y are Minkowskian if $x^+ < x_o^+$ and $y^+ \to +\infty$ respectively. In consequence, these will be called *incoming* and *outgoing* coordinates. In 1+1 dimensional space-times, I define the *horizon* as the curve on which the conformal factor in the incoming coordinates changes its sign, and the *interior* of a black hole as the region of space-time for which this conformal factor is negative. The interior of a black hole is inaccessible to an exterior observer and is bounded by the horizon as it should be. The space-time structures of the described models are shown in figures 1 and 2. They are clearly associated with black holes. The conformal factor is infinite on the horizon for both models and, for the CGHS black-hole model, the horizon also coincides with the singularity of the curvature, and so is given by eq. (2.10). The outgoing coordinates only describe the exterior of the two black holes.

3 Scalar fields in 1+1 dimensional curved space-times

In ref. [10] the dynamics of the massless scalar field was studied in the 1+1 dimensional space-times whose incoming and outgoing coordinates are related by

$$x^{\pm} = x^{\pm}(y^{\pm}), \qquad y^{\pm} \in \mathbb{R}, \tag{3.1}$$

as is the case for the relativistic and CGHS black-hole models. It was shown that the right and left fields are dynamically independent in such space-times, and that it is sufficient to consider only the *right* moving fields if $x^+(y^+) = y^+$. The 1+1 dimensional quantum problem is thus reduced to a one-dimensional quantum problem, so from now on the suffix $\pm$ will be dropped.

The incoming and outgoing right fields, denoted by $\phi(x)$ and $\hat{\phi}(y)$, were defined in ref. [10] from the solutions of the massless Klein-Gordon equations in the incoming and outgoing coordinates respectively. They were considered as kernels of distributions. The set $\mathcal{S}_0(\mathbb{R})$, on which they act, is the set of the Schwartz functions whose Fourier transforms vanish at null momentum,

$$\mathcal{S}_0(\mathbb{R}) = \{ f \in \mathcal{S}(\mathbb{R}) \mid \tilde{f}(0) = 0 \}, \tag{3.2}$$

so the Wightman distribution is positive definite [15]. The incoming and outgoing field distributions are given by

$$\phi[h] = \int_{-\infty}^{+\infty} dx\ h(x)\ \phi(x), \qquad \hat{\phi}[f] = \int_{-\infty}^{+\infty} dy\ f(y)\ \hat{\phi}(y), \tag{3.3}$$

if $h, f \in \mathcal{S}_0(\mathbb{R})$.

The *incoming test function* $\hat{f}(x)$ was defined in ref. [10] in terms of the *outgoing test function* $f(y)$ through the scalar properties of the field distributions:

$$\phi[\hat{f}] = \hat{\phi}[f]. \tag{3.4}$$

The transformation of the test functions describes the propagation of the quantum fields. From eq. (3.4), these are related by the operator U, as $\tilde{\hat{f}} = U\tilde{f}$, whose kernel is given by

$$U(k,p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dy\ e^{-ikx(y)}\ e^{ipy}, \tag{3.5}$$

where k and p are the *incoming* and *outgoing momentum* respectively. For the relativistic and black-hole models, this is given by [10, 11]

$$U_R(k,p) = \frac{e^{-\mathrm{i}k\Delta}\, e^{-\mathrm{i}\Omega(\frac{p}{M})}\, e^{\mathrm{i}\frac{p}{M}\log|k|}}{\sqrt{2\pi}\, M} \left[\frac{\theta(k)}{\sqrt{p\,(1-e^{-\frac{2\pi}{M}p})}} + \frac{\theta(-k)}{\sqrt{p\,(e^{\frac{2\pi}{M}p}-1)}} \right], \tag{3.6}$$

$$U_{CGHS}(k,p) = \frac{e^{\mathrm{i}\,x_o(p-k)}}{2\pi\lambda} \left(\frac{\lambda}{M}\right)^{\mathrm{i}\,\frac{(p-k)}{\lambda}} B\left(\mathrm{i}\,\frac{(p-k)}{\lambda}+0^+, -\mathrm{i}\,\frac{p}{\lambda}+0^+\right), \tag{3.7}$$

where $\Omega(p) = \arg\left[\Gamma(ip)\right]$ and B is the beta function.

The *incoming* and *outgoing Hilbert spaces*, $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$, were defined using the incoming and outgoing fields. The *incoming* and *outgoing wave function spaces* are given by $L^2(\frac{dk}{2k}, \mathbb{R}_+)$ and $L^2(\frac{dp}{2p}, \mathbb{R}_+)$ respectively.

In ref. [10] the mean values of several observables, constructed in the outgoing coordinates, were also computed in the incoming vacuum. These describe the properties of the outgoing radiation in the exterior of the black holes. The two-point function, energy-momentum tensor, number of created particles and current were considered. I repeat here their definitions and the results obtained, since they will be useful below. They are valid for both real and complex scalar fields.

- Incoming and outgoing two-point functions:

$$W_o(x,x') = (\Omega_o, \phi(x)\,\phi(x')^\dagger\,\Omega_o) = -\frac{1}{4\pi}\,\log\left[x'-x+\mathrm{i}\,0^+\right], \tag{3.8}$$

$$\widehat{W}_o(y,y') = (\Omega_o, \hat{\phi}(y)\,\hat{\phi}(y')^\dagger\,\Omega_o) = -\frac{1}{4\pi}\,\log\left[x(y')-x(y)+\mathrm{i}\,0^+\right]. \tag{3.9}$$

Equations involving two-point functions are always and only valid between kernels of distributions on $\mathcal{S}_0(\mathbb{R}) \times \mathcal{S}_0(\mathbb{R})$. In particular,

$$(\Omega_o, \phi[g_1]\,\phi[g_2]^\dagger\,\Omega_o) = \int_0^\infty \frac{dk}{2k}\,\tilde{g}_2(k)^*\,\tilde{g}_1(k), \tag{3.10}$$

where $\tilde{g}_1, \tilde{g}_2 \in \mathcal{S}_0(\mathbb{R})$.

- Energy-momentum tensor:

$$\begin{aligned}\widehat{T}_o(y) &= (\Omega_o, :\widehat{\Theta}\,(y):_{out}\,\Omega_o) \\ &= \lim_{\varepsilon\to 0}\,(\Omega_o, \left[\widehat{\Theta}_\varepsilon(y) - \Theta_\varepsilon(x(y))\right]\Omega_o) = -\frac{1}{24\pi}\,\mathrm{S}_y[x(y)],\end{aligned} \tag{3.11}$$

where S_y is the Schwartz derivative and

$$\Theta(x) = \partial_x\phi(x)^\dagger\,\partial_x\phi(x), \tag{3.12}$$

$$\widehat{\Theta}(y) = \partial_y\hat{\phi}(y)^\dagger\,\partial_y\hat{\phi}(y), \tag{3.13}$$

$$\Theta_\varepsilon(x) = \frac{1}{2}\left[\partial_x\phi(x)^\dagger\,\partial_x\phi(x+\varepsilon) + \partial_x\phi(x+\varepsilon)^\dagger\,\partial_x\phi(x)\right], \tag{3.14}$$

$$\widehat{\Theta}_\varepsilon(y) = \frac{1}{2}\left[\partial_y\hat{\phi}(y)^\dagger\,\partial_y\hat{\phi}(y+\varepsilon) + \partial_y\hat{\phi}(y+\varepsilon)^\dagger\,\partial_y\hat{\phi}(y)\right]. \tag{3.15}$$

• Mean number of spontaneously created particles:

$$\bar{N}_o[f] \;=\; (\Omega_o, \hat{\phi}[f]^\dagger \hat{\phi}[f]\,\Omega_o) \;=\; \int_0^\infty \frac{dk}{2k}\left|(U\tilde{f})(-k)\right|^2, \tag{3.16}$$

if $\tilde{f} \in \mathcal{S}(\mathbb{R}_+)$ is a normalized test function, where $\mathcal{S}(\mathbb{R}_+)$ is set of Schwartz functions whose Fourier transforms vanish for negative momentum.

• Mean current:

$$\begin{aligned}\hat{J}_o(y) &= (\Omega_o, :\hat{\Upsilon}(y):_{out}\ \Omega_o)\\ &= \lim_{\varepsilon\to 0}\,(\Omega_o, \left[\hat{\Upsilon}_\varepsilon(y) - \Upsilon_\varepsilon(x(y))\right]\Omega_o) \;=\; 0,\end{aligned} \tag{3.17}$$

where

$$\Upsilon(x) \;=\; \mathrm{i}\;\phi(x)^\dagger\;\overleftrightarrow{\partial}_x\;\phi(x), \tag{3.18}$$

$$\hat{\Upsilon}(y) \;=\; \mathrm{i}\;\hat{\phi}(y)^\dagger\;\overleftrightarrow{\partial}_y\;\hat{\phi}(y), \tag{3.19}$$

$$\Upsilon_\varepsilon(x) \;=\; \mathrm{i}\left[\;\phi(x+\varepsilon)^\dagger\,\partial_x\phi(x) - \partial_x\phi(x)^\dagger\,.\phi(x+\varepsilon)\;\right], \tag{3.20}$$

$$\hat{\Upsilon}_\varepsilon(y) \;=\; \mathrm{i}\left[\;\hat{\phi}(y+\varepsilon)^\dagger\,\partial_y\hat{\phi}(y) - \partial_y\hat{\phi}(y)^\dagger\,.\hat{\phi}(y+\varepsilon)\;\right]. \tag{3.21}$$

The *thermal mean value* of an observable A in the outgoing Hilbert space $\mathcal{H}_{out}$ was defined in ref. [10] by[3]

$$\langle\, A\,\rangle^{Th}_{\beta,out} \;=\; \frac{\underset{out}{\mathrm{Tr}}\left[e^{-\beta H_{out}}\,A\right]}{\underset{out}{\mathrm{Tr}}\left[e^{-\beta H_{out}}\right]}, \tag{3.22}$$

if the representation of A in $\mathcal{H}_{out}$ is known. For the above observables

$$W^{Th}_{\beta,out}(y,y') \;=\; -\frac{1}{4\pi}\log\left\{\frac{\beta}{\pi}\,\sinh\left[\frac{\pi}{\beta}\left(y'-y+\mathrm{i}\,0^+\right)\right]\right\}, \tag{3.23}$$

$$T^{Th}_{\beta,out}(y) \;=\; T^{Th}_{\beta,out} \;=\; \frac{\pi}{12\beta^2}, \qquad \forall\, y \in \mathbb{R}, \tag{3.24}$$

$$\bar{N}^{Th}_{\beta,out}[f] \;=\; \int_0^\infty \frac{dp}{2p}\frac{|\,\tilde{f}\,(p)\,|^2}{e^{\beta p}-1}, \tag{3.25}$$

$$J^{Th}_{\beta,out}(y) \;=\; 0, \qquad \forall\, y \in \mathbb{R}. \tag{3.26}$$

It has been assumed in eq. (3.25) that $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is normalized and that $f(y)$ exists a.e. and is integrable.

[3]Note that it has to be defined as a limit.

4 Spontaneous creation of particles

Using the results of the preceding section with the relativistic black-hole model [10] and transformation (2.4) gives

$$\widehat{W}_o(y,y') = W^{Th}_{\frac{2\pi}{M},out}(y,y'), \qquad \forall y,y' \in \mathbb{R}, \tag{4.1}$$

$$\widehat{T}_o(y) = T^{Th}_{\frac{2\pi}{M},out}, \qquad \forall y \in \mathbb{R}, \tag{4.2}$$

$$\bar{N}_o[f] = \bar{N}^{Th}_{\frac{2\pi}{M},out}[f], \tag{4.3}$$

$$\widehat{J}_o(y) = 0, \qquad \forall y \in \mathbb{R}. \tag{4.4}$$

It has been assumed in eq. (4.3) that $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is normalized and that $f(y)$ exists a.e. and is integrable. From these results we conclude that the emission of particles is thermal everywhere after the formation of the black hole and that the associated temperature is given by $T = \frac{M}{2\pi}$.

Similarly, for the CGHS black-hole model [13], and transformation (2.12),

$$\widehat{W}_o(y,y') \approx W^{Th}_{\frac{2\pi}{\lambda},out}(y,y'), \qquad \text{if } y,y' \gg 1, \tag{4.5}$$

$$\widehat{W}_o(y,y') \approx W^{Th}_{\infty,out}(y,y'), \qquad \text{if } -y,\ -y' \gg 1, \tag{4.6}$$

$$\lim_{y\to+\infty} \widehat{T}_o(y) = T^{Th}_{\frac{2\pi}{\lambda},out}, \tag{4.7}$$

$$\lim_{y\to-\infty} \widehat{T}_o(y) = T^{Th}_{\infty,out}, \tag{4.8}$$

$$\widehat{J}_o(y) = 0, \qquad \forall y \in \mathbb{R}. \tag{4.9}$$

These results suggest that the emission of particles is thermal near the horizon at late times and that the associated temperature is given by $T = \frac{\lambda}{2\pi}$. However the situation turns out to be more complex after considering the mean number $\bar{N}_o[f]$ of spontaneously created particles.

Asymptotically close to the horizon, the behavior of $\bar{N}_o[f]$ is analyzed in the following way. The translation of the test function f towards the horizon [16] is defined as

$$f_{y_o}(y) = f(y - y_o), \tag{4.10}$$

and we examine whether

$$\lim_{y_o\to+\infty} \left(\bar{N}_o[f_{y_o}] - \bar{N}^{Th}_{\frac{2\pi}{\lambda},out}[f_{y_o}] \right) \stackrel{?}{=} 0. \tag{4.11}$$

Theorem 1 *Let $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ be a normalized wave function such that $f(y)$ exists a.e. and is integrable. If there exist three constants $\underline{\alpha > 1/2}$, $C > 0$ and $L \geq 1$ such that*

$$|f(y)| \leq \frac{C}{|y|^{1+\alpha}}, \qquad \textit{if } y \leq -L, \tag{4.12}$$

then for the CGHS model and for all $y_o > 0$,

$$\left| \bar{N}_o[f_{y_o}] - \bar{N}^{Th}_{\frac{2\pi}{\lambda},out}[f_{y_o}] \right| \;\leq\; \frac{32\, C^2}{\alpha^2(2\alpha - 1)\left(\frac{1}{4}\, y_o + L - 1\right)^{2\alpha - 1}} + e^{2L - y_o/2}\left(\| f \|_{L^1} + \| f' \|_{L^1} \right)^2 ; \tag{4.13}$$

however if

$$f(y) \;\approx\; \frac{C}{(-y)^{1+\alpha}}, \qquad \textit{if } y \ll -1, \tag{4.14}$$

where $\underline{0 < \alpha \leq 1/2}$ *and* $C \in \mathbb{C}$, *then*

$$\bar{N}_o[f_{y_o}] \;=\; \bar{N}^{Th}_{\frac{2\pi}{\lambda},out}[f_{y_o}] \;=\; \infty, \qquad \forall y_o \in \mathbb{R}, \tag{4.15}$$

$$\left| \bar{N}_o[f_{y_o}] - \bar{N}^{Th}_{\frac{2\pi}{\lambda},out}[f_{y_o}] \right| \;=\; \infty, \qquad \forall y_o \in \mathbb{R}. \tag{4.16}$$

This theorem is proved in ref. [13]. Under some conditions, the bound (4.13) shows that eq. (4.11) is true if the modulus of f decreases sufficiently fast *very far from the horizon.* This bound is composed of two terms: the first decreases algebraically in y_o with exponent $2\alpha - 1$, and the second decreases exponentially in y_o. However, if the function f decreases relatively weakly *very far from the horizon* and does not oscillate, eq. (4.16) shows that eq. (4.11) is not true. This result implies that *the behavior of some non-local observables may be non-thermal near the horizon.*

There exists a bound for $\bar{N}_o[f]$ in terms of the Fourier transform $\tilde{f}$.

Theorem 2 *If* $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ *is a normalized wave function such that* $f(y)$ *exists a.e. and is integrable, then for the CGHS model*

$$\bar{N}_o[f] \;\leq\; C \int_0^\infty \frac{dp}{2p} \frac{|\tilde{f}(p)|^2}{1 - e^{-\frac{2\pi}{\lambda}p}}, \tag{4.17}$$

where $C > 0$ *is a constant.*

This theorem is also proved in ref. [13]. The bound (4.17) may be infinite depending on the infrared behavior of $\tilde{f}$.

5 Emission stimulated by a one-particle state

For simplicity I will assume from now on that $M = \lambda = 1$ and $\Delta = x_o^+ = 0$. In this case, the transformation between the (right) incoming and outgoing coordinates is given for the relativistic black-hole model by

$$x(y) \;=\; -e^{-y}, \qquad \forall y \in \mathbb{R}, \tag{5.1}$$

and for the CGHS model by

$$x(y) = -\log\left(1 + e^{-y}\right), \qquad \forall\, y \in \mathbb{R}, \tag{5.2}$$

see eqs (2.4) and (2.12). Both transformations apply $\mathbb{R}$ on $\mathbb{R}_-$ and the first is the asymptotical form of the second for $y \gg 1$.

In this section we assume that the incoming state is a one-particle state $\Omega_g \in \mathcal{H}_{in}$. For the real scalar field this is defined as

$$\Omega_g = \phi[g]^\dagger\, \Omega_o, \tag{5.3}$$

where $\tilde{g} \in \mathcal{S}(\mathbb{R}_+)$ and $\Omega_o \in \mathcal{H}_{in}$ is the incoming vacuum. For the complex scalar field, the one-antiparticle state is also defined as

$$\bar{\Omega}_g = \Omega_{g^*} = \phi[g^*]^\dagger\, \Omega_o, \tag{5.4}$$

where $\tilde{g} \in \mathcal{S}(\mathbb{R}_+)$. The first subsections of this section are devoted to the real scalar field and the last one to the complex scalar field.

5.1 The two-point function

For the incoming state Ω_g, the incoming and outgoing two-point functions denoted by $W_g(x, x')$ and $\widehat{W}_g(y, y')$ respectively, are defined as

$$W_g(x, x') = (\Omega_g, \phi(x)\, \phi(x')^\dagger\, \Omega_g), \tag{5.5}$$

$$\widehat{W}_g(y, y') = (\Omega_g, \hat{\phi}(y)\, \hat{\phi}(y')^\dagger\, \Omega_g), \tag{5.6}$$

and are related by

$$\widehat{W}_g(y, y') = W_g(x(y), x(y')), \tag{5.7}$$

since $\hat{\phi}(y) = \phi(x(y))$ for all $y \in \mathbb{R}$ [10]. These functions may be expressed in terms of the primitive G of g defined as

$$G(x) = \int_{-\infty}^{x} dx'\, g(x'). \tag{5.8}$$

Theorem 3 *Between kernels of distributions on $\mathcal{S}_0(\mathbb{R}) \times \mathcal{S}_0(\mathbb{R})$,*

$$W_g(x, x') = W_o(x, x') + \frac{1}{2}\,\mathrm{Re}\,[\, G(x)\, G(x')^*\,], \tag{5.9}$$

$$\widehat{W}_g(y, y') = \widehat{W}_o(y, y') + \frac{1}{2}\,\mathrm{Re}\,[\, G(x(y))\, G(x(y'))^*\,], \tag{5.10}$$

where $W_o(x, x')$ and $\widehat{W}_o(y, y')$ are the two-point functions (3.8) and (3.9).

Proof. The two-point function (5.5) is computed using Wick's theorem and

$$(\Omega_o, \phi[g]\, \phi(x)\, \Omega_o) \;=\; \frac{\mathrm{i}}{2}\, G(x), \tag{5.11}$$

from which eq. (5.9) is obtained. Equation (5.10) follows from the property (5.7). □

This last theorem applies to both black-hole models.

Theorem 4 *Let $\tilde{g} \in \mathcal{S}(\mathbb{R}_+)$ be an incoming normalized test function. For the relativistic and CGHS models, we have the following asymptotic relation between kernels of distributions on $\mathcal{S}_0(\mathbb{R}) \times \mathcal{S}_0(\mathbb{R})$,*

$$G(0) \;=\; 0 \;\Longleftrightarrow\; \widehat{W}_g(y,y') \;\approx\; W^{Th}_{2\pi,out}(y,y') \qquad \text{if } \; y,\, y' \gg 1; \tag{5.12}$$

furthermore, for the relativistic model,

$$\widehat{W}_g(y,y') \;\approx\; W^{Th}_{2\pi,out}(y,y'), \qquad \text{if } \; -y,\, -y' \gg 1, \tag{5.13}$$

and for the CGHS model,

$$\widehat{W}_g(y,y') \;\approx\; W^{Th}_{\infty,out}(y,y'), \qquad \text{if } \; -y,\, -y' \gg 1, \tag{5.14}$$

where $W^{Th}_{\beta,out}(y,y')$ is given by eq. (3.23).

Proof. For both models, we have $x(y) \approx 0$ if $y \gg 1$. Equation (5.10) implies that

$$\widehat{W}_g(y,y') \;\approx\; \widehat{W}_o(y,y') + \frac{1}{2}\,|\,G(0)\,|^2, \qquad \text{if } \; y,\, y' \gg 1, \tag{5.15}$$

since G is continuous. Equation (5.12) is then deduced from the results (4.1) and (4.5). Equation (5.10) also implies that

$$\widehat{W}_g(y,y') \;\approx\; \widehat{W}_o(y,y'), \qquad \text{if } \; -y,\, -y' \gg 1, \tag{5.16}$$

since G vanishes at infinity. Equations (5.13) and (5.14) are deduced from the results (4.1) and (4.6). □

For both black-hole models, this theorem states that, in the outgoing coordinates, the incoming state Ω_g is thermal [10] close to the horizon *if and only if* $G(0) = 0$, and in this case the temperature is given by $(2\pi)^{-1}$. Very far from the horizon, the incoming state Ω_g is also thermal and the associated temperature is given by $(2\pi)^{-1}$ for the relativistic model and vanishes for the CGHS model.

5.2 The energy-momentum tensor

The energy-momentum observables in the incoming and outgoing coordinates are given respectively by eqs (3.12) and (3.13). Their regularized mean values in the incoming state Ω_g

will be denoted by $T_g(x)$ and $\widehat{T}_g(y)$ respectively. In the normal and subtraction regularization schemes, these are given by

$$\begin{aligned}
T_g(x) &= (\Omega_g, : \Theta(x) :_{in} \Omega_g), & (5.17)\\
T_g(x) &= \lim_{\varepsilon\to 0} \left[(\Omega_g, \Theta_\varepsilon(x)\,\Omega_g) - (\Omega_o, \Theta_\varepsilon(x)\,\Omega_o) \right], & (5.18)\\
\widehat{T}_g(y) &= (\Omega_g, : \widehat{\Theta}(y) :_{out} \Omega_g), & (5.19)\\
\widehat{T}_g(y) &= \lim_{\varepsilon\to 0} \left[(\Omega_g, \widehat{\Theta}_\varepsilon(y)\,\Omega_g) - (\Omega_o, \Theta_\varepsilon(x(y))\,\Omega_o) \right], & (5.20)
\end{aligned}$$

where $\Theta_\varepsilon(x)$ and $\widehat{\Theta}_\varepsilon(y)$ are defined by eqs (3.14) and (3.15) respectively.

Theorem 5 *If $\tilde{g} \in \mathcal{S}(\mathbb{R}_+)$ is a normalized test function, and if $T_g(x)$ and $\widehat{T}_g(y)$ are defined respectively by eqs (5.17) and (5.19), or by eqs (5.18) and (5.20), then*

$$\begin{aligned}
T_g(x) &= \frac{1}{2}\, |g(x)|^2, & (5.21)\\
\widehat{T}_g(y) &= \widehat{T}_o(y) + x'(y)^2\, T_g(x(y)), & (5.22)
\end{aligned}$$

where $\widehat{T}_o(y)$ is given by eq. (3.11).

This theorem is proved is appendix A.1. It applies to both black-hole models.

Theorem 6 *If $\tilde{g} \in \mathcal{S}(\mathbb{R}_+)$ is a normalized incoming test function, then for both models*

$$\lim_{y\to+\infty} \widehat{T}_g(y) = T^{Th}_{2\pi,out}\,; \qquad (5.23)$$

furthermore for the CGHS model

$$\lim_{y\to-\infty} \widehat{T}_g(y) = T^{Th}_{\infty,out}\,, \qquad (5.24)$$

where $T^{Th}_{\beta,out}$ is given by eq. (3.24).

Proof. For both models, we have $x'(y) \approx e^{-y}$ if $y \gg 1$. Equation (5.22) implies that

$$\widehat{T}_g(y) - \widehat{T}_o(y) \approx \frac{1}{2} e^{-2y}\, |g(0)|^2, \qquad \text{if } y \gg 1, \qquad (5.25)$$

from which eq. (5.23) is deduced using the results (4.2) and (4.7). For the CGHS model, we have $x'(y) \approx 1$ if $-y \gg 1$. Since g vanishes at infinity, eq. (5.22) also implies that

$$\widehat{T}_g(y) - \widehat{T}_o(y) \approx \tfrac{1}{2}\, |g(y)|^2 \approx 0, \qquad \text{if } -y \gg 1, \qquad (5.26)$$

from which eq. (5.24) is deduced using the result (4.8). □

For both black-hole models, the behavior of the energy-momentum tensor is thus still thermal close to the horizon for the incoming one-particle state Ω_g, *even if the test function g is well localized in the vicinity of the horizon.* Furthermore, in the case of the CGHS model, an incoming one-particle state does not induce any radiation very far from the horizon.

In a given set of coordinates, the total energy is given by the integral of the energy-momentum tensor. The incoming and outgoing energies, E^{in} and E^{out}, are defined as

$$E^{in} = \int_{-\infty}^{+\infty} dx\ T(x), \qquad E^{out} = \int_{-\infty}^{+\infty} dy\ \hat{T}(y). \tag{5.27}$$

The energy E_g^{out} of the radiation stimulated by the one-particle state Ω_g is always larger than the energy E_o^{out} of the spontaneously emitted radiation, as shown in the theorem below. This theorem gives a necessary condition for the outgoing energy E_g^{out} to be smaller than the sum of the incoming energy E_g^{in} of the one-particle state Ω_g and the spontaneously emitted energy E_o^{out}.

Theorem 7 *If $\tilde{g} \in \mathcal{S}(\mathbb{R}_+)$ is a normalized incoming test function, then for any transformation of coordinates $x = x(y)$,*

$$E_o^{out} < E_g^{out}. \tag{5.28}$$

Moreover if

$$x'(y) < 1, \qquad \forall y \in \mathbb{R}, \tag{5.29}$$

then

$$E_g^{out} < E_o^{out} + E_g^{in}. \tag{5.30}$$

Proof. Together with theorem 5, definitions (5.27) imply that

$$E_g^{out} - E_o^{out} = \frac{1}{2}\int_{-\infty}^{+\infty} dy\ x'(y)^2\ |g(x(y))|^2, \tag{5.31}$$

from which eq. (5.28) is deduced. Since

$$E_g^{in} = \frac{1}{2}\int_{-\infty}^{+\infty} dx\ |g(x)|^2, \tag{5.32}$$

eq. (5.30) is obtained under the hypothesis (5.29). □

Hypothesis (5.29) is not satisfied by the relativistic model but by the CGHS model. For this last model only, eq. (5.30) shows that part of the incoming energy is absorbed by the black hole.

5.3 The mean number of created particles

The mean number $\bar{N}_g[f]$ of particles stimulated by the incoming state Ω_g is given by

$$\bar{N}_g[f] = (\Omega_g, \hat{\phi}[f]^\dagger \hat{\phi}[f]\,\Omega_g), \tag{5.33}$$

where $\tilde{f} \in \mathcal{S}(\mathrm{I\!R}_+)$. The operators A and B are defined respectively as the positive and negative contributions of U to the incoming momentum:

$$(A\tilde{f})(k) = \theta(k)\,(U\tilde{f})(k), \qquad (B\tilde{f})(k) = \theta(k)\,(U\tilde{f})(-k). \tag{5.34}$$

Theorem 8 *If $\tilde{f}, \tilde{g} \in \mathcal{S}(\mathrm{I\!R}_+)$ are two normalized test functions, then for any transformation of coordinates $x(y)$,*[4]

$$\bar{N}_g[f] = \bar{N}_o[f] + \left|\langle \tilde{g}, A\tilde{f}\,\rangle\right|^2 + \left|\langle \tilde{g}^*, B\tilde{f}\,\rangle\right|^2. \tag{5.35}$$

Proof. This theorem is proved using the field transformation (3.4) and Wick's theorem. □

Equation (5.35) may be extended by continuity to the wave functions $\tilde{f} \in L^2(\frac{dp}{2p}, \mathrm{I\!R}_+)$ and $\tilde{g} \in L^2(\frac{dk}{2k}, \mathrm{I\!R}_+)$ if $f(y)$ and $g(x)$ exist a.e. and are integrable.

Some general bounds for $\bar{N}_g[f]$ are now given. The classical incoming function $\tilde{g}_{cl}$ has been defined in ref. [7] in terms of the outgoing test function $f(y)$ by

$$\tilde{g}_{cl}(k) = \|A\tilde{f}\,\|^{-1}(A\tilde{f}\,)(k). \tag{5.36}$$

Theorem 9 *If $\tilde{f} \in L^2(\frac{dp}{2p}, \mathrm{I\!R}_+)$ and $\tilde{g} \in L^2(\frac{dk}{2k}, \mathrm{I\!R}_+)$ are two normalized wave functions such that $f(y)$ and $g(x)$ exist a.e. and are integrable, then for any transformation of coordinates $x(y)$,*[5]

$$\bar{N}_o[f] \le \bar{N}_g[f] \le 1 + 3\,\bar{N}_o[f], \tag{5.37}$$

$$1 + \bar{N}_o[f] \le \bar{N}_{g_{cl}}[f]. \tag{5.38}$$

Proof. To obtain eq. (5.37), the Cauchy-Schwartz inequality is applied to eq. (5.35) using the fundamental relation $A^\dagger A = B^\dagger B + E$, where E is the identity [10]. The inequality (5.38) is deduced from eq. (5.35) and def. (5.36). □

This theorem states that $\bar{N}_g[f]$ is finite if and only if $\bar{N}_o[f]$ is finite, and that $\bar{N}_g[f]$ is always equal to or larger than $\bar{N}_o[f]$. For the classical incoming function it implies, in this particular case, that the mean number of created particles stimulated by the state Ω_g is larger than the sum of the mean number of incoming particles and the mean number of

[4] Equation (5.35) was first obtained by Wald [7].
[5] Equation (5.38) was first obtained by Wald [7].

spontaneously created particles, although the opposite inequality (5.30) is satisfied for the incoming and outgoing energies.

For both black-hole models, bounds for $\bar{N}_g[f]$ and $\bar{N}_g[f] - \bar{N}_o[f]$ exist in terms of the Fourier transform $\tilde{f}$.

Theorem 10 *If $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ and $\tilde{g} \in L^2(\frac{dk}{2k}, \mathbb{R}_+)$ are two normalized wave functions such that $f(y)$ and $g(x)$ exist a.e. and are integrable, then for both models*

$$\bar{N}_g[f] \;\leq\; C \int_0^\infty \frac{dp}{2p} \frac{|\tilde{f}(p)|^2}{1 - e^{-2\pi p}}, \tag{5.39}$$

where $C > 0$ is a constant which does not depend on the incoming function g.

Proof. Equation (5.39) follows from eq. (5.37) using the result (4.3) and theorem 2. □

Theorem 11 *If $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ and $\tilde{g} \in L^2(\frac{dk}{2k}, \mathbb{R}_+)$ are two normalized wave functions such that $f(y)$ and $g(x)$ exist a.e. and are integrable, then for both models,*

$$\left| \bar{N}_g[f] - \bar{N}_o[f] \right| \;\leq\; \frac{1}{\pi} \, \| g \|^2_{L^1(dx, \mathbb{R}_-)} \left(\int_0^\infty \frac{dp}{2p} \, |\tilde{f}(p)| \right)^2. \tag{5.40}$$

This theorem is proved in appendix A.2. It shows in particular that the mean number of created extra particles $\bar{N}_g[f] - \bar{N}_o[f]$ may be finite even if $\bar{N}_o[f]$ and $\bar{N}_g[f]$ are both infinite. The bound (5.40) depends only on the norm of g in $L^1(dx, \mathbb{R}_-)$, i.e. only on the restriction of g to the *exterior* of the black holes.

The outgoing function f_{p_o} of mode $p_o > 0$ is defined as[6]

$$\tilde{f}_{p_o}(p) \;=\; 2p\, \delta(p - p_o), \qquad \forall\, p \geq 0, \tag{5.41}$$

and the outgoing function $\check{g}(y)$ is defined in terms of the incoming function $g(x)$ as

$$\check{g}(y) \;=\; x'(y)\, g(x(y)), \qquad \forall\, y \in \mathbb{R}. \tag{5.42}$$

There is a bound for the mean number of particles in a given outgoing mode. The total mean number of particles, which is defined as [10]

$$\bar{N}^{tot} \;=\; \int_0^\infty \frac{dp}{2p}\, \bar{N}[f_p], \tag{5.43}$$

can also computed for a one-particle incoming state.

[6]The definition of the null mode $f_{p=0}$ is given in ref. [10].

Theorem 12 *If $\tilde{g} \in L^2(\frac{dk}{2k}, \mathbb{R}_+)$ is a normalized incoming test function, then for both black holes*[7] *and for $p \geq 0$,*

$$\bar{N}_g[f_p] \;\leq\; \bar{N}_o[f_p] + \frac{1}{\pi}\, \| g \|^2_{L^1(dx,\mathbb{R}_-)}, \tag{5.44}$$

$$\bar{N}_g^{tot} \;=\; \bar{N}_o^{tot} + \| \overset{\vee}{\tilde{g}} \|^2, \tag{5.45}$$

and

$$\bar{N}_g^{tot} \;=\; \infty; \tag{5.46}$$

moreover if $\tilde{g} \in \mathcal{S}(\mathbb{R}_+)$, the difference $\bar{N}_g^{tot} - \bar{N}_o^{tot}$ is finite if and only if $G(0) = 0$, where G is defined by eq. (5.8).

This theorem is proved in appendix A.3. For both black-hole models, it states that the incoming state Ω_g induces a finite mean number of extra particles in a given mode. The *total* mean number of extra particles created may be finite or infinite depending on g.

5.4 Close to the horizon

The behavior of the mean number $\bar{N}_g[f]$ of stimulated particles is now examined close to the horizon by computing a bound for $\bar{N}_g[f_{y_o}] - \bar{N}_o[f_{y_o}]$ in terms of y_o, where the function f_{y_o} is defined in eq. (4.10).

Theorem 13 *Let $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ and $\tilde{g} \in L^2(\frac{dk}{2k}, \mathbb{R}_+)$ be two normalized wave functions such that $f(y)$ and $g(x)$ exist a.e. and are integrable. If there exist three constants $\underline{\alpha > 0}$, $C > 0$, and $L \geq 0$ such that*

$$| f(y) | \;\leq\; \frac{C}{y^{1+\alpha}}, \qquad \text{if } y \leq -L, \tag{5.47}$$

then for both black holes and for $y_o \gg 1$,

$$\begin{aligned} \left| \bar{N}_g[f_{y_o}] - \bar{N}_o[f_{y_o}] \right| \;\leq\; & \| g \|^2_{L^1(dx,\mathbb{R}_-)} \frac{4\,C^2}{\alpha^2 \left(\frac{1}{2}\, y_o + L \right)^{2\alpha}} \\ & + \; e^{2L - y_o}\, \| f \|^2_{L^1(dy,\mathbb{R})}\, | g(0^-) |^2 . \end{aligned} \tag{5.48}$$

This theorem is proved in appendix A.4. Under the specified conditions, it shows that for both black-hole models

$$\lim_{y_o \to +\infty} \left(\bar{N}_g[f_{y_o}] - \bar{N}_o[f_{y_o}] \right) \;=\; 0. \tag{5.49}$$

[7]Equation (5.45) is in fact true for all transformations $x = x(y)$.

The bound (5.48) is composed of two terms: the first decreases algebraically in y_o with exponent 2α and depends only on the restriction of g to the *exterior* of the black holes; the second decreases exponentially in y_o and is absent if g vanishes at the horizon (more precisely at $x = 0^-$).

The corollaries below follow directly from the result (4.3) and theorems 1 and 13.

Corollary 1 *Under the hypothesis of theorem 13, for the relativistic model and for $y_o \gg 1$,*

$$\left| \bar{N}_g[f_{y_o}] - \bar{N}^{Th}_{2\pi,out}[f_{y_o}] \right| \;\leq\; \| g \|^2_{L^1(dx,\mathrm{IR}_-)} \frac{4\, C^2}{\alpha^2 \left(\frac{1}{2}\, y_o + L \right)^{2\alpha}} + e^{2L-y_o} \| f \|^2_{L^1(dy,\mathrm{IR})} \, | g(0^-) |^2 \,. \tag{5.50}$$

Corollary 2 *Under the hypothesis of theorem 13 and assuming further that $\underline{\alpha > 1/2}$ and $L \geq 1$, for the CGHS model and for $y_o \gg 1$,*

$$\left| \bar{N}_g[f_{y_o}] - \bar{N}^{Th}_{2\pi,out}[f_{y_o}] \right| \;\leq\; \left(1 + \| g \|^2_{L^1(dx,\mathrm{IR}_-)} \right) \frac{64\, C^2}{\alpha\,(2\alpha - 1) \left(\frac{1}{4}\, y_o + L - 1 \right)^{2\alpha-1}} + 2\, e^{2L-y_o/2} \left(\| f \|_{L^1(dy,\mathrm{IR})} + \| f' \|_{L^1(dy,\mathrm{IR})} \right)^2 \left(1 + | g(0^-) |^2 \right). \tag{5.51}$$

Under the specified conditions, these corollaries show that the mean number of particles stimulated by a one-particle state is thermal asymptotically close to the horizon for both black-hole models:

$$\lim_{y_o \to +\infty} \left(\bar{N}_g[f_{y_o}] - \bar{N}^{Th}_{2\pi,out}[f_{y_o}] \right) \;=\; 0. \tag{5.52}$$

The specified conditions are stronger for the CGHS model than for the relativistic model, i.e. in this last case eq. (5.52) is valid for all algebraically decreasing functions. *For the CGHS model,* theorems 1 and 13 imply that *eq. (5.52) is not true for sufficiently weakly decreasing functions.* The exponents in the denominator in eqs (5.50) and (5.51) are 2α for the relativistic model and $2\alpha - 1$ for the CGHS model.

5.5 The complex scalar field

For the complex scalar field, the incoming and outgoing two-point functions are also defined respectively by eqs (5.5) and (5.6), and the regularized energy-momentum tensors of the complex scalar field in the incoming and outgoing coordinates by eqs (5.17) and (5.19), or by eqs (5.18) and (5.20).

Theorem 14 *If $\tilde{g} \in \mathcal{S}(\mathbb{R}_+)$ is a normalized test function, then for the complex scalar field, between kernels of distributions on $\mathcal{S}_0(\mathbb{R}) \times \mathcal{S}_0(\mathbb{R})$,*

$$W_g(x,x') = W_o(x,x') + \frac{1}{4}\,\mathrm{Re}\,[\,G(x)\,G(x')^*\,], \tag{5.53}$$

$$\widehat{W}_g(y,y') = \widehat{W}_o(y,y') + \frac{1}{4}\,\mathrm{Re}\,[\,G(x(y))\,G(x(y'))^*\,], \tag{5.54}$$

where $W_o(x,x')$ and $\widehat{W}_o(y,y')$ are given by eqs (3.8) and (3.9); furthermore

$$T_g(x) = T_{g^*}(x) = \tfrac{1}{4}\,|g(x)|^2, \tag{5.55}$$

$$\widehat{T}_g(y) = \widehat{T}_{g^*}(y) = \widehat{T}_o(y) + x'(y)^2\,T_g(x(y)), \tag{5.56}$$

where $\widehat{T}_o(y)$ is given by eq. (3.11).

The proof of this theorem is similar to that of theorems 3 and 5. Note the appearance of the factor 1/4 in eqs (5.53) to (5.55) instead of 1/2 in eqs (5.9), (5.10) and (5.21). For the complex scalar field, the conclusions this theorem implies are identical to those for the real scalar field, i.e. theorems 4, 6 and 7 are also true in the complex case. The results of subsection 5.3 concerning the mean number of created particles are also valid for the complex scalar field.

In the incoming and outgoing coordinates, the mean currents associated with a test function $g \in \mathcal{S}(\mathbb{R}_+)$ are denoted by $J_g(x)$ and $\widehat{J}_g(y)$ respectively. These are given in the normal and subtraction regularization schemes by

$$J_g(x) = (\Omega_g, : \Upsilon(x) :_{in}\, \Omega_g), \tag{5.57}$$

$$J_g(x) = \lim_{\varepsilon\to 0}\,[\,(\Omega_g, \Upsilon_\varepsilon(x)\,\Omega_g) - (\Omega_o, \Upsilon_\varepsilon(x)\,\Omega_o)\,], \tag{5.58}$$

$$\widehat{J}_g(y) = (\Omega_g, : \widehat{\Upsilon}(y) :_{out}\, \Omega_g), \tag{5.59}$$

$$\widehat{J}_g(y) = \lim_{\varepsilon\to 0}\left[\,(\Omega_g, \widehat{\Upsilon}_\varepsilon(y)\,\Omega_g) - (\Omega_o, \Upsilon_\varepsilon(x(y))\,\Omega_o)\,\right], \tag{5.60}$$

where $\Upsilon(x)$, $\widehat{\Upsilon}(y)$, $\Upsilon_\varepsilon(x)$ and $\widehat{\Upsilon}_\varepsilon(y)$ are defined by eqs (3.18) to (3.21). The incoming and outgoing total charges of the state Ω_g are given by the integrals

$$Q_g^{in} = \int_{-\infty}^{+\infty} dx\, J_g(x), \qquad Q_g^{out} = \int_{-\infty}^{+\infty} dy\, \widehat{J}_g(y). \tag{5.61}$$

Theorem 15 *If $\tilde{g} \in \mathcal{S}(\mathbb{R}_+)$ is a normalized test function, and if $J_g(x)$ and $\widehat{J}_g(y)$ are defined by eqs (5.57) and (5.59), or by eqs (5.58) and (5.60), then*

$$J_g(x) = -J_{g^*}(x) = -\tfrac{\mathrm{i}}{4}\,G(x)^*\,\overleftrightarrow{\partial}_x\,G(x), \tag{5.62}$$

$$\widehat{J}_g(y) \;=\; -\widehat{J}_{g^*}(y) \;=\; -\tfrac{i}{4}\, G(x(y))^* \overleftrightarrow{\partial}_y\, G(x(y)), \tag{5.63}$$

where G is the primitive (5.8) of g; furthermore

$$Q_g^{in} \;=\; -Q_{g^*}^{in} \;=\; 1, \tag{5.64}$$

and for both models,

$$Q_g^{out} \;=\; -Q_{g^*}^{out} \;=\; \int_{-\infty}^{0} dx\, J_g(x). \tag{5.65}$$

This theorem is proved in appendix A.5. It shows that the mean current is a non-local function, and that the mean current of a particle is not positive definite locally, nor is that of an antiparticle negative definite locally[8]. For both models, the outgoing mean charge (5.65) is only part of the incoming mean charge (5.64). If Q^{in} is positive, Q^{out} may be negative, so *it is possible to observe a negative outgoing mean charge for a positive incoming mean charge.*

6 Emission stimulated by a thermal state

In this section we assume that the incoming state is thermal with temperature β^{-1}. This temperature must not be confused with that of the emitted radiation. The mean value of an observable A in this thermal state is given by

$$\langle A \rangle_{\beta,in} \;=\; \frac{\operatorname*{Tr}_{in}\left[e^{-\beta H_{in}} A \right]}{\operatorname*{Tr}_{in}\left[e^{-\beta H_{in}} \right]} \tag{6.1}$$

and is thus a thermal mean value in the incoming Hilbert space $\mathcal{H}_{in}$. The definitions and results of this section are valid for both real and complex scalar fields.

6.1 The two-point function

The incoming and outgoing two-point functions for an incoming thermal state, denoted by $W_{\beta,in}^{Th}(x,x')$ and $\widehat{W}_{\beta,in}^{Th}(y,y')$ respectively, are defined as

$$W_{\beta,in}^{Th}(x,x') \;=\; \langle\, \phi(x)\, \phi(x')^\dagger \,\rangle_{\beta,in}\,, \tag{6.2}$$

$$\widehat{W}_{\beta,in}^{Th}(y,y') \;=\; \langle\, \hat{\phi}(y)\, \hat{\phi}(y')^\dagger \,\rangle_{\beta,in}\,, \tag{6.3}$$

and are related by

$$\widehat{W}_{\beta,in}^{Th}(y,y') \;=\; W_{\beta,in}^{Th}\,(x(y), x(y')). \tag{6.4}$$

[8]In contrast to the case of the Dirac field, see ref. [12].

Theorem 16 *Between kernels of distributions on $\mathcal{S}_0(\mathrm{IR}) \times \mathcal{S}_0(\mathrm{IR})$,*

$$W^{Th}_{\beta,in}(x,x') = -\frac{1}{4\pi}\log\left\{\frac{\beta}{\pi}\sinh\left[\frac{\pi}{\beta}\left(x'-x+\mathrm{i}\,0^+\right)\right]\right\}, \tag{6.5}$$

$$\widehat{W}^{Th}_{\beta,in}(y,y') = -\frac{1}{4\pi}\log\left\{\frac{\beta}{\pi}\sinh\left[\frac{\pi}{\beta}\left(x(y')-x(y)+\mathrm{i}\,0^+\right)\right]\right\}. \tag{6.6}$$

Proof. See eq. (3.23). □

This last theorem applies to both black-hole models.

Theorem 17 *Between kernels of distributions on $\mathcal{S}_0(\mathrm{IR}) \times \mathcal{S}_0(\mathrm{IR})$, for both models and for all $\beta > 0$,*

$$\widehat{W}^{Th}_{\beta,in}(y,y') \approx W^{Th}_{2\pi,out}(y,y'), \qquad \text{if } y,y' \gg 1; \tag{6.7}$$

furthermore for the CGHS model and for all $\beta > 0$,

$$\widehat{W}^{Th}_{\beta,in}(y,y') \approx W^{Th}_{\beta,out}(y,y'), \qquad \text{if } -y,-y' \gg 1, \tag{6.8}$$

where $W^{Th}_{\beta,out}(y,y')$ is given by eq. (3.23).

Proof. For both models, we have $x(y) \approx -e^{-y} \approx 0^-$ if $y \gg 1$. Together with eq. (3.9), eq. (6.6) then implies for all $\beta > 0$, if $y,y' \gg 1$,

$$\widehat{W}^{Th}_{\beta,in}(y,y') \approx -\tfrac{1}{4\pi}\log\left(e^{-y}-e^{-y'}+\mathrm{i}\,0^+\right) \approx \widehat{W}_o(y,y'), \tag{6.9}$$

from which eq. (6.7) is deduced using the results (4.1) and (4.5).

For the CGHS model, we have further that $x(y) \approx y$ if $-y \gg 1$. Equation (6.6) then implies for all $\beta > 0$, if $-y,-y' \gg 1$,

$$\widehat{W}^{Th}_{\beta,in}(y,y') \approx -\tfrac{1}{4\pi}\log\left\{\tfrac{\beta}{\pi}\sinh\left[\tfrac{\pi}{\beta}\left(y'-y+\mathrm{i}\,0^+\right)\right]\right\} \tag{6.10}$$

from which eq. (6.8) is deduced using eq. (3.23). □

For both black-hole models and in the outgoing coordinates, the incoming thermal state is thus thermal close to the horizon. In this region, the associated outgoing temperature *does not depend* on the incoming temperature and coincides with that for the spontaneous emission in the same region. Furthermore, for the CGHS model only, and in the outgoing coordinates, the incoming thermal state of temperature β^{-1} is also thermal very far from the horizon, and the associated outgoing temperature is also β^{-1}.

6.2 The energy-momentum tensor

In the incoming and outgoing coordinates, the energy-momentum tensors for an incoming thermal state of temperature β^{-1} will be denoted by $T^{Th}_{\beta,in}(x)$ and $\widehat{T}^{Th}_{\beta,in}(y)$ respectively. These are given in the normal and subtraction regularization schemes by

$$T^{Th}_{\beta,in}(x) = \langle :\Theta(x):_{out} \rangle_{\beta,in}\,, \tag{6.11}$$

$$T^{Th}_{\beta,in}(x) = \lim_{\varepsilon\to 0}\left[\langle \Theta_\varepsilon(x)\rangle_{\beta,in} - (\Omega_o, \Theta_\varepsilon(x)\,\Omega_o)\right], \tag{6.12}$$

$$\widehat{T}^{Th}_{\beta,in}(y) = \langle :\widehat{\Theta}(y):_{out} \rangle_{\beta,in}\,, \tag{6.13}$$

$$\widehat{T}^{Th}_{\beta,in}(y) = \lim_{\varepsilon\to 0}\left[\langle \widehat{\Theta}_\varepsilon(y)\rangle_{\beta,in} - (\Omega_o, \Theta_\varepsilon(x(y))\,\Omega_o)\right], \tag{6.14}$$

where the observables $\Theta(x)$, $\Theta_\varepsilon(x)$, $\widehat{\Theta}(y)$ and $\widehat{\Theta}_\varepsilon(y)$ are defined by eqs (3.12) to (3.15).

Theorem 18 *If $T^{Th}_{\beta,in}(x)$ and $\widehat{T}^{Th}_{\beta,in}(y)$ are defined by eqs (6.11) and (6.13), or by eqs (6.12) or (6.14), then*

$$T^{Th}_{\beta,in}(x) = T^{Th}_{\beta,in} = \frac{\pi}{12\beta^2}\,, \tag{6.15}$$

$$\widehat{T}^{Th}_{\beta,in}(y) = \widehat{T}_o(y) + x'(y)^2\, T^{Th}_{\beta,in}\,, \tag{6.16}$$

where $\widehat{T}_o(y)$ is given by eq. (3.11).

This last theorem is proved in appendix A.6. Theorem 7, which concerns the incoming and outgoing energies, may be generalized to the incoming thermal states. Theorem 18 applies to both black-hole models.

Theorem 19 *For both models,*

$$\lim_{y\to+\infty} \widehat{T}^{Th}_{\beta,in}(y) = T^{Th}_{2\pi,out}\,, \tag{6.17}$$

furthermore, for the CGHS model,

$$\lim_{y\to-\infty} \widehat{T}^{Th}_{\beta,in}(y) = T^{Th}_{\beta,out}\,, \tag{6.18}$$

where $T^{Th}_{\beta,out}$ is given by eq. (3.24).

Proof. For both models, $x'(y) \approx 0$ if $y \gg 1$. For the CGHS model, it is also true that $x'(y) \approx 1$ if $-y \gg 1$. Equations (6.17) and (6.18) are then deduced from eq. (6.16) using the results (4.2), (4.7) and (4.8). □

For both black-hole models, the behavior of the energy-momentum tensor is thus thermal close to the horizon. In this region, the outgoing temperature is *not affected* by the incoming temperature and coincides with that for the spontaneous emission in the same region. Furthermore, for the CGHS model only, the energy-momentum tensor is also thermal very far from the black hole and its associated temperature is equal to the temperature of the incoming thermal state.

6.3 The mean number of created particles

The mean number $\bar{N}^{Th}_{\beta,in}[f]$ of particles induced by a thermal state of temperature β^{-1} is defined as

$$\bar{N}^{Th}_{\beta,in}[f] = \langle \hat{\phi}[f]^\dagger \hat{\phi}[f] \rangle_{\beta,in} . \tag{6.19}$$

Theorem 20 *If $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is a normalized wave function such that $f(y)$ exists a.e. and is integrable, then*

$$\bar{N}^{Th}_{\beta,in}[f] = \int_{-\infty}^{+\infty} \frac{dk}{2k} \frac{|(U\tilde{f})(k)|^2}{e^{\beta k}-1}, \tag{6.20}$$

and for all $\beta > 0$,

$$\bar{N}_o[f] \leq \bar{N}^{Th}_{\beta,in}[f]. \tag{6.21}$$

Proof. Equation (6.20) is deduced directly from def. (6.19) (see ref. [10] for details), and implies, with eq. (3.16),

$$\bar{N}^{Th}_{\beta,in}[f] = \bar{N}_o[f] + \int_0^\infty \frac{dk}{2k} \frac{1}{e^{\beta k}-1} \left[|(U\tilde{f})(k)|^2 + |(U\tilde{f})(-k)|^2 \right], \tag{6.22}$$

from which eq. (6.21) is deduced. □

The mean number of particles stimulated by a thermal state is thus always equal to or larger than that for the spontaneous emission. Equation (6.22) implies that

$$\lim_{\beta\to\infty} \bar{N}^{Th}_{\beta,in}[f] = \bar{N}_o[f]. \tag{6.23}$$

In the incoming momentum, the integral in eq. (6.20) may be infrared divergent. For the relativistic black-hole model, it is always divergent.

Theorem 21 *If $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is a normalized wave function such that $f(y)$ exists a.e. and is integrable, then for the relativistic model*

$$\bar{N}^{Th}_{\beta,in}[f] - \bar{N}_o[f] = \infty, \qquad \forall \beta > 0, \tag{6.24}$$

and so $\bar{N}^{Th}_{\beta,in}[f]$ is always infinite for all $\beta > 0$.

Proof. For the relativistic black hole, the kernel of U is given by eq. (3.6). Using eq. (6.20), the divergent term,

$$\int_0^\infty \frac{dk}{2k} \frac{e^{i \log k\,(p-p')}}{e^{\beta k}-1} = \infty, \qquad \forall p, p' \in \mathbb{R}_+, \tag{6.25}$$

appears in the kernel of $\bar{N}^{Th}_{\beta,in}[f]$. This implies eq. (6.24). □

For the relativistic black-hole model and for all inverse temperature, this theorem also shows that the incoming thermal state induces an infinite number of extra particles in comparaison with the spontaneous case and for a given test function. If the mean number of stimulated particles is considered for a given mode or in total (see def. (5.43)), this last result is true for *all models*.

Theorem 22 *If the outgoing function f_p is defined by eq. (5.41), then for any transformation of coordinates $x = x(y)$ and for $p \geq 0$*

$$\bar{N}^{Th}_{\beta,in}[f_p] - \bar{N}_o[f_p] \quad = \quad \infty, \tag{6.26}$$

$$\bar{N}^{Th,tot}_{\beta,in} - \bar{N}^{tot}_o \quad = \quad \infty. \tag{6.27}$$

Proof. It is alway true that

$$\hat{\tilde{f}}_p(k) \quad = \quad 2pU(k,p). \tag{6.28}$$

Using eq. (6.20), the divergent term

$$\int_{-\infty}^{+\infty} \frac{dk}{2k} \frac{e^{i\,k[x(y)-x(y')]}}{e^{\beta k}-1} \quad = \quad \infty, \qquad \forall y, y' \in \mathbb{R}, \tag{6.29}$$

appears in the kernel of $\bar{N}^{Th}_{\beta,in}[f_p]$ for all transformations $x = x(y)$. This implies eqs (6.26) and (6.27). □

For the CGHS black-hole model, bounds for $\bar{N}^{Th}_{\beta,in}[f] - \bar{N}_o[f]$ and $\bar{N}^{Th}_{\beta,in}[f]$ exist in terms of the Fourier transform $\tilde{f}$.

Theorem 23 *If $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is a normalized wave function such that $f(y)$ exists a.e. and is integrable, then for the CGHS model*

$$\left| \bar{N}^{Th}_{\beta,in}[f] - \bar{N}_o[f] \right| \quad \leq \quad \frac{\pi}{\beta} \int_0^\infty \frac{dp}{2p} \frac{|\tilde{f}(p)|^2}{1-e^{-2\pi p}}. \tag{6.30}$$

This theorem is proved in appendix A.7 and, together with theorem 2, it implies the corollary below.

Corollary 3 *If $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is a normalized wave function such that $f(y)$ exists a.e. and is integrable, then for the CGHS model*

$$\bar{N}^{Th}_{\beta,in}[f] \quad \leq \quad C\,(1+\beta^{-1}) \int_0^\infty \frac{dp}{2p} \frac{|\tilde{f}(p)|^2}{1-e^{-2\pi p}}, \tag{6.31}$$

where $C > 0$ is a constant.

6.4 Close to the horizon

For the relativistic black-hole model and if f_{y_o} is defined by eq. (4.10), it is clear that $\bar{N}^{Th}_{\beta,in}[f_{y_o}]$ does not approach $\bar{N}_o[f_{y_o}]$ asymptotically near the horizon, since $\bar{N}^{Th}_{\beta,in}[f_{y_o}]$ is alway infinite for this model (see theorem 21). However, for the CGHS black-hole model and under some conditions, $\bar{N}^{Th}_{\beta,in}[f_{y_o}]$ does indeed tend to $\bar{N}_o[f_{y_o}]$ asymptotically near the horizon:

$$\lim_{y_o \to +\infty} \left(\bar{N}^{Th}_{\beta,in}[f_{y_o}] - \bar{N}_o[f_{y_o}] \right) = 0. \tag{6.32}$$

This is shown in the following theorem, which is proved in appendix A.8.

Theorem 24 *Let $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ be a normalized wave function such that $f(y)$ exists a.e. and is integrable. If there exist three constants $\alpha > 1/2$, $C > 0$ and $L \geq 1$ such that*

$$|f(y)| \leq \frac{C}{y^{1+\alpha}}, \qquad if\ y \leq -L, \tag{6.33}$$

then, for the CGHS model and for $y_o > 0$,

$$\begin{aligned} \left| \bar{N}^{Th}_{\beta,in}[f_{y_o}] - \bar{N}_o[f_{y_o}] \right| \leq\ & \frac{C^2}{\beta\, \alpha^2 (2\alpha - 1) \left(\frac{1}{2} y_o + L\right)^{2\alpha-1}} \\ & + \frac{4\pi}{\beta}\, e^{L - y_o/2} \int_0^\infty \frac{dp}{2p} \frac{|\tilde{f}(p)|^2}{1 - e^{-2\pi p}} . \end{aligned} \tag{6.34}$$

For the relativistic black-hole model, theorem 21 implies that

$$\bar{N}^{Th}_{\beta,in}[f_{y_o}] - \bar{N}^{Th}_{2\pi,out}[f_{y_o}] = \infty, \qquad \forall y_o \in \mathbb{R}. \tag{6.35}$$

In this case the mean number of created particles is thus never thermal close to the horizon. For the CGHS black-hole model and under some conditions, the following corollary to theorems 1 and 24 shows that $\bar{N}^{Th}_{\beta,in}[f]$ tends to the thermal value $\bar{N}^{Th}_{2\pi,out}[f]$ close to the horizon:

$$\lim_{y_o \to +\infty} \left(\bar{N}^{Th}_{\beta,in}[f_{y_o}] - \bar{N}^{Th}_{2\pi,out}[f_{y_o}] \right) = 0. \tag{6.36}$$

Corollary 4 *Under the hypothesis of theorem 24, for the CGHS model and for $y_o > 0$,*

$$\begin{aligned} \left| \bar{N}^{Th}_{\beta,in}[f_{y_o}] - \bar{N}^{Th}_{2\pi,out}[f_{y_o}] \right| \leq\ & \frac{32\,(1 + \beta^{-1})\, C^2}{\alpha^2 (2\alpha - 1) \left(\frac{1}{4} y_o + L - 1\right)^{2\alpha-1}} \\ & + e^{2L - y_o/2} \left[\left(\|f\|_{L^1} + \|f'\|_{L^1} \right)^2 + \frac{4\pi}{\beta} \int_0^\infty \frac{dp}{2p} \frac{|\tilde{f}(p)|^2}{1 - e^{-2\pi p}} \right]. \end{aligned} \tag{6.37}$$

For the CGHS black-hole model, theorem 1 implies that *eqs (6.32) and (6.36) are not valid if f decreases sufficiently weakly very far from the horizon and does not oscillate.*

Relativistic BH	$\widehat{W}(y,y')$	$\hat{T}(y)$	$\bar{N}[f]$
vacuum	$\circ^1$	$\circ^1$	$\circ^1$
one-particle state	$\otimes$	$\circ$	$\circ^2$
thermal state	$\circ$	$\circ$	$\times$

CGHS BH	$\widehat{W}(y,y')$	$\hat{T}(y)$	$\bar{N}[f]$
vacuum	$\circ$	$\circ$	$\otimes$
one-particle state	$\otimes$	$\circ$	$\otimes$
thermal state	$\circ$	$\circ$	$\otimes$

Table 1: Behavior of mean values of the outgoing observables in a given incoming state, close to the horizon of the relativistic and CGHS black holes.
$\circ$: thermal; $\times$: non-thermal; $\otimes$: thermal or non-thermal depending on the test functions; 1: everywhere, i.e. not only near the horizon; 2: in general.

7 Conclusions

The emission of massless bosons by the relativistic and CGHS black holes have been studied for one-particle and thermal incoming states. Mean values of observables constructed in the outgoing coordinates were computed in these states. The results obtained in this paper are summarized in table 1. They show that the emitted radiation exhibits both thermal and non-thermal properties close to the horizon for both black-hole models. For all incoming states, the thermal properties are always associated with the temperature $\frac{M}{2\pi}$ for the relativistic black hole and with the temperature $\frac{\lambda}{2\pi}$ for the CGHS black hole.

For the *relativistic* black hole and for non-vacuum incoming states, the emitted radiation has no thermal properties, except close to the horizon, contrary to the spontaneous emission.

For this model, the mean number of created particles stimulated by a thermal state is *non-thermal* for all outgoing wave functions. For the *CGHS* black-hole model and for all the incoming states studied, the mean number of particles may be non-thermal for some outgoing test functions.

For *one-particle* incoming states and for both black-hole models, the outgoing two-point function may be thermal or not, depending on the incoming test function. For incoming *thermal* states and for both models, the outgoing two-point function is thermal close to the horizon for all incoming temperatures.

For both black-hole models, it is remarkable that the *energy-momentum tensor* is thermal close to the horizon for all the incoming states considered. It thus seems to be stable with respect to the incoming state. In particular, the energy of the incoming state must be very large near the horizon to modify it close to the horizon. For both black-hole models, a non-vacuum incoming state amplifies the emitted radiation, in the sense that the energy-momentum tensor and mean number of particles are equal to or larger than those for the spontaneous emission.

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A Appendices

Theorem 25 *If* $\tilde{g} \in L^q(dk, \mathbb{R}_+)$ *with* $q \in [1, 2]$ *satisfies*

$$\tilde{g}(k) = \theta(k)\tilde{g}(k), \qquad \forall k \in \mathbb{R}, \tag{A.1}$$

then

i) g is analytic in the upper complex half-plane $\mathrm{Im}\,(x) > 0$,

ii) there exists a positive constant C such that

$$|g(x)| < C, \qquad \textit{if } \mathrm{Im}\,(x) > 0, \tag{A.2}$$

and so g is regular in the upper complex half-plane $\mathrm{Im}\,(x) > 0$.

This theorem is proved in ref. [17]. The primitives $\hat{F}(x)$ and $F(y)$ of $\hat{f}(x)$ and $f(y)$ are defined respectively as

$$\hat{F}(x) = \int_{x(-\infty)}^{x} dx'\, \hat{f}(x'), \tag{A.3}$$

$$F(y) = \int_{-\infty}^{y} dy'\, f(y'). \tag{A.4}$$

A.1 Proof of theorem 5

Assuming that $T_g(x)$ and $\widehat{T}_g(y)$ are defined by eqs (5.17) and (5.19) respectively, the incoming and outgoing fields are expanded as

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{2k} \left[a_{in}(k)\, e^{-ikx} + a_{in}^\dagger(k)\, e^{ikx} \right], \tag{A.5}$$

$$\hat{\phi}(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dp}{2p} \left[a_{out}(p)\, e^{-ipy} + a_{out}^\dagger(p)\, e^{ipy} \right]. \tag{A.6}$$

Ordering normally the field operators in def. (5.19), using

$$(\Omega_o, a_{out}(p)\, a_{in}^\dagger(k)\, \Omega_o) = 2p\, U(k,p), \tag{A.7}$$

$$(\Omega_o, a_{out}^\dagger(p)\, a_{in}^\dagger(k)\, \Omega_o) = 2p\, U(k,-p), \tag{A.8}$$

and

$$\int_0^\infty dp\, 2p\, U(k,p)\; e^{-iyp} = -\frac{1}{\pi} \int_{-\infty}^{+\infty} dy' \, \frac{e^{-ikx(y')}}{(\, y - y' - i\, 0^+\,)^2}, \tag{A.9}$$

it follows that

$$\widehat{T}_g(y) = (\Omega_o, :\widehat{\Theta}(y):_{out}\, \Omega_o) + \frac{x'(y)^2}{8\pi^2} \left| \int_{-\infty}^{+\infty} dx \, \frac{g(x)}{x - x(y) - i\, 0^+} \right|^2. \tag{A.10}$$

Using the Cauchy theorem and def. (3.11), this last equation implies that

$$\widehat{T}_g(y) = \widehat{T}_o(y) + \frac{x'(y)^2}{2} \, |g(x(y))|^2, \tag{A.11}$$

since g is a regular function in the upper complex half-plane Im $(x) > 0$ (see theorem 25). Equations (5.21) and (5.22) then follow from eq. (A.11) with $x(y) = y$.

Alternatively, if $T_g(x)$ and $\widehat{T}_g(y)$ are defined by eqs (5.18) and (5.20), eqs (5.21) and (5.22) are directly obtained using def. (3.11), Wick's theorem and

$$(\Omega_o, \phi[g]\, \partial_y \hat{\phi}(y)\, \Omega_o) = \frac{i}{2}\, \partial_y G(x(y)). \tag{A.12}$$

□

A.2 Proof of theorem 11

Considering the term $\langle\, \tilde{g}\, , A\tilde{f}\ \rangle$ in eq. (5.35), we have

$$\begin{aligned} \int_0^\infty \frac{dk}{2k}\, \tilde{g}\,(k)^*\, \tilde{\hat{f}}\,(k) &= \frac{i}{2} \int_{-\infty}^{+\infty} dk\, \tilde{g}\,(k)^*\, \tilde{\hat{F}}(k) = \frac{i}{2} \int_{-\infty}^{0} dx\, g(x)^*\, \hat{F}(x) \\ &= \frac{i}{2} \int_{-\infty}^{+\infty} dy\, \check{g}\,(y)^*\, F(y) = \int_0^\infty \frac{dp}{2p}\, \tilde{\check{g}}\,(p)^*\, \tilde{f}\,(p), \end{aligned} \tag{A.13}$$

where $\hat{F}(x)$ and $\check{g}(y)$ are defined by eqs (A.3) and (5.42) respectively. Note that $\tilde{\check{g}}(0)$ may not be zero. A bound for $\tilde{\check{g}}(p)$ is given by

$$\sqrt{2\pi}\ |\tilde{\check{g}}(p)| \ \leq \ \|\check{g}\|_{L^1(dy,\mathbb{R})} \ \leq \ \|g\|_{L^1(dx,\mathbb{R}_-)}\,. \tag{A.14}$$

Equation (A.13) then shows that

$$\sqrt{2\pi}\ |\langle \tilde{g}, A\tilde{f}\rangle| \ \leq \ \|g\|_{L^1(dx,\mathbb{R}_-)} \int_0^\infty \frac{dp}{2p}\ |\tilde{f}(p)|\,. \tag{A.15}$$

Since the bound (A.15) is also valid for $\langle \tilde{g}^*, B\tilde{f}\rangle$, eq. (5.40) is true. □

A.3 Proof of theorem 12

Using def. (5.41), eqs (5.35) and (A.13) imply that

$$\bar{N}_g[f_p] \ = \ \bar{N}_o[f_p] + 2\ |\tilde{\check{g}}(p)|^2\,. \tag{A.16}$$

This last equation implies eqs (5.44) and (5.45) using the bound (A.14) and def. (5.43).

As $\sqrt{2\pi}\,\tilde{\check{g}}(0) = G(0)$, it is clear from eq. (5.45) that $\bar{N}_g^{tot} - \bar{N}_o^{tot}$ diverges if $G(0) \neq 0$. If $G(0) = 0$, integrating by parts gives

$$\sqrt{2\pi}\,\tilde{\check{g}}(p) \ = \ i\,p \int_{-\infty}^0 dx\, G(x)\, y'(x)\, e^{-ipy(x)}. \tag{A.17}$$

For $g \in \mathcal{S}(\mathbb{R}_+)$, since $G(x) = \mathcal{O}(x)$ if $x \approx 0^-$ and $G(x) = \mathcal{O}(1/x)$ if $-x \gg 1$, we conclude that $\tilde{\check{g}}(p) = \mathcal{O}(p)$ if $p \approx 0$ (since $y'(x) = \mathcal{O}(1/x)$ if $x \approx 0^-$ and $-x \gg 1$ for both black-hole models). Furthermore $\tilde{\check{g}}(p) = \mathcal{O}(1/p)$ at infinity. This shows that the norm $\|\tilde{\check{g}}\|$ is finite if $G(0) = 0$ and $g \in \mathcal{S}(\mathbb{R}_+)$. □

A.4 Proof of theorem 13

Let ξ be a function such that

$$\lim_{y_o \to +\infty} \xi(y_o) \ = \ \lim_{y_o \to +\infty} [y_o - \xi(y_o)] \ = \ +\infty. \tag{A.18}$$

Considering the term $\langle \tilde{g}, A\tilde{f}_{y_o}\rangle$ in eq. (5.35), we have

$$\left| \int_0^\infty \frac{dk}{k}\, \tilde{g}(k)^*\, \hat{\tilde{f}}_{y_o}(k) \right| \ = \ \left| \int_{-\infty}^{+\infty} dy\, x'(y)\, g(x(y))^*\, F(y - y_o) \right| \tag{A.19}$$

$$\leq \ \left| \int_{-\infty}^{-\xi(y_o)} dy\, x'(y + y_o)\, g(x(y + y_o))^*\, F(y) \right| + \left| \int_{-\xi(y_o)}^{+\infty} dy\, x'(y + y_o)\, g(x(y + y_o))^*\, F(y) \right|.$$

The two terms in this last equation are treated separately. On one hand, the hypothesis (5.47) implies, after an integration by parts,

$$\left| \int_{-\infty}^{-\xi(y_o)} dy\, x'(y+y_o)\, g(x(y+y_o))^*\, F(y) \right|$$

$$\begin{aligned} &\leq \; |G(x(y_o - \xi(y_o)))|\,|F(-\xi(y_o))| + \|g\|_{L^1(dx,\mathbb{R}_-)} \int_{-\infty}^{-\xi(y_o)} dy\; |f(y)| \\ &\leq \; 2\|g\|_{L^1(dx,\mathbb{R}_-)} \frac{C}{\alpha\, \xi(y_o)^\alpha}, \qquad \text{if } \xi(y_o) \geq L, \end{aligned} \tag{A.20}$$

and on the other hand,

$$\left| \int_{-\xi(y_o)}^{+\infty} dy\, x'(y+y_o)\, g(x(y+y_o))^*\, F(y) \right| \;\leq\; \|f\|_{L^1(dy,\mathbb{R})} \int_{x(y_o-\xi(y_o))}^{0} dx\; |g(x)|\,. \tag{A.21}$$

An identical bound is obtained for the term $\langle \tilde{g}^*, B\tilde{f}_{y_o} \rangle$.

With $\xi(y_o) = \frac{1}{2}\, y_o + L$, eqs (A.20) and (A.21) imply if $y_o \geq 0$

$$\begin{aligned} \left| \bar{N}_g[f_{y_o}] - \bar{N}_g[f_{y_o}] \right| \;&\leq\; 4\,\|g\|^2_{L^1(dx,\mathbb{R}_+)} \frac{C^2}{\alpha^2 \left(\frac{1}{2}\, y_o + L\right)^{2\alpha}} \\ &\quad + \|f\|^2_{L^1(dy,\mathbb{R})}\, x(L - y_o/2)^2\, M_g(x(L - y_o/2))^2, \end{aligned} \tag{A.22}$$

where $M_g(x)$ $(x < 0)$ is the average of $|g|$ on the interval $[x, 0]$:

$$M_g(x) \;=\; \frac{1}{|x|} \int_x^0 dx\; |g(x)|\,. \tag{A.23}$$

The bound (5.48) follows from eq. (A.22) if $y_o \gg 1$ by noting that for both models $x(y) \approx e^{-y}$, if $y \gg 1$. □

A.5 Proof of theorem 15

The proofs of eqs (5.62) and (5.63) are similar to that of theorem 5. From defs (5.58) and (5.60) the mean current transforms as a tensor:

$$\hat{J}_g(y) \;=\; x'(y)\, J_g(x(y)). \tag{A.24}$$

Using the Parseval identity and def. (5.61),

$$Q_g^{in} \;=\; \int_0^\infty \frac{dk}{2k}\; |\tilde{g}(k)|^2, \tag{A.25}$$

which implies eq. (5.64). Definitions (5.61) with transformation (A.24) imply eq. (5.65). □

A.6 Proof of theorem 18

Here we only give the proof for $T^{Th}_{\beta,in}(x)$ and $\widehat{T}^{Th}_{\beta,in}(y)$ defined by eqs (6.12) and (6.14) respectively. Equation (6.15) follows from the result (3.24). Differentiating eq. (6.6) twice gives

$$\langle\, \partial_y \hat{\phi}(y)\, \partial_y \hat{\phi}(y+\varepsilon)^\dagger \,\rangle_{\beta,in} \;=\; -\frac{\pi}{4\beta^2}\, \frac{x'(y+\varepsilon)\, x'(y)}{\sinh^2\left\{ \frac{\pi}{\beta}\, [x(y+\varepsilon) - x(y)] \right\}}\,. \tag{A.26}$$

Expanding this last equation at $\varepsilon = 0$,

$$\widehat{T}^{Th}_{\beta,in}(y) \;=\; \widehat{T}_o(y) + x'(y)^2\, \frac{\pi}{12\beta^2}\,, \tag{A.27}$$

which implies eq. (6.16). □

A.7 Proof of theorem 23

Equation (6.22) implies

$$\left|\, \bar{N}^{Th}_{\beta,in}[f] - \bar{N}_o[f] \,\right| \;\leq\; \frac{1}{2\,\beta} \int_{-\infty}^{+\infty} \frac{dk}{k^2}\, |\,\hat{\tilde{f}}\,(k)\,|^2\,. \tag{A.28}$$

This last term is more easily treated than the original expression (6.22) (the decreasing exponential does not play a crucial role here). If $\hat{F}(x)$ and $F(y)$ are respectively defined by eqs (A.3) and (A.4), then

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{dk}{k^2}\, |\,\hat{\tilde{f}}\,(k)\,|^2 \;&=\; \int_{-\infty}^{0} dx\; |\,\hat{F}(x)\,|^2 \;<\; \int_{-\infty}^{+\infty} dy\; |\,F(y)\,|^2 \\ &=\; \int_{0}^{\infty} \frac{dp}{p^2}\, |\,\tilde{f}\,(p)\,|^2, \end{aligned} \tag{A.29}$$

where it has been assumed that $x'(y) < 1$, $\forall y \in \mathbb{R}$ (the proof is thus not valid for the relativistic black-hole model). Equation (A.29) implies the bound (6.30). □

A.8 Proof of theorem 24

Let ξ be a function such that

$$\lim_{y_o \to +\infty} \xi(y_o) \;=\; \lim_{y_o \to +\infty} [\, y_o - \xi(y_o)\,] \;=\; +\infty. \tag{A.30}$$

With hypothesis (6.33) eq. (6.22) implies that

$$\left|\, \bar{N}^{Th}_{\beta,in}[f_{y_o}] - \bar{N}_o[f_{y_o}] \,\right| \;\leq\; \frac{1}{\beta} \int_{-\infty}^{+\infty} \frac{dk}{k^2}\, |\,\hat{\tilde{f}}_{y_o}(k)\,|^2$$

$$
\begin{aligned}
&= \frac{1}{\beta} \int_{-\infty}^{+\infty} dy\ x'(y + y_0)\ |F(y)|^2 \\
&\leq \frac{1}{\beta} \left[\int_{-\infty}^{-\xi(y_o)} dy\ |F(y)|^2 + \frac{1}{1 + e^{y_o - \xi(y_o)}} \int_{-\xi(y_o)}^{\infty} dy\ |F(y)|^2 \right] \\
&\leq \frac{1}{\beta} \left[\frac{C^2}{\alpha^2 (2\alpha - 1)\ \xi(y_o)^{2\alpha - 1}} + e^{\xi(y_o) - y_o} \int_0^{\infty} \frac{dp}{p^2}\ |\tilde{f}(p)|^2 \right], \quad \text{(A.31)}
\end{aligned}
$$

if $\xi(y_o) \geq L$. With $\xi(y_o) = \frac{1}{2} y_o + L$, eq. (A.31) implies the bound (6.34).

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