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A lattice model of local algebras of observables and fields with braid group statistics

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Dedicated to Klaus Hepp and Walter Hunziker on the occasion of their 60^{th} birthday

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Abstract

Using the 6j-symbols and the R-matrix for the quantum group $Sl_q(2,\mathbb{C})$ at roots of unity we construct local algebras of observables and fields with braid group statistics on the lattice $\mathbb{Z}$. These algebras are closely related to the XXZ-Heisenberg model and the RSOS models thus exhibiting the quantum group symmetry of these models. Our discussion relates the theory of integrable lattice models to the Doplicher-Haag-Roberts theory of superselection sectors. The construction of these algebras is a variant of the path space construction of Ocneanu and Sunder which replaces the usual tensor product construction of lattice models in statistical mechanics and extends previous discussions by Pasquier. Our construction is based on the theory of coloured graphs on S^2 and the associated Wigner-Eckhart theorem obtained previously by the authors.

1 Introduction

Lattice models of quantum statistical models are usually based on the concept of the tensor product and variants thereof like the antisymmetric (=fermionic) and symmetric (=bosonic) tensor product (see e.g. [1]). Thus to each site x on a finite lattice Λ one associates a Hilbert space h_x (usually finite dimensional) and the Hilbert space associated to Λ is then given as $\mathcal{H}_\Lambda = \bigotimes_x h_x$. As an example for the Heisenberg model one takes $h_x = \mathbb{C}^2$ and for the Hubbard model one takes $h_x = \mathbb{C}+\mathbb{C}^2+\mathbb{C} \cong \mathbb{C}^4$ which is the Fockspace for two spin 1/2 particles (spin-up and spin-down). In this last case $\mathcal{H}_\Lambda = \bigwedge_{x\in\Lambda} h_x$ and in both cases it makes sense to speak of global $SU(2)$ invariance for lattice models given in terms of local Hamiltonians and where on each h_x one has the canonical representation of SU(2).

The aim of this article is to provide a general set-up for one-dimensional lattice models which is based on the so-called path space formulation. This is similar to Baxters [2] formulation of the eight-vertex model in terms of the SOS-model. Using the quantum group $SL_q(2,\mathbb{C})$ [3] at roots of unity ($q = \exp\frac{\pi i}{r}$, $3 \leq r \in \mathbb{N}^+$) Pasquier [4] introduced this concept (see e.g. Ocneanu [5] and Sunder [6] (see also [7])), for lattice models of quantum statistical mechanics. The path space we will use is the set of all paths in the Bratteli diagram obtained by tensoring the "good" representations of $SL_q(2,\mathbb{C})$ (see e.g. [8]) with the fundamental spin $\frac{1}{2}$ representation.

Thus to each "interval" $I \subset \mathbb{Z}$ we will associate a finite dimensional Hilbert space V_I with basis elements labelled by paths of length $|I|+1$, where $|I|$ denotes the number of lattice points in I. Intuitively to each lattice point there is associated one copy of the fundamental representation. For generic q not a root of unity and up to multiplicities this formulation is essentially equivalent to the tensor product formulation. In particular this basis replaces a basis in the tensor product representation labelled by magnetic quantum numbers. For q being a root of unity, the case we will consider, the path space V_I has a dimension smaller than $2^{|I|}$, is the dimension of the corresponding tensor product version. To each V_I we will associate a $*$-subalgebra of $\mathrm{End}(V_I)$, the so-called observable algebra $\mathcal{A}_I$ localized in I. By construction it is the linear space of elements in $\mathrm{End}(V_I)$ whose matrix elements with respect to the path basis are given in terms of invariants of planar coloured graphs (with colours indexed by the "good" irreducible representations of $SL_q(2,\mathbb{C})$) on the boundary $S^2 = \partial D^3$ of the unit ball. Such invariants in terms of partition functions were obtained in [9] as generalizations of the combinatorial Turaev-Viro approach to topological quantum field theory using the $6j$ symbols of $SL_q(2,\mathbb{C})$ [10].

These constructions of V_I and $\mathcal{A}_I$ for varying I are related in the following way. There is a canonical bilinear map $\circ : V_{I_1} \times V_{I_2} \to V_{I_1 \cup I_2}$ for neighboring intervals I_1 and I_2. This map $\circ$ replaces the tensor product and is not injective since $\dim V_{I_1} \cdot \dim V_{I_2} > \dim V_{I_1 \cup I_2}$, which is again related to the fact that we work at roots of unity. Furthermore this map $\circ$ induces a map from $\mathcal{A}_{I_1} \times \mathcal{A}_{I_2}$ into $\mathcal{A}_{I_1 \cup I_2}$, also denoted by $\circ$. In particular $\circ$ leads to an injective $*$-homomorphism $\iota_{I',I}$ of $\mathcal{A}_I$ into $\mathcal{A}_{I'}$ for any $I \subset I'$. Now $\iota_{I,I_1}(\mathcal{A}_{I_1})$ and $\iota_{I,I_2}(\mathcal{A}_{I_2})$ commute wherever I_1 and I_2 are disjoint subintervals of I. Thus our construction exhibits all the properties required for local observable algebras on a lattice. In particular one may introduce dynamics in the form of models given by local Hamiltonians. Moreover, motivated by the theory of superselection theory of Doplicher, Haag and Roberts [11] (see also [12]) we construct a local endomorphism associated to the fundamental representation and whose index of inclusion is given by the square of the q-dimension $d_q(\frac{1}{2})$ of the fundamental representation.

Furthermore we may even construct field algebras $\mathcal{F}_I$ which are $*$-subalgebras of $\mathrm{End}(V_I)$ and which contain $\mathcal{A}_I$. $\mathcal{F}_I$ is the analogue of the algebra in the tensor product formulation with a basis given by magnetic quantum numbers generated by irreducible tensor operators T^j of spin j. More precisely this analogue looks as follows. Recall that such operator T^j are completely determined by their reduced matrix elements $< k_r \| T^j \| k_s >$ due to the Wigner-Eckhart theorem for $SU(2)$ by which

$$< k_r, m_r \mid T^j_m \mid k_s, m_s >= C \begin{pmatrix} k_r & j & k_s \\ m_r & m & m_s \end{pmatrix} < k_r \| T^j \| k_s >$$

Here m_r, m and m_s are magnetic quantum numbers, k_r and k_s are total angular momentum (s = source, r = range) and C is the resulting Clebsch-Gordon coefficient. In our context $\mathcal{F}_I$ is indeed generated by elements $\psi_j(k_s, k_r)$ replacing these reduced matrix elements. As a $*$-subalgebra of $\mathcal{F}_I$, the algebra $\mathcal{A}_I$ may then be viewed as the algebra of invariant operators, i.e. those for which $j = 0$. In fact the Jones index of the inclusion $\mathcal{A}_I \subset \mathcal{F}_I$ is essentially given by $\sum_j d_q^2(j)$, i.e. the sum of the squares of the q-dimension squared of the representation space for the spins j.

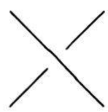
Moreover, the inclusion map $\iota_{I',I}$ ($I \subset I'$) extends to an inclusion map from $\mathcal{F}_I$ into $\mathcal{F}_{I'}$, such that the tensor product construction for the $\mathcal{A}_I$'s extends to the $\mathcal{F}_I$'s. Also there is a family of traces tr_I on these field algebras compatible with these inclusion maps. This allows one to discuss the thermodynamic limit, i.e. the inductive limit, of these algebras, which then carries a trace. This in turn makes a GNS construction possible.

Our path space formulation finally allows a discussion of quantum group symmetry, thus providing an alternative to the discussion given by Mack and Schomerus [13]. In fact, we construct an algebra which we call a path Hopf algebra, where the tensor product operation $\otimes$ is replaced by $\circ$, but which otherwise exhibits all properties usually valid in Hopf algebras.

This article is organized as follows. In Section 2 we briefly review the theory of planar, coloured graphs as given in [9] and which we will need in what follows. Section 3, where we introduce the spaces V_I and the local observable algebras $\mathcal{A}_I$, is a review in our language of well known results as far as index theorems (see e.g. Wenzl [14]) and the tensor product construction (see e.g. [15]) is concerned. Thus 3 serves as an introduction of techniques which will be used in the following chapters. In Section 4 we introduce the field algebra. In Section 5 we make a further generalization by introducing so-called generalized field algebras, which in Section 6 will allow us to introduce these path Hopf algebras and to discuss the associated quantum group symmetry. These (generalized) fields also exhibit a structure which may be considered to be a path space version of what is called the concept of auxiliary space by the St. Petersburg school (see e.g. [16]). In Section 7 we discuss the thermodynamic limit. Finally in Section 8 we show how to construct models by defining local Hamiltonians, exemplified in the definition of RSOS models in this path space formulation. In particular we give the order parameters for the RSOS model.

2 The theory of coloured graphs on S^2 revisited

In this section we briefly review the theory of coloured graphs on the boundary ∂M of a compact oriented 3-manifold M given in terms of the $6j$-symbols and the R-matrix of the quantum group $Sl_q(2,\mathbb{C})$ at roots of unity ($q = \exp(i\pi/r)$, $r \in \mathbb{N}+2$) (see [9]). We will only need the case $M = D^3$ (the unit ball in $\mathbb{R}^3$) such that $\partial M = S^2$ (the unit disc). Let $|G|$ be the topological space associated to a 1-dimensional simplicial complex G (see e.g. [17]). By assumption on G, every vertex $\sigma^0 \in G$ is contained in the boundary of $n = n(\sigma^0)$ 1-simplexes with $3 \le n \le 4$ and we will say that σ^0 is an n-vertex. Again by assumption every 4-vertex σ^0 is given an additional structure by pairing the 4 1-simplexes meeting at σ^0 into two unordered pairs. The 1-simplexes in an pair are called opposite to each other. In addition one of the pairs is given the name "above" and the other pair the name "below" as depicted in (1.1). We say that the pair "below" undercrosses the pair "above".

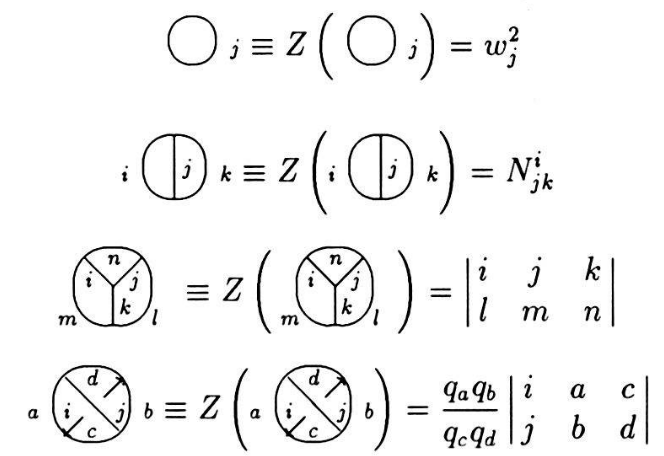


(2.1)

By abuse of notation we continue to denote by $|G|$ this topological space with this additional structure. By definition a coloured graph $|G|_{\underline{x}}$ is such a space $|G|$ and a map $\underline{x} : \sigma^1 \mapsto x(\sigma^1)$ from the set of nonoriented 1-simplexes in G into the set $\mathcal{I} = \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{r}{2} - 1\}$ ($r \in \mathbb{N}+2$) with the following property: If two 1-simplexes σ_1^1 and σ_2^1 are opposite to each other at a 4-vertex then $x(\sigma_1^1) = x(\sigma_2^1)$ (compare Fig. (2.1)).

By definition a coloured graph on S^2 is a pair $(|G|_{\underline{x}}, \varphi)$ where φ is a homeomorphism of $|G|$ into S^2 with the following local property near a 4-vertex σ^0. In a neighbourhood in S^2 of $\varphi(\sigma^0)$, the images of the two open opposite 1-simplexes in one pair are separated by the images of the closed 1-simplexes in the other pair (compare again (2.1)). Two coloured graphs $(|G|_{\underline{x}}, \varphi)$ and $(|G|_{\underline{x}}, \varphi')$ are called homotopic if there is a homotopy φ_t $(0 \le t \le 1)$ of the maps φ and φ' such that $(|G|_{\underline{x}}, \varphi_t)$ is a coloured graph for all $0 \le t \le 1$. Using the $6j$-symbols and the R-matrices of the quantum group $Sl_q(2,\mathbb{C})$ at roots of unity ($q = \exp(i\pi/r)$, $r \in \mathbb{N}+2$) and by a generalization of the state sum of Turaev and Viro [10] one may associate complex numbers $Z(|G|_{\underline{x}}, \varphi)$ to coloured graphs $(|G|_{\underline{x}}, \varphi)$ on S^2.

These numbers are homotopy invariant. To simplify notation we will identify $|G|$ with its image $\varphi(|G|)$ in S^2 and we will call $|G|$ a planar graph. Furthermore we will even use the symbol $|G|_{\underline{\mathcal{I}}}$ to denote its state sum. Thus one has in particular



$$= w_j^2 \tag{2.2}$$

$$= N^i_{jk} \tag{2.3}$$

$$= \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \tag{2.4}$$

$$= \frac{q_a q_b}{q_c q_d} \begin{vmatrix} i & a & c \\ j & b & d \end{vmatrix} \tag{2.5}$$

Here w_j^2 is up to a sign the q-dimension

$$w_j^2 = (2j+1)_{-q} = (-1)^{2j} d_q(j) = (-1)^{2j} \frac{\sin \frac{\pi}{r}(2j+1)}{\sin \frac{\pi}{r}}. \tag{2.6}$$

For later purpose we make the convention that $w_j = i^{2j}|w_j|$. Hence the w_j are real for integer j and purely imaginary for half-integer j.

The fusion matrix is for $i, j, k \in \mathcal{I}$)

$$N^i_{jk} = \begin{cases} 1 & \text{if } k \leq i+j,\ j \leq i+k,\ i \leq j+k,\ r-2 \geq i+j+k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

The expression (2.4) is the $6j$-symbol and (2.5) is the R-matrix (see [18]) with

$$q_a = (-1)^a q^{a(a+1)} \quad (a \in \mathcal{I}). \tag{2.8}$$

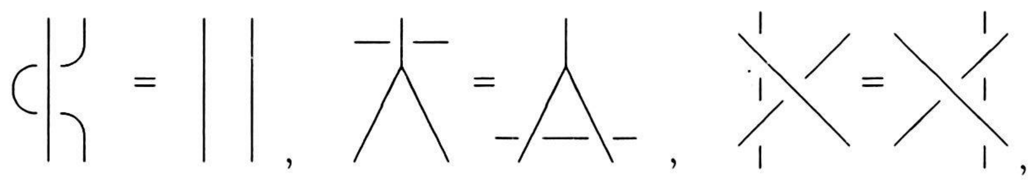
The $6j$-symbol is normalized such that it has the symmetry

$$\begin{vmatrix} i & j & k \\ \ell & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & \ell & n \end{vmatrix} = \begin{vmatrix} i & k & j \\ \ell & n & m \end{vmatrix} = \begin{vmatrix} i & m & n \\ \ell & j & k \end{vmatrix} \tag{2.9}$$

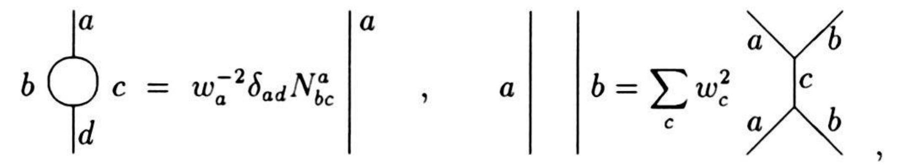
and is nonvanishing exactly when $N^i_{jk} = N^i_{mn} = N^j_{\ell n} = N^k_{\ell m} = 1$. One has

$$\sum_c N^a_{bc} w_c^2 = w_a^2 w_b^2. \tag{2.10}$$

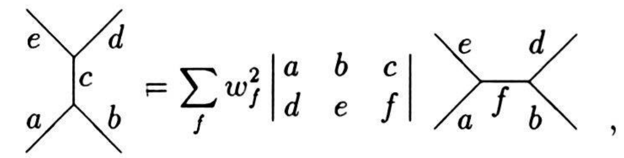
With respect to the state sum the following local rules are valid:



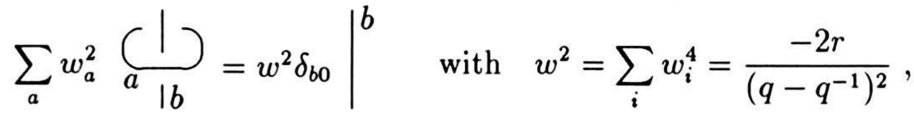
(2.11)



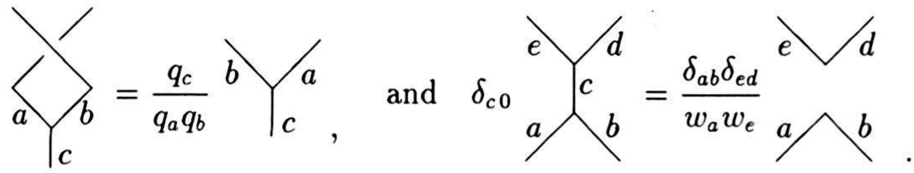
(2.12)



(2.13)

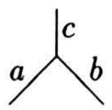


(2.14)



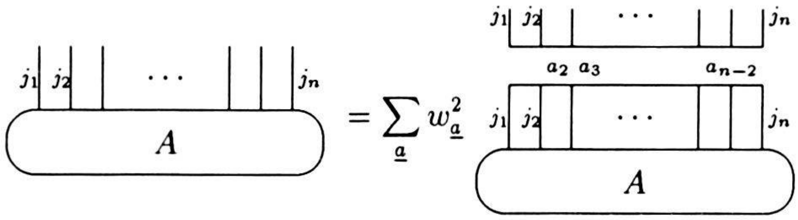
(2.15)

Also for a local colouring near a 3-vertex of the form



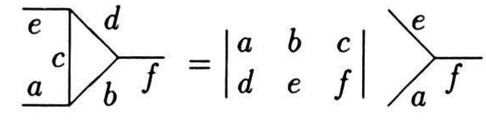
(2.16)

one has the fusion rule in the sense that the partition function vanishes unless $N^a_{bc} = 1$. Finally for a local part A of $|G|_{\underline{x}}$ with n "external legs" of colours $j_1, \ldots, j_n$ one has a Wigner-Eckhart theorem in the form



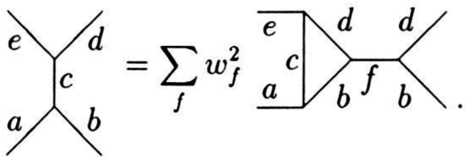
(2.17)

with the weight factor $w_{\underline{a}}^2 = \prod_{\nu=2}^{n-2} w_{a_\nu}^2$. For $n = 2$ one has $j_1 = j_2$ and the weight factor is $w_{j_1}^{-2}$. For $n = 1$ the single colour j_1 has to be zero. Note that the first relation in (2.12) may be derived from (2.17) and (2.3). Furthermore we have



(2.18)

such that (2.13) may be rewritten as

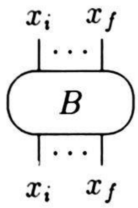


(2.19)

Note that (2.19) alternatively follows from the second relation in (2.12). Relation (2.13) is called a Fierz transformation in the present context.

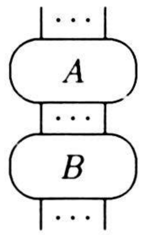
3 The local observable algebra

In this section we will associate to each interval $I = \{x_i, x_i+1, \ldots, x_f = x_i+|I|-1\}$ (x_i= initial point, x_f=final point), of $|I|$ lattice points in $\mathbb{Z}$ a finite dimensional $*$-algebra $\mathcal{A}_I$ which will be a subalgebra of $\mathrm{End}(V_I)$, where V_I is a finite dimensional vector space over $\mathbb{C}$. $\mathcal{A}_I$ will be the linear space spanned by elements A whose matrix elements will be given in terms of (partition functions of) certain coloured graphs. First we recall some well knows fact on the braid group. Let B be a braid in the braid group $\mathfrak{B}_{|I|}$ of $|I|$ elements. This means B is a graph containing only 4-vertices (in the sense of (2.1) and satisfying the rules (2.11)) with $2|I|$ external legs whose colours are all 1/2, where $|I|$ of these legs point upward and $|I|$ legs point downward. We depict this as follows

$B =$ 

(3.1)

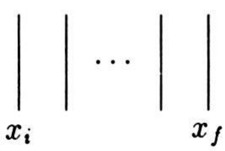
Unless stated otherwise, in what follows vertical legs will always carry colour 1/2 (they are depicted by thin lines whereas lines of arbitrary colour are depicted by thick lines). The multiplication of two braids is given putting them on top of each other

$AB =$ 

. (3.2)

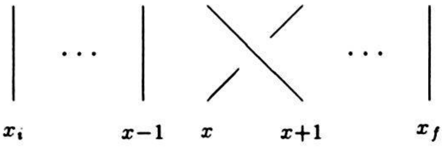
Definition 3.1 *The linear space spanned by these braids (with coefficients in $\mathbb{C}$) is the braid group algebra $\mathcal{B}_I$*

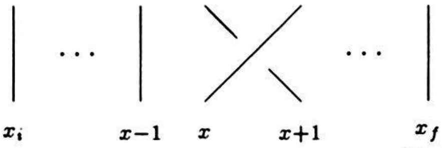
Lemma 3.2 *The braid group algebra $\mathcal{B}_I$ is generated by the unit braid*

$\mathbf{1}_I =$ 

(3.3)

and the simple braids

$r_x^I =$ 

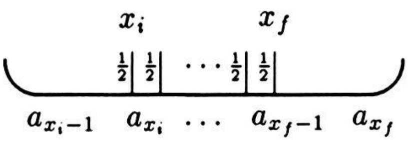
and $(r_x^I)^{-1} =$ 

(3.4)

with $x_i \leq x \leq x_f - 1$.

Next we introduce the path space representations of the braid group algebra $\mathcal{B}_I$. We start with a construction of vector spaces V_I for any interval I.

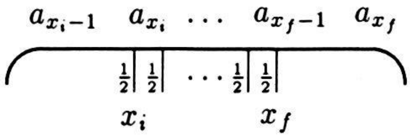
Definition 3.3 *The basis of the path space V_I is given by the symbols $|\underline{a}\rangle$ with the path $\underline{a} = (a_{x_i-1}, a_{x_i}, \ldots a_{x_f})$ $(a_x \in \mathcal{I})$ such that the fusion rule $N^{\frac{1}{2}}_{a_x a_{x+1}} = 1$ $(x_i - 1 \leq x < x_f)$ holds. We depict this graphically as*

$|\underline{a}\rangle = |a_{x_i-1}, \ldots, a_{x_f}\rangle = w_{\underline{a}}$  (3.5)

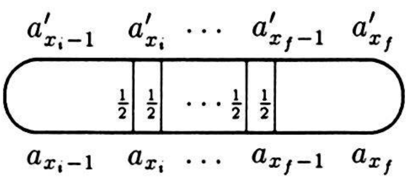
where

$$w_{\underline{a}} = \frac{1}{w_{a_{x_f}}} \prod_{x=x_i}^{x_f} w_{a_x}. \tag{3.6}$$

This is the path space version of the usual tensor product $\mathbb{C}^{\otimes 2|I|}$ in the context of lattice models in statistical mechanics (see e.g. [19]). (For later purpose we also allow $I = \emptyset$, with $\underline{a} = (a)$ and $w_{\underline{a}} = 1/w_a$.) The "bra"-vector corresponding to (3.5) is depicted by

$\langle \underline{a}| = \langle a_{x_i-1}, \ldots, a_{x_f}| = w_{\underline{a}}$  (3.7)

and by definition the pairing is given in terms of a graph as

$\langle \underline{a}'|\underline{a}\rangle = w_{\underline{a}}^2$  $= \delta_{\underline{a}'\underline{a}} = \prod_{x=x_i-1}^{x_f} \delta_{a'_x a_x}.$ (3.8)

where the second relation follows from iterative application of (2.12). This allows us to introduce a scalar product on V_I by

$$\langle v'|v\rangle = \sum_{\underline{a}} \bar{v}'_{\underline{a}} v_{\underline{a}} \tag{3.9}$$

for $|v\rangle = \sum_{\underline{a}} |\underline{a}\rangle v_{\underline{a}}$ and $|v'\rangle = \sum_{\underline{a}} |\underline{a}\rangle v'_{\underline{a}}$ making V_I a finite dimensional Hilbert space. These spaces V_I form the ingredients for our path space approach and replace the tensor product stuctures usually employed in lattice models (see e.g. [19], [1]). In analogy to the usual tensor product, however, there is a bilinear map $\bigcirc : V_{I_1} \times V_{I_2} \to V_{I_1 \cup I_2}$ written as $v_1 \times v_2 \mapsto v_1 \bigcirc v_2$ ($v_i \in V_{I_i}$, $i = 1,2$) and defined on the basis vectors as follows. For

$$\begin{aligned} |\underline{a}_1\rangle &= |a_{x_i^1-1}, \ldots, a_{x_f^1}\rangle \in V_{I_1} \\ |\underline{a}_2\rangle &= |a_{x_i^2-1}, \ldots, a_{x_f^2}\rangle \in V_{I_2} \end{aligned} \tag{3.10}$$

with $x_f^1 = x_i^2 - 1$ and hence $a_{x_f^1} = a_{x_i^2-1}$ we set

$$|\underline{a}_1 \bigcirc \underline{a}_2\rangle = |a_{x_i^1-1}, \ldots, a_{x_f^1} = a_{x_i^2-1}, \ldots, a_{x_f^2}\rangle \in V_{I_1 \cup I_2} \tag{3.11}$$

if $a_{x_f^1} = a^2_{x_i-1}$ and $|\underline{a}_1 \bigcirc \underline{a}_2\rangle = 0$ otherwise. Note that in contrast to the tensor product set-up we have $\dim(V_{I_1})\dim(V_{I_2}) > \dim(V_{I_1 \cup I_2})$. This inequality is related to the fact that the tensor product of two "good" representations of $sl_q(2,\mathbb{C})$ contain "bad" representations. This is also visible in the fact that

$$\langle \underline{a}'_1 \bigcirc \underline{a}'_2 | \underline{a}_1 \bigcirc \underline{a}_2\rangle = \delta_{a'_{x_f^1}, a_{x_f^1}} \langle \underline{a}'_1 | \underline{a}_1\rangle \langle \underline{a}'_2 | \underline{a}_2\rangle \ . \tag{3.12}$$

Concerning the dimension of the spaces we have

$$\dim V_I = \sum_{a,b} (N^{1/2})^{|I|}_{ab}. \tag{3.13}$$

In particular with $\tilde{I} = I \cup \{x_f + 1\}$

$$\dim V_{\tilde{I}} < 2 \dim V_I \tag{3.14}$$

such that by complete induction

$$\dim V_I < \text{const.}\, 2^{|I|} \tag{3.15}$$

for all I. We can even say more. Since the Verlinde matrix S diagonalizes each fusion matrix N^l with eigenvalues S_{ld}/S_{0d}, we have (see e.g. [9])

$$\dim V_I = \sum_{a,b,d} \left(\frac{S_{1/2d}}{S_{0d}}\right)^{|I|} S_{ad} S_{bd} = \sum_d \left(\frac{S_{1/2d}}{S_{0d}}\right)^{|I|} f_d^2 \tag{3.16}$$

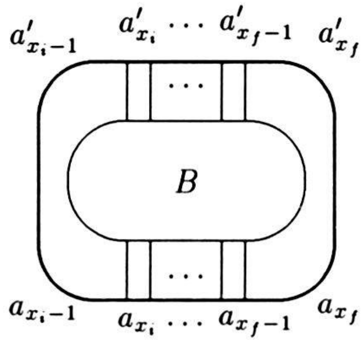
with $f_d = \sum_a S_{ad}$. Since the largest absolute value of the eigenvalues of $N^{1/2}$ is $d_q(1/2)$ and since $f_{1/2} \neq 0$ we obtain

$$\lim_{\substack{|I| \to \infty \\ |I| \text{ even}}} \ln \dim V_I = \ln d_q(1/2) < \ln 2. \tag{3.17}$$

Note that the corresponding number in the tensor product formulation is $\ln 2$, when the space at each lattice site is $\mathbb{C}^2$. The above number may be interpreted as the entropy per site at temperature $T = \infty$ (see also [20]).

To each braid $B \in \mathcal{B}_I$ we associate an element of $\mathrm{End}(V_I)$ also denoted by B. Its matrix elements w.r.t. the basis $|\underline{a}\rangle$ (i.e. $B|\underline{a}\rangle = \sum_{\underline{a}'} |\underline{a}'\rangle\langle\underline{a}'|B|\underline{a}\rangle$) are given as

$$\langle\underline{a}'|B|\underline{a}\rangle = w_{\underline{a}} w_{\underline{a}'} \tag{3.18}$$



where $a_{x_i-1} = a'_{x_i-1}, a_{x_f} = a'_{x_f}$.

Definition 3.4 *The subalgebra $\mathcal{A}_I \subset \mathrm{End}(V_I)$ spanned by all graphs of the form (3.18) is the algebra of local observables.*

By the Wigner-Eckart theorem (2.17) the matrix multiplication of the elements in $\mathcal{A}_I$ as transformations on the vector space V_I is compatible with the multiplication in the braid group algebra $\mathcal{B}_I$.

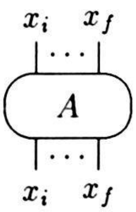
Lemma 3.5 *The algebra $\mathcal{A}_I \subset \mathrm{End}(V_I)$ yields a representation of the braid group algebra $\mathcal{B}_I$*

$$\pi_I : \mathcal{B}_I \to \mathcal{A}_I \ . \tag{3.19}$$

Since we do not associate a numerical value to an open graph, in the following all relations containing open graphs in $\mathcal{B}_I$ have to be understood in terms of relations in $\mathcal{A}_I$ via eq. (3.18).

Let A be an arbitrary planar graph with $2|I|$ external legs whose colours are all $1/2$ (note that the colour $1/2$ corresponds to the fundamental representation of $Sl_q(2,\mathbb{C})$). We assume $|I|$ of these legs point upward and that $|I|$ legs point downward and depict this also as follows

$$A = \tag{3.20}$$

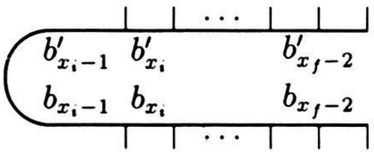


Lemma 3.6 *Such graph A can be written as a linear combination of braids. Therefore we may identify the linear space spanned by all such planar graphs A with the braid group algebra $\mathcal{B}_I$.*

The proof of this lemma will be given in Appendix A. Note that $\mathcal{B}_I \cong \mathbb{C}$ if $|I| = 1$.

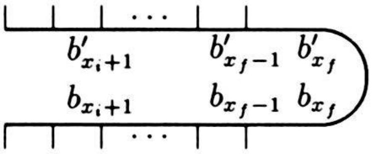
Theorem 3.7 *$\mathcal{A}_I$ is spanned by the coloured graphs of the form*

$$e^{I,l}_{\underline{b}'\underline{b}} = \Big(\prod_{x=x_i-1}^{x_f-2} w_{b'_x} w_{b_x} \Big) \tag{3.21}$$



with $\underline{b} = (b_{x_{i-1}}, b_{x_i}, \ldots, b_{x_{f-1}})$, $\underline{b}' = (b'_{x_{i-1}}, b_{x_i}, \ldots, b_{x_{f-1}})$ and $b'_{x_i-1} = b_{x_i-1}$ or alternatively by the elements of the form

$$e^{I,r}_{\underline{b}'\underline{b}} = \Big(\prod_{x=x_i+1}^{x_f} w_{b'_x} w_{b_x} \Big) \tag{3.22}$$



with $b'_{x_f} = b_{x_f}$. We have the orthogonality relations

$$e^{I,r}_{\underline{b}'\underline{b}} e^{I,r}_{\underline{b}'''\underline{b}''} = \delta_{\underline{b},\underline{b}'''}\, e^{I,r}_{\underline{b}'\underline{b}''} \tag{3.23}$$

and the completeness relation

$$\sum_{\underline{b}} e^{I,r}_{\underline{b}\underline{b}} = \mathbf{1}_I\ . \tag{3.24}$$

Similar relations are valid for $e^{I,l}_{\underline{b}'\underline{b}}$. The center of $\mathcal{A}_I$ is spanned by the minimal projectors

$$p^I_j = \sum_{\substack{\underline{b} \\ b_{x_f}=j}} e^{I,r}_{\underline{b}\underline{b}} = \sum_{\substack{\underline{b} \\ b_{x_i-1}=j}} e^{I,l}_{\underline{b}\underline{b}} \tag{3.25}$$

such that $\mathcal{A}_I$ decomposes as

$$\mathcal{A}_I = \bigoplus_j \mathcal{A}_{I,j}\ , \quad \text{where} \quad \mathcal{A}_{I,j} = p^I_j \mathcal{A}_I\ . \tag{3.26}$$

The proof is a trivial consequence of the Wigner-Eckhart theorem. Note that the factors in front of eqs. (3.21) and (3.22) are real by the fusion rules and eq. (2.6).

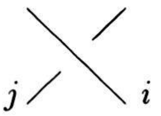
Example 3.8 *The matrix elements of $e^{I,r}_{\underline{b}'\underline{b}}$ are given by*

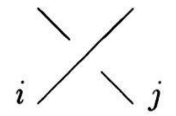
$$\langle \underline{a}'|e^{I,r}_{\underline{b}'\underline{b}}|\underline{a}\rangle = w_{\underline{a}'} w_{\underline{a}} \prod_{x=x_i+1}^{x_f} w_{b'_x} w_{b_x} \begin{vmatrix} a_{x_1} & a_{x-1} & b_{x-1} \\ \frac{1}{2} & b_x & a_x \end{vmatrix} \begin{vmatrix} a'_{x_1} & a'_{x-1} & b'_{x-1} \\ \frac{1}{2} & b'_x & a'_x \end{vmatrix} \tag{3.27}$$

with $b_{x_i} = \frac{1}{2} = b'_{x_i}$. *The matrix elements of the elementary braids* r_x^I *are obtained from (2.5) as*

$$\langle \underline{a}' | r_x^I | \underline{a} \rangle = \Big(\prod_{\substack{y=x_i-1 \\ y \neq x}}^{x_f} \delta_{a_y, a'_y} \Big) w_{a'_x} w_{a_x} \frac{q_{a_{x-1}} q_{a_{x+1}}}{q_{a_x} q_{a'x}} \begin{vmatrix} 1/2 & a_{x-1} & a_x \\ 1/2 & a_{x+1} & a'_x \end{vmatrix}. \qquad (3.28)$$

The algebra $\mathcal{A}_I$ is also a $*$-algebra, whose involution of a graph A in $\mathcal{A}_I$ by definition is given by associating to A^* the graph obtained by mirroring the graph A along an arbitrary horizontal axis, such that for example a 4-vertex

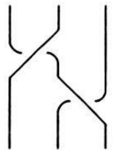
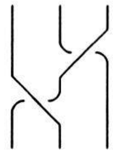


is mirrored into


and vice versa. (3.29)

Extending this antilinearly to all of $\mathcal{A}_I$, it is easy to see that with respect to the scalar product on V_I introduced above, A^* is indeed the adjoint of A.

Example 3.9 ($|I| = 3$)*:*

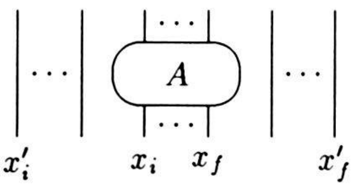
$*:$  $\mapsto$  (3.30)

Obviously, we have $e_{\underline{b}'\underline{b}}^{I,r^*} = e_{\underline{b}\underline{b}'}^{I,r}$ and by definition $(r_x^I)^{-1} = (r_x^I)^*$. In particular the algebra $\mathcal{A}_I \subseteq \mathrm{End}(V_I)$ defines a $*$-representations of $\mathcal{A}_I$ and also of the group algebra of the braid group $\mathfrak{B}_{|I|}$ for $|I|$ elements.

Remark 3.10 *It is easy to see that* $B \in \mathfrak{B}_{|I|} \subset \mathcal{B}_I$ *given by a braid is unitary* $BB^* = \mathbf{1}_I$. *Hence we have a unitary representation of the braid group.*

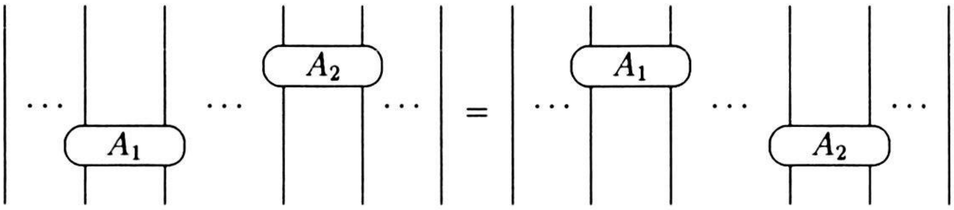
Moreover, $\mathcal{A}_I$ is also a C^*-algebra, since the identity transformation $\mathbf{1}_I$ of $\mathrm{End}(V_I)$ belongs to $\mathcal{A}_I$ and is given in terms of the graph (3.3). In particular (see e.g. [7] proposition II.1) $\mathcal{A}_I$ is (isomorphic to) a direct sum of full matrix algebras. We will characterize these full matrix algebras in Section 4.

In the following part of this section we analyse the inclusion properties of the local observable algebra under enlarging the lattice I. On the braid group algebras we now define canonical injective $*$-homomorphisms $\iota_{I',I} : \mathcal{A}_I \to \mathcal{A}_{I'}$ whenever $I \subseteq I'$ and satisfying $\iota_{I'',I'} \circ \iota_{I',I} = \iota_{I'',I}$ for $I \subseteq I' \subseteq I''$ and $\iota_{I,I} = id_{\mathcal{A}_I}$ as follows. With $I' = \{x'_i, \ldots, x'_f\}$ such that $x'_i \leq x_i$ and $x_f \leq x'_f$ and for a graph $A \in \mathcal{A}_I$ (and extended by linearity to all of $\mathcal{A}_I$) we let

$\iota_{I',I}(A) =$ 

(3.31)

i.e. $\iota_{I',I}(A)$ is A with $x_i - x_i'$ vertical lines with colour $1/2$ added to the left and $x_f' - x_f$ vertical lines with colour $1/2$ added to the right. If we denote the corresponding maps on the observable algebras by the same letter, then in particular the maps $\iota_{I',I}$ are isometric, when the algebras $\mathcal{A}_I$ are viewed as C^*-algebras (see e.g. [7], prop. II.4). With this construction we have that $\mathcal{A}_{I_1}$ and $\mathcal{A}_{I_2}$ commute whenever I_1 and I_2 are disjoint in the sense that $\iota_{I',I_1}(\mathcal{A}_{I_1})$ and $\iota_{I',I_2}(\mathcal{A}_{I_2})$ are commuting subalgebras of $\mathcal{A}_{I'}$ for any $I' \supset I_1 \cup I_2$. For graphs $A_1 \in \mathcal{A}_{I_1}$ and $A_2 \in \mathcal{A}_{I_2}$ this is written pictorially as (I_1 to the left of I_2)

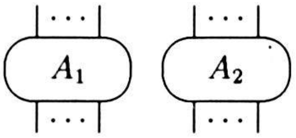


(3.32)

For later purposes we will rewrite the construction (3.31) in another way. Write $I' = I_\ell' \cup I \cup I_r'$ with $I_\ell' = \{x_i', \dots, x_i - 1\}$, $I_r' = \{x_f + 1, \dots, x_f'\}$. Then we write (3.31) as

$$\iota_{I',I}(A) = \mathbf{1}_{I_\ell'} \circ A \circ \mathbf{1}_{I_r'} \tag{3.33}$$

and more generally we introduce a multiplication $\circ : \mathcal{A}_{I_1} \times \mathcal{A}_{I_2} \to \mathcal{A}_{I_1 \cup I_2}$, with I_1 and I_2 forming neighboring intervals, on graphs $A_i \in \mathcal{A}_{I_i}$ $(i = 1, 2)$ and extending bilinearly as

$A_1 \circ A_2 =$ 

. (3.34)

This operation " $\circ$ " replaces the tensor product of operators and is compatible with the corresponding operation on the vector spaces V_I. In fact by the Wigner-Eckhart theorem we have

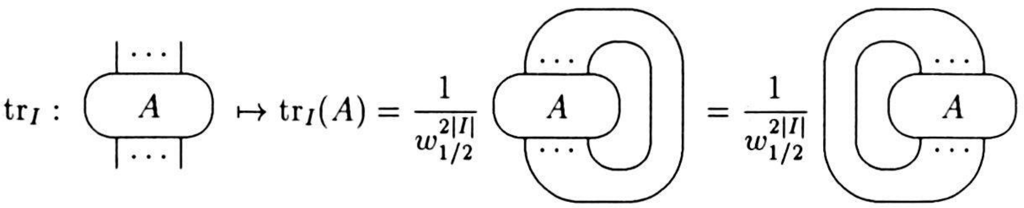
$$\langle \underline{a}_1' \circ \underline{a}_2' | A_1 \circ A_2 | \underline{a}_1 \circ \underline{a}_2 \rangle = \delta_{a'^1_{x_f}, a^1_{x_f}} \langle \underline{a}_1' | A_1 | \underline{a}_1 \rangle \langle \underline{a}_2' | A_2 | \underline{a}_2 \rangle \tag{3.35}$$

for $|\underline{a}_i\rangle, |\underline{a}_i'\rangle \in V_{I_i}$ $(i = 1, 2)$, generalizing eq. (3.12). Obviously the product $\circ$ is associative, compatible with the $*$-operation and

$$(A_1 B_1) \circ (A_2 B_2) = (A_1 \circ A_2)(B_1 \circ B_2) \tag{3.36}$$

holds for $A_1, B_1 \in \mathcal{A}_{I_1}$ and $A_2, B_2 \in \mathcal{A}_{I_2}$. This tensor product construction in the path space picture is of course well known ([15]) using the juxtaposition of braids.

We also note there is a trace tr_I on $\mathcal{A}_I$, given on graphs by



(3.37)

with $\mathrm{tr}_I(\mathbf{1}_I) = 1$. Note that the second relation in eq. (3.37) follows from the homotopy invariance of the partition functions of such coloured graphs on S^2. More explicitly

$$\mathrm{tr}_I(A) = \frac{1}{w^2 w_{1/2}^{2|I|}} \sum_{\underline{a}} w_{a_{x_i-1}}^2 w_{a_{x_f}}^2 \langle \underline{a}|A|\underline{a}\rangle \ , \quad A \in \mathcal{A}_I. \tag{3.38}$$

This trace is faithful, i.e. the quadratic form defined by the trace via $(A, B) = \mathrm{tr}_I(AB)$ $(A, B \in \mathcal{A}_I)$ is non degenerate. This follows easily from eq. (3.23) and the fact that

$$\mathrm{tr}_I(e_{\underline{b}',\underline{b}}^{I,r}) = \frac{w_{b_{x_f}}^2}{w_{1/2}^{2|I|}} \delta_{\underline{b}'\underline{b}} \ . \tag{3.39}$$

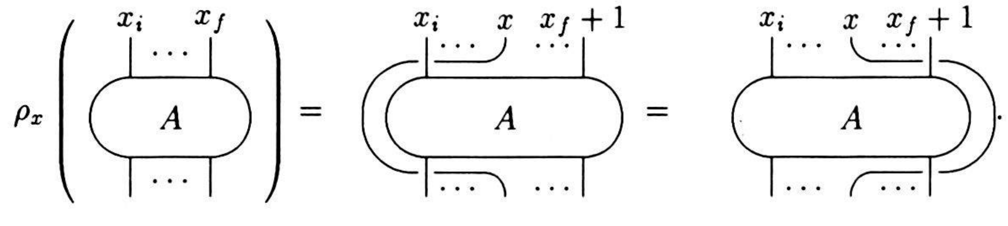
Note that by eq. (2.6) and the fusion rules $w_{a_{x_i-1}}^2 w_{a_{x_f}}^2 / w_{1/2}^{2|I|} > 0$. Therefore we have that $\mathrm{tr}_I(A) \geq 0$ for $A \geq 0$ with strict inequality if $A \neq 0$ since $\mathrm{tr}_I(\cdot)$ is faithful.

Note that we have compatibility of these local traces in the form

$$\mathrm{tr}_{I'}(\iota_{I',I}(A)) = \mathrm{tr}_I(A) \tag{3.40}$$

for all $A \in \mathcal{A}_I$ and $I \subseteq I'$.

Motivated by the concept of superselection sectors in algebraic quantum field theory we turn to a construction of certain local endomorphisms. They will be denoted by ρ_x $(x \in \mathbb{Z})$ and will be maps from $\mathcal{A}_I$ into $\mathcal{A}_{\tilde{I}}$ with $\tilde{I} = I \cup \{x_f + 1\}$. The construction of ρ_x goes as follows. If $x \notin I$ we set $\rho_x(A) = A \circ \mathbf{1}_{\{x_f+1\}} = \iota_{\tilde{I},I}(A)$. If $x \in I$ we define ρ_x on graphs $A \in \mathcal{A}_I$ (and extend by linearity to all of $\mathcal{A}_I$) by



(3.41)

Note that if $x = x_f + 1$ then

$$\rho_x(A) = A \circ \mathbf{1}_{\{x\}} \ .$$

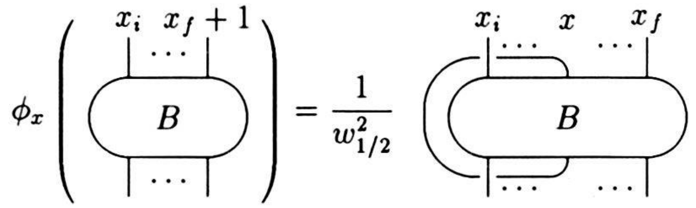
Obviously the first relation in (2.11) implies

$$\rho_x(\mathbf{1}_I) = \mathbf{1}_{\tilde{I}} \quad \text{and} \quad \rho_x(AB) = \rho_x(A)\rho_x(B). \tag{3.42}$$

Also one has

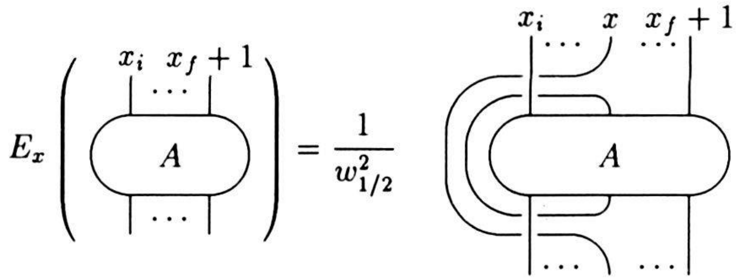
$$\rho_x(A^*) = \rho_x(A)^* \tag{3.43}$$

such that ρ_x is a $*$-homomorphism. The map ρ_x has a left inverse $\phi_x : \mathcal{A}_{\bar{I}} \to \mathcal{A}_I$ given by a partial trace in the form



(3.44)

on graphs $B \in \mathcal{A}_{\bar{I}}$. Indeed, it follows directly from eqs. (2.2) and (2.11) that $\phi_x(\rho_x(A)) = A$ for all $A \in \mathcal{A}_I$. Since ϕ_x is a left inverse or ρ_x the operator $E_x = \rho_x \circ \phi_x : \mathcal{A}_{\bar{I}} \to \mathcal{A}_{\bar{I}}$ defines a conditional expectation on each $\mathcal{A}_{\bar{I}}$ with range $\rho_x(\mathcal{A}_I)$. More precisely on graphs $A \in \mathcal{A}_{\bar{I}}$ $E_x(A)$ takes the form



(3.45)

and it fulfills for $A, C \in \rho_x(\mathcal{A}_I)$ and $B \in \mathcal{A}_{\bar{I}}$

$$E_x(ABC) = AE_x(B)C \ . \tag{3.46}$$

E_x is compatible with the trace $\mathrm{tr}_{\bar{I}}$ in the sense that

$$\mathrm{tr}_{\bar{I}}(E_x(A)) = \mathrm{tr}_{\bar{I}}(A) \tag{3.47}$$

holds for all $A \in \mathcal{A}_{\bar{I}}$.

Again by the braiding relations (2.11) for any graph $A \in \mathcal{A}_I$ and $x \in I$ we also have

$$\rho_x(A) = (r^{\bar{I}}_{x,[x+1,x_f]})^{-1}\ \iota_{\bar{I},I}(A)\ r^{\bar{I}}_{x,[x+1,x_f]} = (r^{\bar{I}}_{x,[x+1,x_f]})^{-1}\ (A \bigcirc \mathbf{1}_{\{x_f+1\}})\ r^{\bar{I}}_{x,[x+1,x_f]} \tag{3.48}$$

with

$$r^{\bar{I}}_{x,[x+1,x_f]} = \begin{cases} r^{\bar{I}}_x r^{\bar{I}}_{x+1} \cdots r^{\bar{I}}_{x_f} & \text{if } x_i \leq x < x_f \\ \mathbf{1}_{\bar{I}} & \text{if } x = x_f + 1 \end{cases} \tag{3.49}$$

such that $r^{\bar{I}}_{x,[x+1,x_f]} \in \mathcal{B}_{|I|} \subset \mathcal{A}_{\bar{I}}$ is unitary for $x \in I$.

Theorem 3.11 *For all I with $|I| \geq r-2$ and all $x \in \mathbb{Z}$ the Jones index of the inclusion $\rho_x(\mathcal{A}_I) \subset \mathcal{A}_{\bar{I}}$ of finite dimensional C^*-algebras satisfies*

$$\left[\ \mathcal{A}_{\bar{I}} : \rho_x(\mathcal{A}_I)\ \right] = \left[\ \mathcal{A}_{\bar{I}} : \iota_{\bar{I},I}(\mathcal{A}_I)\ \right] = w^4_{1/2} = d^2_q(\tfrac{1}{2}). \tag{3.50}$$

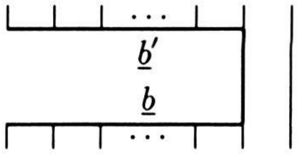
Also $\mathrm{tr}_{\bar{I}}$ is a Markov trace for this inclusion.

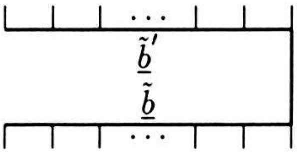
Proof: The first equality follows from the general theory of the Jones index for multimatrix algebras, since by (3.48) and (3.49) $\rho_x(\mathcal{A}_I)$ and $\mathcal{A}_I \circ \mathbf{1}_{\{x_f+1\}}$ are related by an inner automorphism in $\mathcal{A}_{\bar{I}}$. We present two proofs of the second equality.

a) We determine the inclusion matrix and consider the projectors (3.25)

$$p^I_j = \sum_{\substack{\underline{b} \\ b_{x_f}=j}} e^I_r(\underline{b}, \underline{b})$$

which are either zero or minimal central idempotents of $\mathcal{A}_I$ such that the $\mathcal{A}_{I,j} = p^I_j \mathcal{A}_I$ are either zero or full matrix algebras for all $j \in \mathcal{I}$. If $|I| \geq r-2$ then for all $j \in \mathcal{I}$ they are nonvanishing exactly when $2j + |I|$ is even. This follows easily from the fusion rules. Furthermore by eq. (2.12)

$$e^{I,r}_{\underline{b}'\underline{b}} \circ \mathbf{1}_{\{x_f+1\}} = \prod_{x=x_i+1}^{x_f} w_{b'_x} w_{b_x}$$ 

$$= \prod_{x=x_i+1}^{x_f} w_{b'_x} w_{b_x} \sum_{\tilde{b}_{x_f+1}} N^{1/2}_{b_{x_f}\tilde{b}_{x_f+1}} w^2_{\tilde{b}_{x_f}}$$ 

$$= \sum_{\tilde{b}_{x_f+1}} N^{1/2}_{b_{x_f}\tilde{b}_{x_f+1}} e^{\bar{I},r}_{\underline{\tilde{b}},\underline{\tilde{b}}'} \tag{3.51}$$

with

$$\begin{aligned} \underline{\tilde{b}} &= (b_{x_i+1}, \ldots, b_{x_{f-1}}, b_{x_f}, \tilde{b}_{x_f+1}) \\ \underline{\tilde{b}}' &= (b'_{x_i+1}, \ldots, b'_{x_{f-1}}, b'_{x_f}, \tilde{b}_{x_f+1}). \end{aligned}$$

Since the minimal central idempotents of $\mathcal{A}_I \circ \mathbf{1}_{\{x_f+1\}}$ are the $q^{\bar{I}}_j = p^I_j \circ \mathbf{1}_{\{x_f+1\}}$, (with j as above) relation (3.51) shows that the inclusion matrix λ for the pair $\mathcal{A}_I \circ \mathbf{1}_{\{x_f+1\}} \subset \mathcal{A}_{\bar{I}}$ is given by ($j, \tilde{j} \in \mathcal{I}$, $2j + |I|$ even and $2\tilde{j} + |I|$ odd)

$$\lambda_{\tilde{j}j} = N^{\frac{1}{2}}_{\tilde{j}j} . \tag{3.52}$$

In fact both algebras $q^{\bar{I}}_j p^{\bar{I}}_{\tilde{j}} (\mathcal{A}_I \circ \mathbf{1}_{\{x_f+1\}}) q^{\bar{I}}_j p^{\bar{I}}_{\tilde{j}}$ and $q^{\bar{I}}_j p^{\bar{I}}_{\tilde{j}} \mathcal{A}_{\bar{I}} q^{\bar{I}}_j p^{\bar{I}}_{\tilde{j}}$ are equal and spanned by the elements $e^{\bar{I},r}_{\underline{\tilde{b}}'\underline{\tilde{b}}}$ with $b_{x_f} = j = b'_{x_f}$ and $\tilde{b}_{x_f+1} = \tilde{j} = \tilde{b}'_{x_f+1}$ when $N^{\frac{1}{2}}_{\tilde{j},j} = 1$. By rearranging the indices such that the integer indices come first and the halfinteger indices next when $|I|$ is odd (and vice versa if $|I|$ is even), we have $N^{\frac{1}{2}} = \begin{pmatrix} 0 & \lambda \\ \lambda^t & 0 \end{pmatrix}$. Now the index of the inclusion equals $\|\lambda\|^2 = \|\lambda^t\|^2 = \|N^{\frac{1}{2}}\|^2$ (see [7], prop. 1.2.4) and the claim follows since

$\|N^{\frac{1}{2}}\| = |w^2_{\frac{1}{2}}|$. In fact in e.g. [9] it was shown that $|w_a^2|$ is the largest eigenvalue with eigenvector $\{(-1)^{2c}w_c^2\}_{c\in\mathcal{I}}$ for the fusion matrix N^a (compare eq. (2.10)).

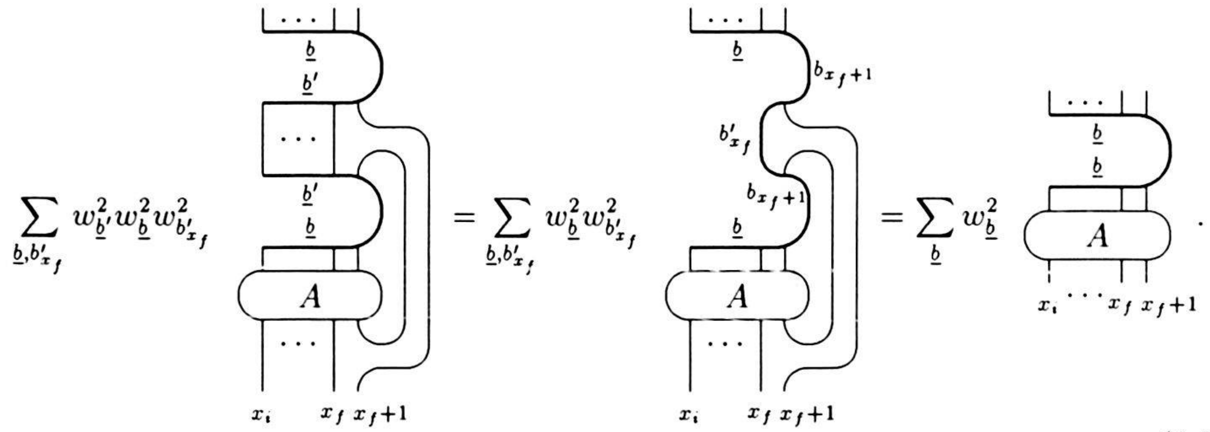
b) For the second proof we introduce a quasi basis (see e.g. [21]). We recall that in the context of C^*-algebras a quasibasis serves to introduce the notion of an index. In the context of type II_1 von Neumann algebras the analogue is the so called Pimsner-Popa basis [22] (see e.g. [7]). Therefore this proof will become important in Section 7 where we perform the thermodynamic limit $I \to \mathbb{Z}$. First we construct a quasi basis for the map $E_x : \mathcal{A}_{\tilde{I}} \to \mathcal{A}_{\tilde{I}}$ when $x = x_f + 1$ (and $|I| \geq r - 2$). We define

$$v^{\tilde{I}}_{x_f+1}(b'_{x_f}, \underline{b}) = w_{1/2} w_{b'_{x_f}} e^{\tilde{I},r}_{\underline{b}'\underline{b}}. \tag{3.53}$$

with a suitable but fixed $\underline{b}'$ satisfying the fusion rule $\prod_{x=x_i+1}^{x_f} N^{1/2}_{b'_x b'_{x+1}} = 1$. Then for any $\underline{b}$ satisfying a similar fusion rule and *any* $b_{x_f+1} = b'_{x_f+1}$ with $|\tilde{I}| + 2b_{x_f}$ even we have $v^{\tilde{I}}_{x_f+1}(b'_{x_f}, \underline{b}) \neq 0$. Now for any graph $A \in \mathcal{A}_{\tilde{I}}$ the following equality holds

$$\sum_{b'_{x_f},\underline{b}} (-1)^{|I|} (v^{\tilde{I}}_{x_f+1}(b'_{x_f}, \underline{b}))^* \, E_x\Big(v^{\tilde{I}}_{x_f+1}(b'_{x_f}, \underline{b})\, A\Big) = A \tag{3.54}$$

which follows from the graphical representation of its left hand side using eqs. (3.23) and (3.45) and the fact that $w_{1/2} w_{b'_{x_f}}$ equals its complex conjugate times $(-1)^{|I|}$



(3.55)

The relations (2.10) and (2.12) imply the first equality and the second one after summation over b'_{x_f} (see also (3.24)). Finally, we may sum over all $\underline{b}$ by using again (2.12) to obtain eq. (3.54) for all graphs A in $\mathcal{A}_{\tilde{I}}$ and hence by linearity for all $A \in \mathcal{A}_{\tilde{I}}$. Thus the $v^{\tilde{I}}_{x_f+1}(b'_{x_f}, \underline{b})$ and their adjoints (modulo a sign) form a quasibasis. On the other hand an easy calculation using again eqs. (2.10)and (3.24) shows that

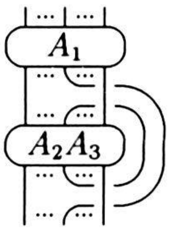
$$\sum_{b'_{x_f},\underline{b}} (-1)^{|I|} (v^{\tilde{I}}_{x_f+1}(b'_{x_f}, \underline{b}))^* v^{\tilde{I}}_{x_f+1}(b'_{x_f}, \underline{b}) = w^4_{1/2}\, \mathbf{1}_{\tilde{I}} \;. \tag{3.56}$$

To cover the general case $x_i < x \le x_f$ we set (see eq. (3.49))

$$v_x^{\bar I}(b'_{x_f},\underline{b}) = \left(r^{\bar I}_{x,[x+1,x_f]}\right)^{-1} v^{\bar I}_{x_f+1}(b'_{x_f},\underline{b})\; r^{\bar I}_{x,[x+1,x_f]} \tag{3.57}$$

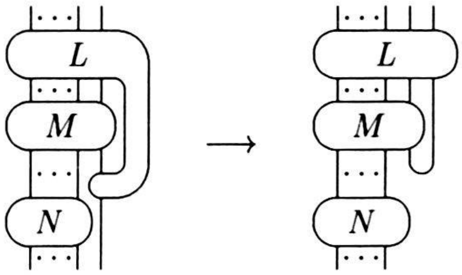
Then again these elements and their adjoints (modulo a sign) form a quasibasis with a relation analogous to eq. (3.56). It remains to prove that $\mathrm{tr}_{\bar I}$ is a Markov trace w.r.t. this inclusion. Let $\mathcal{A}_{\bar I,x}$ be the algebra obtained from the fundamental construction, i.e. the algebra of endomorphisms of $\mathcal{A}_{\bar I}$, viewed as a right $\rho_x(\mathcal{A}_I)$ module. Then $\mathcal{A}_{\bar I,x}$ is spanned linearly by elements of the form $A_1 E_x A_2$ where A_1 and A_2 are graphs in $\mathcal{A}_{\bar I}$, which we may give the graphical representation in the form (see (3.45))

$$A_1 E_x A_2(A_3) = A_1 E_x (A_2 A_3) = \tag{3.58}$$



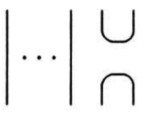
Jones' fundamental construction and the Markov trace may be demonstrated in the context of this section very nicely. For simplicity we take again $x = x_f + 1$ and write $\mathcal{A}_I = N$, $\mathcal{A}_{\bar I} = M$, $\mathcal{A}_{\bar I,x} = \mathcal{A}_{\bar{\bar I}} = L$, i.e. N, M and L are algebras of observables on lattices of $|I|$, $|I|+1$ and $|I|+2$ sites, respectively. Then $N \subset M \subset L$ yields Jones' fundamental construction, i.e. the first step of a Jones tower. The construction $L \cong \mathrm{End}^r_N(M) \cong M \otimes_N M$ considered as a right N-module may be depicted as

$$L(MN) = L(M)N = \tag{3.59}$$



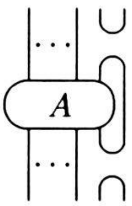
The conditional expectation $E = E_x$ is expressed by the Jones projector

$$e = \frac{1}{d} \quad \in L, \quad (d = w^2_{1/2}) \tag{3.60}$$



for $A \in M$ by

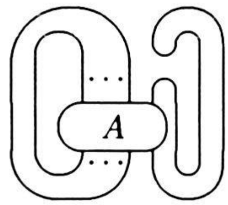
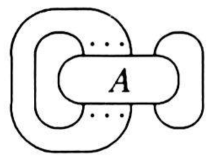
$$E(A)e = eE(A) = eAe = \frac{1}{d^2} \quad . \tag{3.61}$$



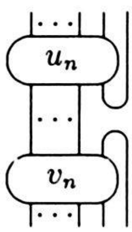
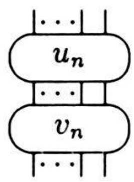
The trace tr_I defined by eq. (3.37) for any I gives the traces tr^N, tr^M and tr^L on N, M and L, respectively. This yields a Markov trace of modulus $[M : N] = d^2$ for the inclusion $N \subset M$, since for $A \in M$

$$\begin{aligned} \mathrm{tr}^L(A) &= \mathrm{tr}^M(A) \\ d^2\mathrm{tr}^L(eA) &= \mathrm{tr}^M(A). \end{aligned} \tag{3.62}$$

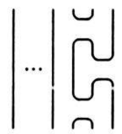
The first equation follows from eq. (3.40) the second one is obvious from

$d^2\mathrm{tr}^L(eA) = d^2 d^{-|I|-2} d^{-1}$  $= d^{-|I|-1}$  $= \mathrm{tr}^M(A)$ (3.63)

For the quasi basis $v_n = v^{\bar{I}}_{x_f+1}(b'_{x_f}, \underline{b})$ of eq. (3.53) the relations (3.54) and (3.56) $\sum_n u_n e v_n = \mathbf{1}^L$ and $\sum_n u_n v_n = d^2\mathbf{1}^M$ with $u_n = (-1)^{|I|}v_n^*$ may be depicted as

$\sum_n \frac{1}{d}$  $= |\cdots|\,|\,| = \mathbf{1}^L$, $\sum_n$  $= |\cdots|\,| = d^2\mathbf{1}^M$. (3.64)

We may even continue Jones' construction one step further $N \subset M \subset L \subset K$ and introduce the Jones projector e' for the inclusion $M \subset L$. The Temperly-Lieb algebra property is easily seen from

$ee'e =$  $= e$, $e'ee' =$  $= e'$ (3.65)

Remark 3.12 *If $|I| < r - 2$ the inclusion matrix is again given by $N^{\frac{1}{2}}_{j\tilde{j}}$. Now however (in addition to the above restrictions on j and $\tilde{j}$) $j \in \mathcal{I}$ is restricted by $0 \le 2j \le 2|I|$ and $\tilde{j}$ by $0 \le 2\tilde{j} \le$ Min $(2|I| + 2, r - 2)$ whenever $|I| > 1$. If $|I| = 1$, then as an additional restriction the value $j = 0$ is excluded, such that j can only take the value $1/2$. Thus for small $|I|$ the index is* not $w^4_{1/2}$. *In particular for $|I| = 1$ the index is 2.*

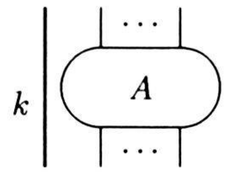
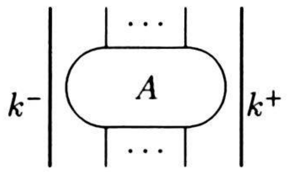
4 The local field algebra

In this section we will construct local field algebras. For this purpose we will extend and thus generalize the concept of local observables as discussed in the previous section. We will be guided by the concept of superselection sectors in algebraic quantum field theory. Indeed, the idea is to introduce structures at infinity. In the present context of the one-dimensional lattice $\mathbb{Z}$ we will choose the left infinity $-\infty$, the construction for $+\infty$ being analogous. Moreover, in the next section we will introduce structures at both infinities simultaneously.

Note that if $V_I(k)$ and $V_I(k^-,k^+)$ denotes the linear space spanned by all $|\underline{a}\rangle$ with fixed $a_{x_i-1}=k$ and $a_{x_i-1}=k^-$, $a_{x_f}=k^+$, respectively such that

$$V_I=\bigoplus_{k\in\mathcal{I}}V_I(k)=\bigoplus_{k^-,k^+\in\mathcal{I}}V_I(k^-,k^+) \tag{4.1}$$

then the endomorphism associated to a graph A by eq. (3.18) leaves each $V_I(k)$ and $V_I(k^-,k^+)$ invariant. We denote these element of $\mathrm{End}(V_I(k))$ and $\mathrm{End}(V_I(k^-,k^+))$ by $A(k)$ and $A(k^-,k^+))$, respectively, and depict them as

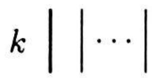
$A(k)=$  , $A(k^-,k^+)=$  (4.2)

with k, k^- and k^+ fixed. Here the rule (3.18) has to be used for defining matrix elements of the corresponding graphs. In particular we have

$$A=\sum_k A(k)=\sum_{k^-,k^+}A(k^-,k^+) \tag{4.3}$$

with $A(k)A(k')=0$ for $k\neq k'$ and $A(k^-,k^+)A(k^{-\prime},k^{+\prime})=0$ for $(k^-,k^+)\neq(k^{-\prime},k^{+\prime})$.

Example 4.1 *The projectors onto the spaces $V_I(k)$ and $V_I(k^-,k^+)$ are depicted by*

$\mathbf{1}_I(k)=$  , $\mathbf{1}_I(k^-,k^+)=$  . (4.4)

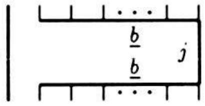
By definition these endomorphims $A(k)$ and $A(k^-,k^+)$ fulfil $A(k)|\underline{a}\rangle=0$ if $k\neq a_{x_i-1}$ and $A(k^-,k^+)|\underline{a}\rangle=0$ if $(k^-,k^+)\neq(a_{x_i-1},a_{x_f})$, respectively. Moreover the algebras $\mathcal{A}_I(k)$ and $\mathcal{A}_I(k^-,k^+)$ defined as the linear hulls of all these endomorphisms yield $*$-representations $\pi_I(k)$ and $\pi_I(k^-,k^+)$ of $\mathcal{A}_I$ (or $\mathcal{B}_I$). For later convenience we also introduce the algebra

$$\hat{\mathcal{A}}_I=\bigoplus_k\mathcal{A}_I(k)=\bigoplus_k\pi_I(k)(\mathcal{A}_I)\ . \tag{4.5}$$

It is easy to see that (see 3.25) $\pi_I(k)(p_j^I)\neq 0$. Hence the representation $\pi_I(k)$ of $\mathcal{A}_I$ is faithful. Note that $\mathbf{1}_I(k)$ are the unit operators in $\mathcal{A}_I(k)$ and that $\mathbf{1}_I=\sum_k\mathbf{1}_I(k)$.

If we combine this decomposition with that of $\mathcal{A}_I$ given by (3.26) we obtain the minimal projectors

$$p_j^I(k) = \mathbf{1}_I(k) p_j^I = \sum_{\substack{\underline{b} \\ b_{x_f}=j}} \prod_{x=x_i+1}^{x_f} w_{b_x}^2 \quad k \qquad (4.6)$$



Lemma 4.2 *The decomposition of the vector space V_I*

$$V_I = \bigoplus_{k,j} V_{I,j}(k) \ , \quad \text{where} \quad V_{I,j}(k) = p_j^I(k) V_I = p_j^I V_I(k) \qquad (4.7)$$

is invariant and irreducible under the action of the observable algebra $\hat{\mathcal{A}}_I$. The corresponding decomposition of the algebra is

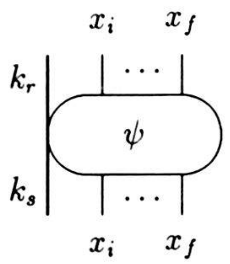
$$\hat{\mathcal{A}}_I = \bigoplus_{k,j} \mathcal{A}_{I,j}(k) \ , \quad \text{where} \quad \mathcal{A}_{I,j}(k) = p_j^I(k) \mathcal{A}_I = p_j^I \mathcal{A}_I(k). \qquad (4.8)$$

The algebras $\mathcal{A}_{I,j}(k)$ are $*$-subalgebra of $\hat{\mathcal{A}}_I$, all of whose elements leave each $V_{I,j}(k)$ $(k, j \in \mathcal{I})$ invariant. (In fact they are zero on $V_{I,j'}(k')$ with $(k,j) \neq (k',j')$.) Moreover, the $\mathcal{A}_{I,j}(k)$ yield $*$-representations of the braid group algebra $\mathcal{B}_I$ and also of the local algebra of observables $\mathcal{A}_I$. (By a similar construction as above the algebras $\mathcal{A}_{I,j}(k^-, k^+) = \mathbf{1}_I(k^-, k^+) p_j^I A_I$ can be obtained using the projectors $\mathbf{1}_I(k^-, k^+)$.)

Since we view each $\mathcal{A}_{I,j}(k)$ as a subalgebra of $\mathrm{End}(V_{I,j}(k))$, we indeed have an interpretation of $\mathcal{A}_{I,j}(k)$ as a representation of $\mathcal{A}_I$ in the "sector" where the charge at minus infinity is k and the charge on the lattice I is j. These structures reflect some of the basic ideas of the theory of superselection sectors in algebraic quantum field theory [11]. However, the situation in the path space formulation here is somewhat different from that in the tensor formulation which is usually used in this context. The quantum number j corresponds to the superselection charge, whereas the quantum number k is the analogue of the "magnetic" quantum number which counts the multiplicities. In the following we investigate this structure in detail and introduce fields which change these quantum numbers.

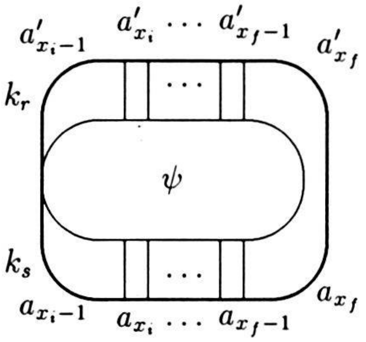
Let now $\psi(k_r, k_s)$ be a planar graph with $2|I| + 2$ external legs, half of them pointing upwards and downwards respectively. All external legs are supposed to carry the colour $1/2$ except the two left ones, whose colours will be denoted by k_s and k_r respectively. k_s will be called the source colour of $\psi(k_r, k_s)$ and k_r the range colour:

$$\psi(k_r, k_s) = \sqrt{w_{k_r} w_{k_s}} \qquad (4.9)$$



To each such graph $\psi(k_r, k_s)$ we first associate a linear map from V_{I,k_s} into V_{I,k_r} also denoted by $\psi(k_r, k_s)$, whose matrix elements w.r.t. to the basis $|\underline{a}\rangle$ are given by

$$\langle \underline{a}'|\psi(k_r, k_s)|\underline{a}\rangle = w_{\underline{a}} w_{\underline{a}'} \sqrt{w_{k_r} w_{k_s}} \tag{4.10}$$

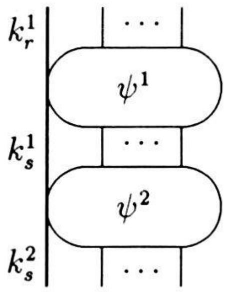


where $k_r = a'_{x_i-1}, k_s = a_{x_i-1}, a_{x_f} = a'_{x_f}$ and $w_{\underline{a}} = \prod_{x=x_i}^{x_f-1} w_{a_x}$. We extend $\psi(k_r, k_s)$ to a linear map from V_I into V_I, again denoted by $\psi(k_r, k_s)$, by setting $\psi(k_r, k_s)$ equal to zero on all $V_I(k)$ with $k \neq k_s$. Let $\mathcal{F}_I$ denote the linear hull in $\mathrm{End}(V_I)$ of all linear transformations obtained in this way.

Lemma 4.3 *The field algebra $\mathcal{F}_I$ is a C^*-algebra.*

Proof: Consider two graphs $\psi^1(k_r^1, k_s^1)$ and $\psi^2(k_r^2, k_s^2)$ with sources and ranges k_s^1, k_r^1 and k_s^2, k_r^2 respectively. It suffices to consider the case $k_r^2 = k_s^1$ since otherwise $\psi^1(k_r^1, k_s^1)\psi^2(k_r^2, k_s^2) = 0$. In analogy to the discussion in section 3 it follows easily from the Wigner-Eckart theorem that we have the correspondence

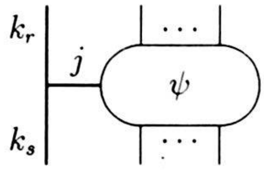
$$\psi^1(k_r^1, k_s^1)\ \psi^2(k_r^2, k_s^2) = \delta_{k_s^1 k_r^2} \sqrt{w_{k_r^1} w_{k_s^2}} w_{k_s^1} \tag{4.11}$$



such that indeed $\mathcal{F}_I$ is an algebra. Analogously the $*$-operation is given on graphs as in section 3 by mirroring along a horizontal axis with the same rules at 4-vertices as in (3.29). In particular the source and range of $\left(\psi(k_r, k_s)\right)^*$ are k_r and k_s respectively, concluding the proof of the lemma.

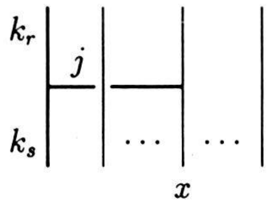
Example 4.4 *Graphs in $\mathcal{F}_I$ of the form*

$$\psi_j(k_r, k_s) = \sqrt{w_{k_r} w_{k_s}} \tag{4.12}$$



will be called field operators. Note that $\psi_j(k_r, k_s)$ vanishes unless j is an integer. Using (2.13), it is easy to see that the linear hull of such operators is $\mathcal{F}_I$. In particular for $x \in I$ we set

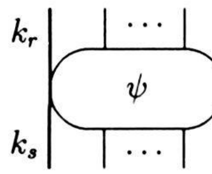
$$\psi_j(k_r, k_s)(x) = \sqrt{w_{k_r} w_{k_s}} \tag{4.13}$$

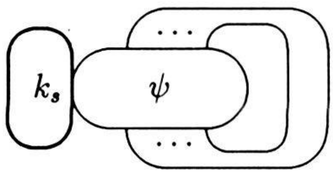


We will discuss the commutation relations for these operators in the next section.

The map $\mathrm{tr}_I : \mathcal{F}_I \to \mathbb{C}$

$$\mathrm{tr}_I : \quad \mapsto \quad \frac{(-1)^{2k_s}\delta_{k_r k_s}}{\beta^{1/2}(w_{1/2})^{2|I|}} \tag{4.14}$$





with $\beta^{1/2} = \sum_k d_q(k) = \sum_k |w_k^2|$ defines a trace, which again is easily seen to be faithful and which extends the trace on $\mathcal{A}$. Cyclicity is easy to see from (4.11) and the fact that $(-1)^{2k_s} = (-1)^{2k_r}$ holds by the fusion rules. Also for any $\psi \in \mathcal{F}_I$ one has

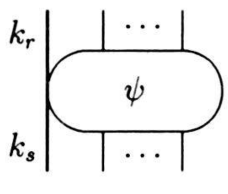
$$\mathrm{tr}_I(\psi) = \frac{1}{\beta^{1/2} w_{1/2}^{2|I|}} \sum_{\underline{a}} (-1)^{2a_{x_i}-1} w_{a_{x_f}}^2 \langle \underline{a}|\psi|\underline{a}\rangle \ . \tag{4.15}$$

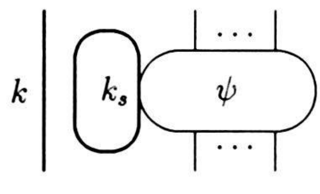
Note that again by the fusion rules $(-1)^{2a_{x_i}-1} w_{a_{x_f}}^2 w_{1/2}^{-2|I|} > 0$. In particular tr_I is positive on positive elements $\psi \in \mathcal{F}_I$ since the trace is nondegenerate. Also (see (4.4))

$$\mathrm{tr}_I(\mathbf{1}_I(k)) = \frac{d_q(k)}{\beta^{1/2}} \ . \tag{4.16}$$

Now the linear map $E_I : \mathcal{F}_I \to \mathcal{A}_I \subset \mathcal{F}_I$ defined on graphs in $\mathcal{F}_I$ by

$$E_I : \quad \mapsto \quad \frac{(-1)^{2k_s}\delta_{k_r k_s}}{\beta^{1/2}} \sum_k \tag{4.17}$$





and extended by linearity to all of $\mathcal{F}_I$ is obviously a conditional expectation with range $\mathcal{A}_I$. Also E_I is compatible with the trace in the sense that $\mathrm{tr}_I(E_I(\psi)) = \mathrm{tr}_I(\psi)$.

Theorem 4.5 *Let $|I| \geq r-2$. Then the Jones index of the inclusion $\hat{\mathcal{A}}_I \subset \mathcal{F}_I$ is given by*

$$[\mathcal{F}_I : \hat{\mathcal{A}}_I] = \sum_j{}' w_j^4 = \sum_j{}' d_q^2(j) = \frac{1}{2} w^2 \ , \tag{4.18}$$

where w^2 was defined in (2.14) and $\sum'$ denotes the restriction that $|I|+2j$ has to be even. The Jones index of the inclusion $\mathcal{A}_I \subset \mathcal{F}_I$ is given by

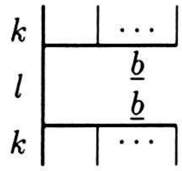
$$[\mathcal{F}_I : \mathcal{A}_I] = \Big(\sum_k (-1)^{2k} w_k^2\Big)^2 = \Big(\sum_k d_q(k)\Big)^2 = \beta. \tag{4.19}$$

Note that w_j^2 is (up to $(-1)^{2j}$) the q-version of the dimension of the irreducible representation of $SL_q(2,\mathbb{C})$ labelled by a such that w^2 is the q-analogue of the order of a finite group. Hence apart from the factor 1/2 relation (4.18) is what one would expect.

Indeed the observable algebra is the quantum symmetry invariant subalgebra of the field algebra (see the discussion in section 6). The analogue in mathematics is well known to be provided by Galois theory. Note that the factor 1/2 reflects the fact that the fields defined by eq. (4.9) do not connect integer and half-integer representations. For an extended field algebra containing also such fields (which map between lattices of different size) the factor 1/2 in eq. (4.18) is absent.

Proof: To prove (4.18) we proceed as follows. The minimal central idempotents of $\mathcal{F}_I$ are obviously of the form

$$P_l^I = w_l^2 \sum_{k,\underline{b}} \prod_{x=x_i-1}^{x_f-2} w_{b_x}^2$$

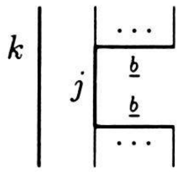


(4.20)

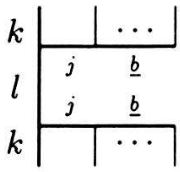
indexed by $l \in \mathcal{I}$. Here $\underline{b} = (b_{x_i-1}, \ldots, b_{x_f})$. Since $|I| \geq r-2$, these P_l^I are all nonvanishing.

The minimal central idempotents of $\hat{\mathcal{A}}_I$ are of the form

$$q_{kj}^I = w_j^2 \sum_{\underline{b}} \prod_{x=x_i}^{x_f-2} w_{b_x}^2$$

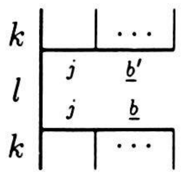


$$= w_j^2 \sum_{l,\underline{b}} w_l^2 \prod_{x=x_i}^{x_f-2} w_{b_x}^2$$



(4.21)

indexed by k and j. These q_{kj}^I are nonzero exactly when $2j+|I|$ is even. Therefore the algebras $P_l^I q_{kj}^I \mathcal{F}_I P_l^I q_{kj}^I$ and $P_l^I q_{kj}^I \hat{\mathcal{A}}_I P_l^I q_{kj}^I$ are both equal and spanned by the elements



. (4.22)

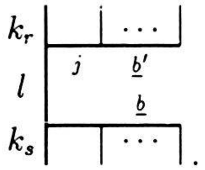
Thus the inclusion matrix λ is again given in terms of the fusion matrix in the form

$$\lambda_{l,(kj)} = N_{lk}^j \quad \text{and} \quad (\lambda\lambda^t)_{l,l'} = \sum_{j\in\mathcal{I}}{}' (N^j N^j)_{l,l'} \tag{4.23}$$

where $\sum'$ refers to the above restriction on j. Now the theorem follows from e.g. Corollary 4.4 in [9]. Since the largest eigenvalue of the matrix $N^j N^j$ is w_j^4 and since the matrices $N^j N^j$ for different j all commute, the norm $\|\lambda\lambda^t\|$ is equal to $\sum_j' w_j^4$. Now it is easy to see (c.f. e.g. [9]) that $\sum_j' w_j^4 = 1/2 \sum_j w_j^4$ and hence the claim (4.18) follows from (2.14).

To prove (4.19) we construct a quasibasis as follows. Fix an interval I with $|I| \geq r-2$. We set

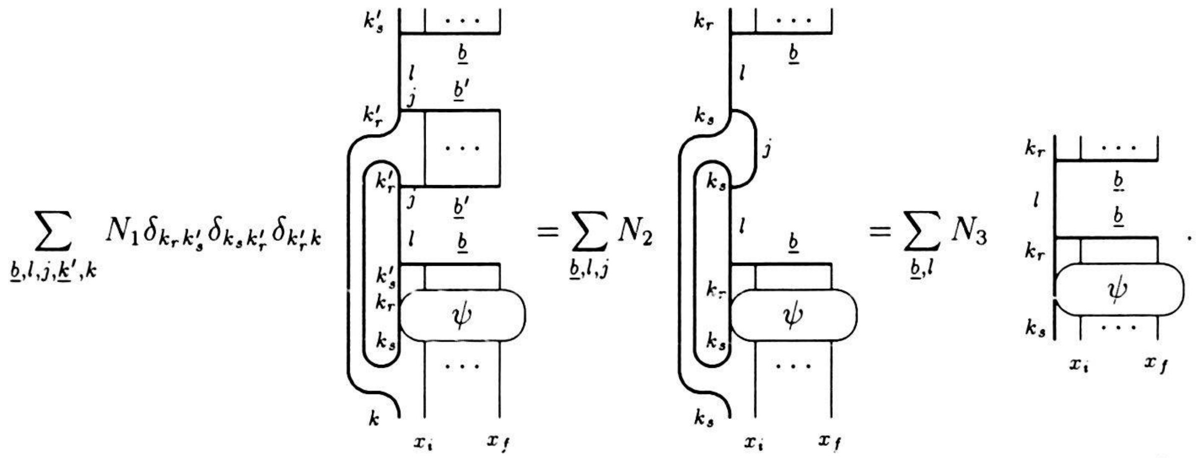
$$v_I(\underline{b}, l, \underline{k}, j) = \beta^{1/4} w_{\underline{b}'} w_{\underline{b}} w_l w_j^2 \tag{4.24}$$



Here $\underline{b} = (b_{x_i-1}, \ldots, b_{x_f})$, $\underline{k} = (k_r, k_s)$, l and j may vary. Also, for given $\underline{b}, \underline{k}, l$ and j satisfying the usual fusion rules, $\underline{b}' = (b'_{x_i}, \ldots, b'_{x_f})$ is a fixed path chosen such that the corresponding fusion rules are satisfied making $v_I(\underline{b}, l, \underline{k}, j) \neq 0$. Then for any graph $\psi(\underline{k}) \in \mathcal{F}_I$ with source k_s and range k_r using (4.17) and the fact that by the fusion rules and eq. (2.6) the factor in eq. (4.24) equals its conjugate complex times $(-1)^{2l+2k_r} = (-1)^{2j}$ we have

$$\sum_{\underline{b},l,\underline{k}',j} (-1)^{2j} v_I^*(\underline{b}, l, \underline{k}', j)\, E_I\big(v_I(\underline{b}, l, \underline{k}', j)\, \psi(\underline{k})\big) = \psi(\underline{k}) \tag{4.25}$$

which can be seen from the graphical representation



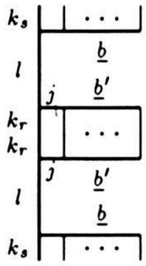
(4.26)

Here the normalization factors are $N_1 = w_{\underline{b}'}^2 w_j^4 w_l^2 w_{\underline{b}}^2$, $N_2 = w_j^2 w_l^2 w_{\underline{b}}^2$ and $N_3 = w_l^2 w_{\underline{b}}^2$. Hence by the completeness relation eq. (4.25) holds for all graphs $\psi(\underline{k}) \in \mathcal{F}_I$ and therefore by linearity for all $\psi(\underline{k}) \in \mathcal{F}_I$. Thus the $v_I(\underline{b}, l, \underline{k}', j)$ and their adjoints (modulo a sign) form a quasibasis for the conditional expectation $E_I : \mathcal{F}_I \to \hat{\mathcal{A}}_I$. The corresponding index follows from the calculation

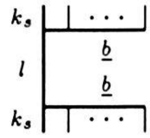
$$\sum_{\underline{b},l,\underline{k},j} (-1)^{2j} v_I^*(\underline{b}, l, \underline{k}, j)\, v_I(\underline{b}, l, \underline{k}, j) = \beta\, \mathbf{1}_I \tag{4.27}$$

which again can be seen from the graphical representation

$$\beta^{1/2} \sum_{\underline{b},l,j,\underline{k}} (-1)^{2k_r} w_{\underline{b}'}^2 w_j^4 w_l^2 w_{\underline{b}}^2$$



$$= \beta^{1/2} \sum_{k_r} (-1)^{2k_r} w_{k_r}^2 \sum_{\underline{b},l,k_s} w_l^2 w_{\underline{b}}^2$$



$$= \beta \, \mathbf{1}_I. \tag{4.28}$$

Here the last equality follows from the completeness relation (2.12) and arguments used in the proof of Theorem 3.11.

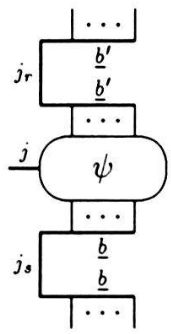
Remark 4.6 *By the first proof we have also established the relation*

$$[\mathcal{F}_I : \pi_{I,k}(\mathcal{A}_I) = \mathcal{A}_I(k)] = w_k^4. \tag{4.29}$$

It is easy to check that tr_I is not a Markov trace w.r.t. the inclusion $\hat{\mathcal{A}}_I \subset \mathcal{F}_I$.

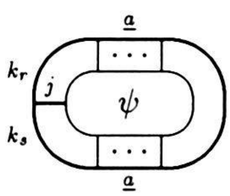
Remark 4.7 *The fields $\psi_j(k_r, k_s) \in \mathcal{F}_I$ introduced above are similar but not to be confused with the fields usually called projected fields, vertex fields or exchange fields. The later are of the form*

$$p_{j_r}^I \psi_j p_{j_s}^I = \sum_{\underline{b}'\underline{b}} w_{\underline{b}'}^2 w_{\underline{b}}^2 \tag{4.30}$$

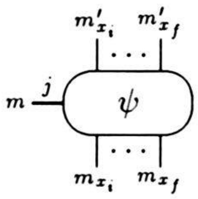


where j_s and j_r denote the total spin (charge) of the states in the corresponding Hilbert spaces. Our fields $\psi_j(k_r, k_s)$ are analogues of tensor fields $\psi_j(m)$ (m = magnetic quantum number associated to the spin j) in the sense that in the path space picture k_r and k_s replace m.

$$\langle \underline{a}' | \psi_j(k_r, k_s) | \underline{a} \rangle =$$



versus $\langle \underline{m}' | \psi_j(m) | \underline{m} \rangle =$ (4.31)



with the graphical representation in the tensor picture for $\psi_j(m)$. The spins (charges)k_s and k_r at $-\infty$ are not necessarily equal to the j_s and j_r of eq. (4.30) since there is also a charge at $+\infty$.

5 Generalized fields

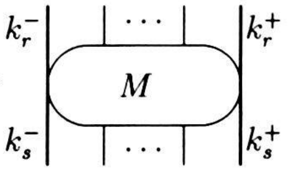
As announced in the previous section, we will work in the context of sectors defined by colours at $-\infty$ and $+\infty$.

Analogously to section 4, let $V_I(k^-, k^+) \subseteq V_I$, $k^-, k^+ \in \mathcal{I}$ (see eq. (4.1)) be the linear space spanned by all symbols $|\underline{a}\rangle$ with $\underline{a} = (k^- = a_{a_i-1}, \ldots, a_{x_f} = k^+)$ such that the fusion rules $N^{1/2}_{a_x a_{x+1}} = 1$ $(x_i - 1 \leq x < x_f)$ hold. Again the case $I = \phi$ is allowed. By construction V_I is the direct sum of all $V_I(k^-, k^+)$

$$V_I = \bigoplus_{k^+, k^- \in \mathcal{I}} V_I(k^-, k^+). \tag{5.1}$$

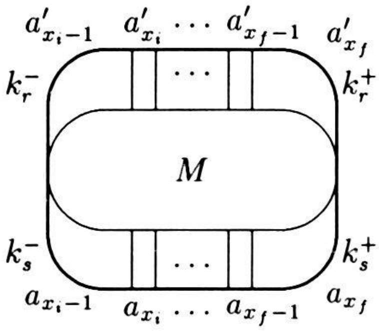
Let now M be a planar graph with $2|I| + 4$ external legs, half of them pointing upwards and half of them pointing downwards. The four corner legs have the colours $k_s^-, k_s^+, k_r^-, k_r^+$ respectively, all other $2|I|$ legs having colour $1/2$:

$$M(k_r^-, k_s^-; k_r^+, k_s^+) = \sqrt{w_{k_r^-} w_{k_s^-} w_{k_r^+} w_{k_s^+}} \tag{5.2}$$



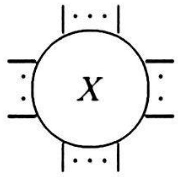
To such a graph we associate a linear map from $V_I(k_s^-, k_s^+)$ into $V_I(k_r^-, k_r^+)$ whose matrix elements w.r.t. to the basis $|\underline{a}\rangle$ is given by

$$\langle \underline{a}' | M(k_r^-, k_s^-; k_r^+, k_s^+) | \underline{a} \rangle = w_{\underline{a}} w_{\underline{a}'} \sqrt{w_{k_r^-} w_{k_s^-} w_{k_r^+} w_{k_s^+}} \quad , \tag{5.3}$$



where $k_r^- = a'_{x_i-1}, k_s^- = a_{x_i-1}, k_r^+ = a'_{x_f}, k_s^+ = a_{x_f}$. We extend this to a linear map from V_I into itself by setting it to zero on each $V_I(k^-, k^+)$ with $(k^-, k^+) \neq (k_s^-, k_s^+)$. The linear hull of all such linear transformations form a $*$-algebra, denoted by $\tilde{\mathcal{F}}_I$ and contained in $\mathrm{End} V_I$.

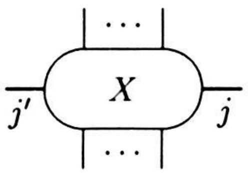
With these generalizations of the discussion in section 4, we are now prepared to introduce new concepts. These concepts are path space formulations of what is called the concept of auxiliary spaces by the St.Petersburg school used in the context of the quantum inverse problem [16]. This will put us into a position to associate algebraic objects called *generalized fields* to planar graphs which also have several horizontal lines, i.e. which are of the form

$X =$ 

(5.4)

For these objects we will extend the notion of the path tensor product $\circ$. Having given an algebraic meaning to such objects, we will be able to discuss the concept of quantum group symmetry and to construct concrete models like the RSOS model in the path space picture.

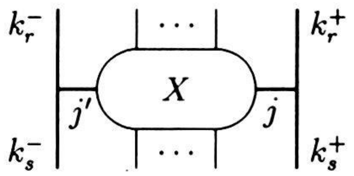
To motivate our procedure we start with coloured graphs of the form

$X_{j',j} =$ 

(5.5)

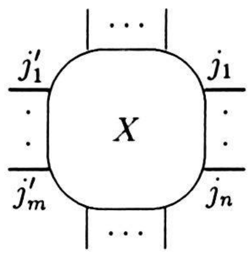
having two horizontal lines, one with colour j' pointing to the left and one with colour j pointing to the right. Again in addition there are $|I|$ lines with colours 1/2 pointing upward and downward respectively.

By definition the object (5.5) associates to each quadruple $(\underline{k}^-, \underline{k}^+) = (k_r^-, k_s^-, k_r^+, k_s^+)$ the element in $\tilde{\mathcal{F}}_I$ given by the graph

$X_{j'j}(\underline{k}^-, \underline{k}^+) = \sqrt{w_{k_r^-} w_{k_s^-} w_{k_r^+} w_{k_s^+}}$ 

. (5.6)

We now generalize to graphs with $|I|$ vertical lines with colours 1/2 pointing upward and downward respectively. In addition there are m horizontal lines pointing to the left with colours $\underline{j}' = (j'_1, \ldots, j'_m)$ and n horizontal lines pointing to the right with colours $\underline{j} = (j_1, \ldots, j_n)$

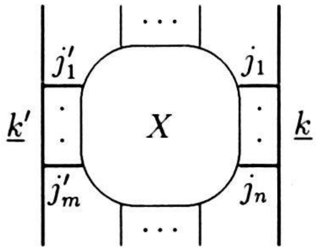
$X_{\underline{j}'\underline{j}} =$ 

. (5.7)

The case $I = \phi$ is allowed. To each such graph and to each $m + n + 2$-tuple $(\underline{k}', \underline{k})$

$$\begin{aligned} \underline{k}' &= (k'_0 = k_r^-, k'_1, \ldots, k'_m = k_s^-) \\ \underline{k} &= (k_0 = k_r^+, k_1, \ldots, k_n = k_s^+) \end{aligned}$$

we associate the element in $\tilde{\mathcal{F}}_I$ given by

$$X_{\underline{j}'\underline{j}}(\underline{k}',\underline{k}) = \tilde{w}_{\underline{k}'}\tilde{w}_{\underline{k}}$$ 

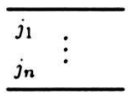
(5.8)

where we have introduced the normalization factors

$$\tilde{w}_{\underline{k}'} = \sqrt{w_{k'_0} w_{k'_m}} \prod_{i=1}^{m-1} w_{k'_i}, \quad \tilde{w}_{\underline{k}} = \sqrt{w_{k_0} w_{k_n}} \prod_{i=1}^{n-1} w_{k_i} \tag{5.9}$$

(for $n = 0 : \tilde{w}_{\underline{k}} = 1$ see eqs. (4.2).) Here $\underline{k}'$ gives the colours on the left vertical line, read from top to bottom. The convention for $\underline{k}$ is similar.

Example 5.1 *a)* $m = n,\ I = \emptyset$

$$\epsilon^n_{\underline{j}} =$$ 

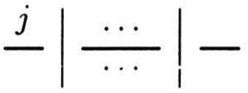
(5.10)

which is the "horizontal" unit braid. As a matrix element with respect to $V_\emptyset$ *we have (for* $\underline{k}'$ *and* $\underline{k}$ *fulfilling the fusion rules)*

$$\langle a'|\epsilon^n_{\underline{j}}(\underline{k}',\underline{k})|a\rangle = \delta_{\underline{k}'\underline{k}}\delta_{a'k_0}\delta_{ak_n} \tag{5.11}$$

(see also eq. (3.6)).

b) $m = n = 1,\ I$ *arbitrary*

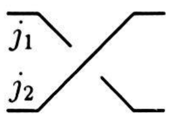
$$L^-_j =$$ 

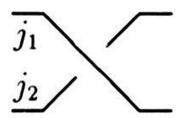
. (5.12)

L^+_j *is defined similarly with the horizontal line overcrossing all vertical lines. For* $I = \{1\}$ *as matrix elements we have*

$$\langle \underline{a}'|L^\pm_j(\underline{k}',\underline{k})|\underline{a}\rangle = \delta_{a'_0k'_0}\delta_{a_0k'_1}\delta_{a'_1k_0}\delta_{a_1k_1}\sqrt{w_{k'_0}w_{k'_1}w_{k_0}w_{k_1}} \left(\frac{q_{k'_0}q_{k_1}}{q_{k'_1}q_{k_0}}\right)^{\pm} \begin{vmatrix} j & k'_0 & k'_1 \\ 1/2 & k_1 & k_0 \end{vmatrix} . \tag{5.13}$$

c) $m = n = 2,\ I = \emptyset$

$$R_{j_1j_2} =$$ 

, $$R^*_{j_1j_2} = (R_{j_2j_1})^* =$$ 

(5.14)

As a matrix element with respect to $V_\emptyset$ we have (for $\underline{k}'$ and $\underline{k}$ fulfilling the fusion rules)

$$\langle a'|R_{j_1j_2}(\underline{k}',\underline{k})|a\rangle = \delta_{a'k'_0}\delta_{ak'_2}\delta_{a'k_0}\delta_{ak_2}w_{k'_1}w_{k_1}\frac{q_{a'}q_{a'}}{q_{k'_1}q_{k_1}}\begin{vmatrix} j_1 & a & k_1 \\ j_2 & a' & k'_1 \end{vmatrix}\,. \tag{5.15}$$

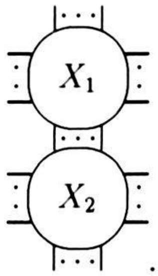
The linear structure on $\tilde{\mathcal{F}}_I$ allows us to view such X given by eq. (5.7) (for fixed m,n and I) as elements of a linear space $\mathcal{G}_I^{mn}$ which is spanned by these X's. Again by the Wigner-Eckhart theorem it is easy to see that $\mathcal{G}_I^{mn}$ is finite dimensional. Now $\mathcal{G}_I^{mn}$ may be viewed as a linear subspace of $\mathcal{G}_I^{m'n'}$ whenever $m \le m', n \le n'$. Indeed, by the rule (2.15) to any $X \in \mathcal{G}_I^{mn}$ we may add $m'-m$ and $n'-n$ additional horizontal lines with colour 0 to the left and to the right respectively by hooking them somewhere up to X (through a 3-vertex) without changing the associated elements of $\tilde{\mathcal{F}}_I$. Therefore it makes sense to introduce the filtered linear space

$$\mathcal{G}_I = \bigcup_{m,n}\mathcal{G}_I^{mn} \tag{5.16}$$

This linear space is actually an algebra with a product • which is compatible with the filtration in the sense that

$$\bullet : \mathcal{G}_I^{m_1n_1}\otimes\mathcal{G}_I^{m_2n_2} \to \mathcal{G}_I^{m_1+m_2\,n_1+n_2}. \tag{5.17}$$

Indeed for graphs $X_1 \in \mathcal{G}_I^{m_1n_1}$ and $X_2 \in \mathcal{G}_I^{m_2n_2}$ of the form (5.7) the product is given as

$X_1\bullet X_2 =$  (5.18)

The symbol • means both a "path tensor" product w.r.t. to the horizontal (auxiliary) space and a matrix product w.r.t. to the vertical space V_I. As a relation in $\tilde{\mathcal{F}}_I$ (see eq. (5.8)) we have

$$(X_1\bullet X_2)(\underline{k}'_1\bullet\underline{k}'_2,\underline{k}_1\bullet\underline{k}_2) = X_1(\underline{k}'_1,\underline{k}_1)X_2(\underline{k}'_2,\underline{k}_2) \tag{5.19}$$

where $\underline{k}_1\bullet\underline{k}_2 = (k_{10},\dots,k_{1n_1} = k_{20},\dots,k_{2n_2})$. By construction $\mathcal{G}_I^{m=0n=0}$ may be identified with the observable algebra $\mathcal{A}_I$ (compare eq. (4.3)). Also $\mathcal{G}_I$ is a $*$-algebra respecting the filtering in the sense that $\mathcal{G}_I^{mn*} = \mathcal{G}_I^{mn}$. Indeed this $*$-operation is the obvious extension of the $*$-operations considered in the previous sections, such that as a relation in $\tilde{\mathcal{F}}_I$

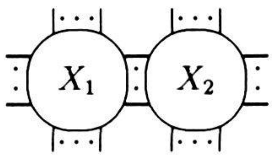
$$(X(\underline{k}',\underline{k}))^* = X^*(\underline{k}'^*,\underline{k}^*) \tag{5.20}$$

where X^* is the graph X reflected w.r.t. a horizontal line and $\underline{k}^*$ is the path $\underline{k}$ in reserved order.

Example 5.2 *With the choice $I = \emptyset$ in we may write Example 5.1 as*

$$\epsilon^n_{\underline{j}} = \epsilon^1_{j_1} \bullet \cdots \bullet \epsilon^1_{j_n} \tag{5.21}$$

Let now I_1 and I_2 be two neighbouring intervals. We extend the map $\circ$ of section 3 to a bilinear map from $\mathcal{G}_{I_1} \times \mathcal{G}_{I_2}$ into $\mathcal{G}_{I_1 \cup I_2}$ as follows. Let $X_i \in \mathcal{G}^{m_i n_i}_{I_i}$ with horizontal colours $(\underline{j}'_i, \underline{j}_i)$ $(i = 1, 2)$. If $n_1 = m_2$ and $\underline{j}_1 = \underline{j}'_2$ we set

$X_1 \odot X_2 =$  (5.22)

and zero otherwise. The symbol "$\odot$" means both a path tensor product "$\circ$" as in eq. (3.34) w.r.t. the spaces V_{I_1} and V_{I_2} and a matrix product "$\cdot$" w.r.t. the horizontal (auxiliary) space. It follows directly that $(X_1 \odot X_2)^* = X_1^* \odot X_2^*$ and

$$(X_1 \bullet X_1') \odot (X_2 \bullet X_2') = (X_1 \odot X_2) \bullet (X_1' \odot X_2') \tag{5.23}$$

with $X_i \in \mathcal{G}^{m_i n_i}_{I_i}$, $X_i' \in \mathcal{G}^{m_i' n_i'}_{I_i}$ provided $n_1 = m_2$, $n_1' = m_2'$. Again by the Wigner-Eckhart theorem, as a relation in $\tilde{\mathcal{F}}_{I_1 \cup I_2}$ and with the conventions used in (3.35) we have in terms of matrix elements

$$\langle \underline{a}_1' \circ \underline{a}_2' | (X_1 \odot X_2)(\underline{k}', \underline{k}) | \underline{a}_1 \circ \underline{a}_2 \rangle = \sum_{\underline{k}''} \langle \underline{a}_1' | X_1(\underline{k}', \underline{k}'') | \underline{a}_1 \rangle \langle \underline{a}_2' | X_2(\underline{k}'', \underline{k}) | \underline{a}_2 \rangle . \tag{5.24}$$

We will write this in a suggestive way as

$$(X_1 \odot X_2)(\underline{k}', \underline{k}) = \sum_{\underline{k}''} X_1(\underline{k}', \underline{k}'') \circ X_2(\underline{k}'', \underline{k}) \tag{5.25}$$

Example 5.3 *a) Using Examples 5.1 a) and c) we can write*

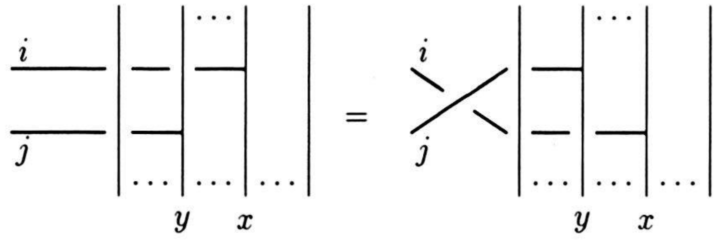
$$R_{ij} \cdot R^*_{ji} = \epsilon^1_i \bullet \epsilon^1_j = \epsilon^2_{ij}. \tag{5.26}$$

Note that $\sum_{\underline{j}} \epsilon^n_{\underline{j}}$ is the unit operator in $\mathcal{G}^{nn}_{\emptyset}$ w.r.t. the horizontal multiplication "$\cdot$".

b) The commutation relations of the local fields $\psi_j(x) \in \mathcal{G}^{10}_I$ defined by eq. (4.13) read

$$\psi_i(x) \bullet \psi_j(y) = R_{ij} \cdot (\psi_j(y) \bullet \psi_i(x)) \quad \text{for } x > y \tag{5.27}$$

or graphically



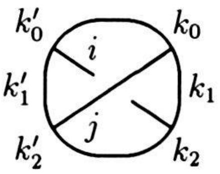
(5.28)

or in terms of matrix elements of the fields $\psi(\underline{k})(x) \in \mathcal{F}_I$ *and the "R-matrix"* $R_{ij}(\underline{k}', \underline{k}) \in \tilde{\mathcal{F}}_\emptyset$

$$\langle \underline{a}'|\psi_i(k_0'k_1')(x)\,\psi_j(k_1'k_2')(y)|\underline{a}\rangle = \sum_{\underline{k}} R_{ij}(\underline{k}', \underline{k})\langle \underline{a}'|\psi_j(k_0k_1)(y)\,\psi_i(k_1k_2)(x)|\underline{a}\rangle \quad (5.29)$$

where $\underline{k}' = (k_0' = k_0, k_1', k_2' = k_2)$. *and* $\underline{k} = (k_0 = a'_{x_i'-1}, k_1, k_2 = a_{x_i-1})$ *The sum is over* k_1 *and the R-matrix is*

$$R_{ij}(\underline{k}', \underline{k}) = \tilde{w}_{k_1'}\tilde{w}_{k_1}$$



$$= w_{k_1'} w_{k_1} \frac{q_{k_0}q_{k_2}}{q_{k_1'}q_{k_1}} \left| \begin{matrix} k_1' & j & k_2 \\ k_1 & i & k_0 \end{matrix} \right|. \quad (5.30)$$

For $x < y$ *the R-matrix R is replaced by* R^*.

c) The commutation relations of the $L_j^\pm$ *of Example 5.1 b) read*

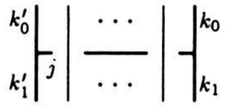
$$R_{ij} \cdot (L_j^\pm \bullet L_i^\pm) = (L_i^\pm \bullet L_j^\pm) \cdot R_{ij}. \quad (5.31)$$

6 Quantum symmetry in the path space formulation

In this section we will exhibit the notion of quantum group symmetry $U(R) \cong U_q(sl(2, \mathbb{C}))$ in the present path version context on the lattice I. The corresponding tensor version has been given by Faddeev, Reshetikhin and Takhtajan [23] (see also [24]).

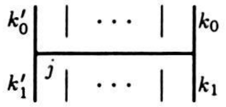
Definition 6.1 *The set* $U_I^p(R)$ *is a* $*$*-subalgebra of* $\tilde{F}_I$ *with unit* $\mathbf{1}_I$. *It has the generators* $L_j^\pm(\underline{k}', \underline{k}) \in \tilde{\mathcal{F}}_I$

$$L_j^-(\underline{k}', \underline{k}) = \tilde{w}_{\underline{k}'}\tilde{w}_{\underline{k}}$$



,

$$L_j^+(\underline{k}', \underline{k}) = \tilde{w}_{\underline{k}'}\tilde{w}_{\underline{k}}$$



. (6.1)

where $\underline{k}' = (k_0', k_1')$, $\underline{k} = (k_0, k_1)$, $k_{0,1}^{(\prime)} \in \mathcal{I}$, $N_{k_0'k_1'}^{1/2} \neq 0$, $N_{k_0k_1}^{1/2} \neq 0$. *For all colours* j *of the horizontal lines they correspond to the elements* $L_j^\pm \in \mathcal{G}_I^{11}$ *(see (5.10)).*

Example 6.2

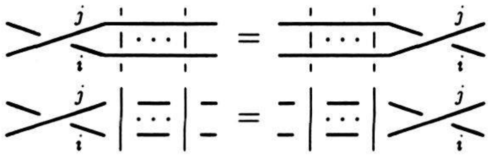
$$u = c_0 \mathbf{1}_I + c_1 L_j^-(\underline{k}', \underline{k}) + c_2 (L_{j_1}^- \bullet L_{j_2}^+)(\underline{l}', \underline{l}) \ \in U_I^p(R) \tag{6.2}$$

with $c_n \in \mathbb{C}$ *and* $\underline{l} = (l_0, l_1, l_2)$.

The generators $L_j^\pm$ satisfy the Yang-Baxter equations (cf. eq. (5.14))

$$R_{ij} \cdot (L_j^\pm \bullet L_i^\pm) = (L_i^\pm \bullet L_j^\pm) \cdot R_{ij} \quad , \quad R_{ij} \cdot (L_j^+ \bullet L_i^-) = (L_i^- \bullet L_j^+) \cdot R_{ij} \tag{6.3}$$

or in terms of graphs



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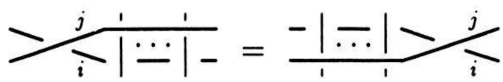


which may also serve as defining relations for the generators of $U_I^p(R)$. As usual we sometimes write these relations also as $R_{12} L_2^\pm L_1^\pm = L_1^\pm L_2^\pm R_{12}$ and $R_{12} L_2^+ L_1^- = L_1^- L_2^+ R_{12}$.

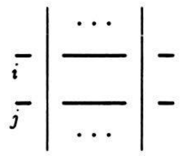
The algebra $U_I^p(R)$ has to be considered as the path version of the algebra of linear functionals on the algebra $A(R)$ of q-functions on the quantum group $SL_q(2, \mathbb{C})$ (see [23] for the tensor version of this construction). The algebra $U_I^p(R)$ is a "path" Hopf algebra with the following structure:

- The algebra product $m : U_I^p(R) \times U_I^p(R) \to U_I^p(R)$ is given by the product in $\tilde{\mathcal{F}}_I$ (see eq. (5.19))

$$m(L_i^\pm(\underline{k}', \underline{k}), L_j^\pm(\underline{l}', \underline{l})) = (L_i^\pm \bullet L_j^\pm)(\underline{k}' \bullet \underline{l}', \underline{k} \bullet \underline{l}) \tag{6.4}$$

  which vanishes for $k_1' \neq l_0'$ or $k_1 \neq l_0$. Graphically as a relation in $\mathcal{G}_I$ this means

$$L_i^- \bullet L_j^- = $$



. (6.5)

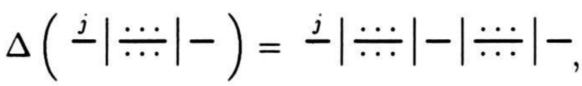
- The coproduct is a map $\Delta : U_I^p(R) \to U_{I_1}^p(R) \circ U_{I_2}^p(R) \subseteq U_{I_1 \cup I_2}^p(R)$ with two neighboring intervals I_1 and I_2 ($I_1 < I_2$) and $I_1 \cup I_2$ an interval of length $|I_1| + |I_2|$. The coproduct for the generating elements is

$$\Delta \mathbf{1} = \mathbf{1} \circ \mathbf{1} \quad \text{and} \quad \Delta(L_j^\pm(\underline{k}', \underline{k})) = (L_j^\pm \odot L_j^\pm)(\underline{k}', \underline{k}) = \sum_{\underline{l}} L_j^\pm(\underline{k}', \underline{l}) \circ L_j^\pm(\underline{l}, \underline{k}) \ . \tag{6.6}$$

As a map of $\mathcal{G}_I \to \mathcal{G}_{I_1 \cup I_2}$ the second relation reads $\Delta L_j^{\pm} = L_j^{\pm} \odot L_j^{\pm}$, and graphically we have

$$\Delta(\mathbf{1}) = \mathbf{1} \bigcirc \mathbf{1} = \mathbf{1} \quad \Leftrightarrow \quad \Delta\Big(\,|\cdots|\,\Big) = |\cdots| \;\; |\cdots| \;, \tag{6.7}$$

$$\Delta(L_j^-) = L_j^- \odot L_j^- \quad \Leftrightarrow \tag{6.8}$$



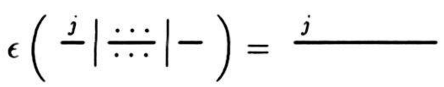
and correspondingly for L_j^+.

- The counit is a linear map $\epsilon : U_I^p(R) \to \mathbb{C}$. It is given for the generating elements by

$$\epsilon(\mathbf{1}) = 1 \quad \text{and} \quad \epsilon(L_j^{\pm}(\underline{k}', \underline{k})) = \delta_{\underline{k}'\underline{k}} N_{k_0 k_1}^j \tag{6.9}$$

or graphically as a map of $\mathcal{G}_I \to \mathcal{G}_\emptyset$ e.g.

$$\epsilon(L_j^-) = \epsilon \tag{6.10}$$



$$= \epsilon_j^1 \, .$$

The counit fulfills by eqs. (6.6) and (6.9)

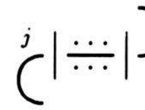
$$(\epsilon \bigcirc \mathrm{id})(\Delta(L^{\pm}(\underline{k}', \underline{k})) = (\mathrm{id} \bigcirc \epsilon)(\Delta(L^{\pm}(\underline{k}', \underline{k})) = \Delta(L^{\pm}(\underline{k}', \underline{k})). \tag{6.11}$$

- The antipode is an antilinear map $S : U_I^p(R) \to U_I^p(R)$, it is given for the generating elements by

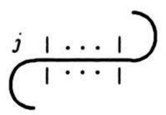
$$S(\mathbf{1}) = \mathbf{1} \quad \text{and} \quad S(L_j^{\pm}(\underline{k}', \underline{k})) = \frac{w_{k_1'} w_{k_0}}{w_{k_0'} w_{k_1}} L_j^{\pm}(\underline{k}^*, \underline{k}'^*) \tag{6.12}$$

where $\underline{k}^*$ is the inverted path as in eq. (5.20). We depict this graphically as a map of $\mathcal{G}_I \to \mathcal{G}_I$

$$S(L_j^-) = \tag{6.13}$$



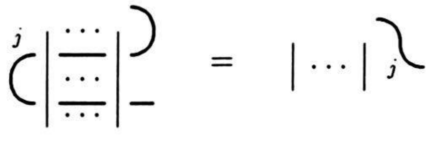
$$, \quad S(L_j^+) =$$



The usual relation for the antipode $m(S \otimes id)\Delta = \mathbf{1} \otimes \epsilon$ in the tensor picture now becomes a relation in $\mathcal{G}_I$ in the path space version of the form

$$S(L_j^{\pm}) \cdot L_j^{\pm} = \mathbf{1} \times^{\neg} \epsilon(L_j^{\pm}) = \mathbf{1} \times^{\neg} \epsilon_j^1 \tag{6.14}$$

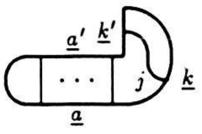
or graphically



(6.15)

where the symbol $\times^{\neg}$ denotes another "path tensor product", whose meaning is obvious from its matrix elements

$$\langle \underline{a}' | \mathbf{1} \times^{\neg} \epsilon^1_j(\underline{k}', \underline{k}) | \underline{a} \rangle = w_{\underline{a}'} w_{\underline{a}} \tilde{w}_{\underline{k}'} \tilde{w}_{\underline{k}} \frac{w_{k_1}}{w_{k_0}}$$



$$= \delta_{\underline{a}'\underline{a}} \delta_{\underline{k}'\underline{k}} \delta_{a'_{x_f} k'_1} \delta_{a_{x_f} k_1} \ . \quad (6.16)$$

Thus $\mathbf{1} \times^{\neg} \epsilon^1_j$ is again not the usual tensor product $\mathbf{1} \otimes \epsilon^1_j$ because of the condition $a_{x_f} = k_1$. Note that in eqs. (6.14) and (6.15) on the right hand side a "double" product appears, with respect to both the vertical and the horizontal spaces: $\cdot : \mathcal{G}^{11}_I \times \mathcal{G}^{11}_I \to \mathcal{G}^{11}_I$. It is defined for graphs $X_{j'j''}$ and $Y_{j''j} \in \mathcal{G}^{11}_I$, in terms of fields in $\tilde{\mathcal{F}}_I$

$$(X_{j'j''} \cdot Y_{j''j})(\underline{k}', \underline{k}) = \sum_{\underline{k}''} X_{j'j''}(\underline{k}', \underline{k}'') Y_{j''j}(\underline{k}'', \underline{k}), \quad (6.17)$$

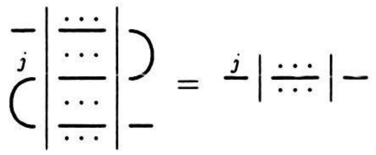
and extended bilinearly. As opposed to eqs. (5.22) and (5.25) there is no "path tensor" product w.r.t. V_I on the r.h.s. but only an operator product w.r.t. V_I. Of course, if the horizontal colours of X and Y do not match appropriately, the product vanishes. The equations (6.14) and (6.16) suggest to write the matrices of the antipodes of the generator as inverse matrices

$$S(L^{\pm}_j) = (L^{\pm}_j)^{-1} \ . \quad (6.18)$$

The usual relation in the tensor picture for the antipode $m(id \otimes m)(id \otimes S \otimes id)(\Delta \otimes id)\Delta = \mathbf{1}$ now becomes a relation in $\mathcal{G}_I$ in the path picture

$$L^{\pm}_j \cdot S(L^{\pm}_j) \cdot L^{\pm}_j = L^{\pm}_j \quad (6.19)$$

or graphically

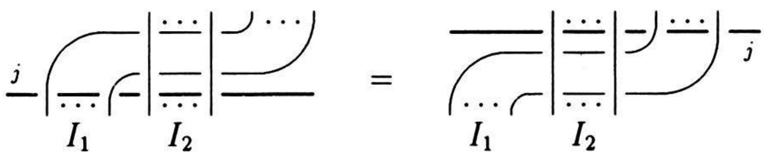


(6.20)

The path Hopf algebra $U^p_I(R)$ is also quasi triangular. The fundamental R-matrix $\mathcal{R}_{12} : V_{I_1} \circ V_{I_2} \to V_{I'_2} \circ V_{I'_1}$ ($|I'_{1,2}| = |I_{1,2}|$, $I'_2 < I'_1$, $|I'_2 \cup I'_1| = |I_1| + |I_2|$) braiding the spaces V_{I_1} and V_{I_2} fulfills

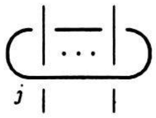
$$\mathcal{R}_{12}\Delta_{12} = \Delta_{21}\mathcal{R}_{12} \quad (6.21)$$

or in terms of graphs for L^-_j



(6.22)

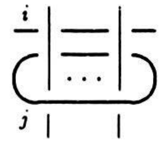
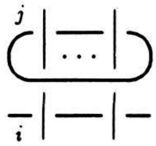
The Casimir elements in $U_I^p(R)$ are given by

$$C_j = \mathrm{tr}_h(S(L_j^-)\cdot L_j^+) = \frac{1}{w_j^2}$$  $(j \in \mathcal{I})$ (6.23)

where the horizontal trace tr_h is defined by

$$\mathrm{tr}_h(S(L_j^-)\cdot L_j^+) = \frac{1}{w_j^2}\sum_{\underline{k}} \frac{w_{k_0}^2}{w_{k_1}^2}(S(L_j^-)\cdot L_j^+)(\underline{k},\underline{k}) \tag{6.24}$$

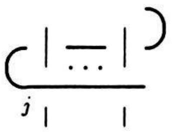
The commutation relations $[L_i^\pm, C_j] = 0$ are obvious, e.g.

$[L_i^-, C_j] =$  $-$  $= 0.$ (6.25)

Similarly, it follows that all observables in $\mathcal{A}_I$ commute with the Casimir operators. Note that (for finite I) the Casimirs are also observables,i.e. in $\mathcal{A}_I$.

Next we discuss properties of special elements in $U_I^p(R)$, which will be used for transformations of states and fields.

Definition 6.3 *The q-symmetry algebra $\mathcal{L}_I$ is a $*$-subalgebra of $U_I^p(R)$. It is generated by the elements*

$L_j = S(L_j^-)\cdot L_j^+ =$  $(j \in \mathcal{I}).$ (6.26)

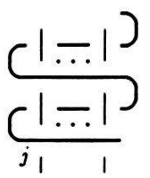
where the product of the two factors is the double product of eq. (6.17) w.r.t. both the vertical and the horizontal space.

For simplicity we list the properties of the L_j's in terms of relations in $\mathcal{G}_I$, $\mathcal{G}_{I_1\cup I_2}$ and $\mathcal{G}_\emptyset$.

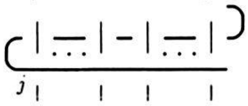
- The algebra product is given by

$$m(L_i, L_j) = L_i \bullet L_j\ . \tag{6.27}$$

Note that for L_j's the double product of eq. (6.17) makes sense

$L_i \cdot L_j = \delta_{ij}$  . (6.28)

- The coproduct is $\Delta(L_j) = \Delta(S(L_j^-) \cdot L_j^+) = \Delta(S(L_j^-)) \cdot \Delta(L_j^+)$ or graphically

$\Delta(L_j) =$  . (6.29)

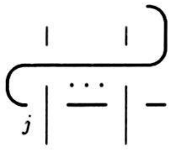
Note that, as opposed to relation (6.8) $\Delta(L_j)$ is not equal to $L_j \odot L_j$. In order to express this relation in terms of L_j only one has to use the fundamental R-matrix $\mathcal{R}_{12} : V_{I_1} \circ V_{I_2} \to V_{I_2'} \circ V_{I_1'}$ braiding the spaces V_{I_1} and V_{I_2} (see also eq. (6.21))

$$\Delta(L) = \mathcal{R}_{21}^{-1} L_1 \mathcal{R}_{12} L_2 \tag{6.30}$$

(The tensor version of this structure is some times denoted by "braided tensor algebra" [25].)

- The counit is given as a map of $\mathcal{G}_I \to \mathcal{G}_\emptyset$ by $\epsilon(L_j) = \epsilon_j^1$ (see also Example 5.1a).

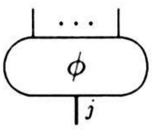
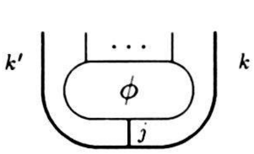
- Analogously to eq. (6.18) we introduce the inverse of L_j as

$L_j^{-1} = S(L_j^+) \cdot L_j^- =$  . (6.31)

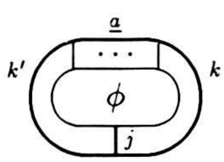
Note that as opposed to relation (6.18) L_j^{-1} is not the antipode $S(L_j) = S(S(L_j^-) \cdot L_j^+) = (S(L_j^+)^{tr} \cdot S^2(L_j^-)^{tr})^{tr}$, where X^{tr} means the transposed matrix: $X^{tr}(\underline{k}', \underline{k}) = X(\underline{k}, \underline{k}')$.

- Finally, note that the Casimir operators of $U_I^p(R)$ defined above may be written as $C_j = \mathrm{tr}_h(L_j)$.

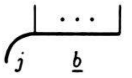
In the second part of this section we investigate the transformation properties of states and field operators under the action of the quantum symmetry algebra $\mathcal{L}_I$. First we discuss the transformation properties of states in V_I under the symmetry transformations. The state space is the span of all states $|\phi_j; k', k\rangle$ obtained from graphs with $|I|$ legs of colour 1/2 pointing upward and one leg of colour j pointing downward

 $\mapsto$ $|\phi_j; k', k\rangle =$  (6.32)

with matrix elements w.r.t. the basis $\{|\underline{a}\rangle\}$

$\langle \underline{a} | \phi_j; k', k \rangle = w_{\underline{a}}$  . (6.33)

Especially, the states $|\phi_j(\underline{b}); k', k\rangle$ with

$\phi_j(\underline{b}) =$  (6.34)

span the space V_I, since they are related to the states $|\underline{a}\rangle$ by a unitary transformation (a product of Fierz transformations (2.13)).

Analogously to Section 5, where we introduced generalized fields $X \in \mathcal{G}_I$ (see eqs. (5.4)-(5.8)) as maps $X : (\underline{k}', \underline{k}) \mapsto X(\underline{k}', \underline{k}) \in \tilde{\mathcal{F}}_I$ we introduce generalized states $|\phi\rangle$ which define a linear space W_I^1 and which are given as maps

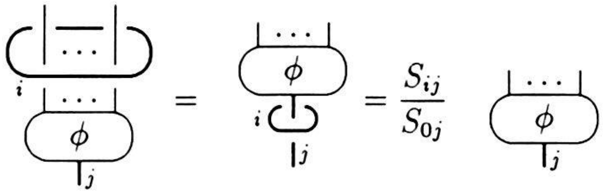
$$|\phi\rangle : (k', k) \mapsto |\phi; k', k\rangle \in V_I \tag{6.35}$$

given by eq. (6.32).

For a state $|\phi_j; k', k\rangle$ and a generalized state $|\phi_j\rangle$ given by eq. (6.32) we call the colour $j \in \mathcal{I}$ the q-spin of the state. This notation makes sense. Indeed, the generalized states $|\phi_j\rangle$ are eigenstates of the Casimir elements C_i

$$C_i|\phi_j\rangle = |\phi_j\rangle \frac{((2i+1)(2j+1))_q}{(2i+1)_q(2j+1)_q}. \tag{6.36}$$

This equation follows as a relation in W_I^1 from



(6.37)

where $S_{ij} = (-1)^{2i+2j} w^{-1} \sin\frac{\pi}{r}(2i+1)(2j+1)/\sin\frac{\pi}{r}$ is the Verlinde matrix (see e.g. [9]). We have the relation (compare eq. (3.25))

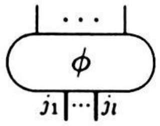
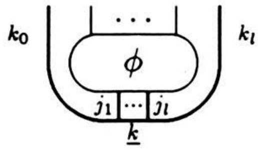
$$p_j^I = S_{0j} \sum_i S_{ji} w_i^2 C_i. \tag{6.38}$$

Indeed the right hand side projects onto the states with q-spin equal to j, because S_{ij} is an orthogonal matrix. The states $|\phi_j; k', k\rangle$ yield the decomposition of the state space (4.7)

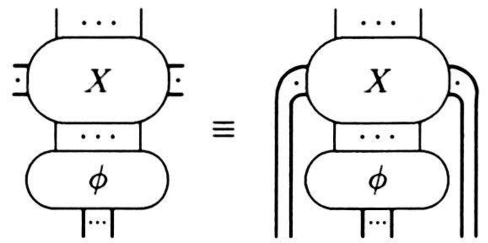
$$V_I = \bigoplus_{k',j,k} V_{I,j}(k', k) \quad \text{with} \quad V_{I,j}(k', k) = p_j^I V_I(k', k)$$

(see also eqs. (4.1) and (4.21)).

Analogously to Section 5, we also introduce the vector spaces W_I^l (for $l = 0, 1, \ldots$) as the span of maps $|\phi_{\underline{j}}\rangle : \underline{k} \mapsto |\phi_{\underline{j}}; \underline{k}\rangle \in V_I$ associated to graphs with $|I|$ legs of colour $1/2$ pointing upward and l lines of colours $\underline{j} = (j_1, \ldots, j_l)$ pointing downward

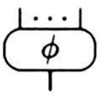
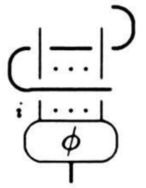

$$: \underline{k} = (k_0, \ldots, k_l) \mapsto |\phi_{\underline{j}}; \underline{k}\rangle = w_{\underline{k}}$$

$$\in V_I \,. \tag{6.39}$$

We set $W_I = \bigcup_l W_I^l$ as a filtered vector space. The generalized fields act as operators on this space: $\mathcal{G}_I^{mn} \times W_I^l \to W_I^{m+l+n}$ defined graphically by



(6.40)

Definition 6.4 *The transformation law for generalized states in W_I under the q-symmetry algebra $\mathcal{L}_I$ is given by*

$$|\phi\rangle \mapsto L_i|\phi\rangle \quad \text{or}$$

$\mapsto$


(6.41)

for all L_i. This means that the q-transformation law for states in V_I is given by

$$|\phi; \underline{l}\rangle \mapsto L_i(\underline{k}', \underline{k})|\phi; \underline{l}\rangle \tag{6.42}$$

Definition 6.5 *A generalized state $|\phi\rangle \in W_I$ is q-invariant, if it is an eigenvector for all elements L_i, more precisely, if*

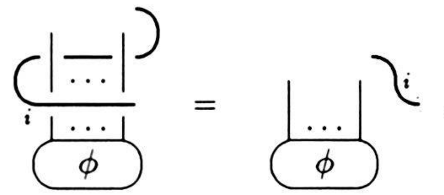
$$L_i|\phi\rangle = |\phi\rangle \times\!\urcorner\, \epsilon_i^1 \tag{6.43}$$

with the notation of eq. (6.15). In terms of matrix elements of a state in V_I this means that

$$\langle \underline{a}|L_i(\underline{k}', \underline{k})|\phi; \underline{l}\rangle = \langle \underline{a}|\phi; \underline{l}\rangle \delta_{\underline{k}'\underline{k}} \delta_{a_{x_f}, k_0} N^i_{k_0' k_0} \,. \tag{6.44}$$

Theorem 6.6 *A state in V_I (or a generalized state $|\phi\rangle \in W_I$) is invariant, if and only if its q-spin is zero.*

Proof: Let $|\phi\rangle$ be equal to $|\phi_{j=0}\rangle$ of relation (6.32), then $L_i|\phi_0\rangle = |\phi_0\rangle \times^{\urcorner} \epsilon_i^1$, which is of the form



(6.45)

Conversely, let $|\phi_j\rangle$ be invariant. Using $\mathrm{tr}_h \epsilon_i^1 = 1$ we obtain

$$C_i|\phi_j\rangle = \mathrm{tr}_h(L_i)|\phi_j\rangle = \mathrm{tr}_h(|\phi_j\rangle \times^{\urcorner} \epsilon_i^1) = |\phi_j\rangle$$

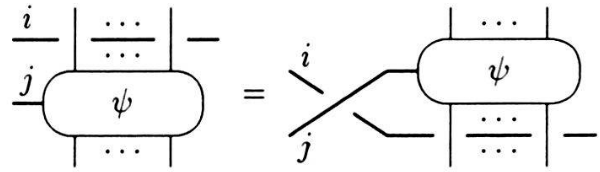
which implies combined with eq. (6.36) that the q-spin j is zero.

The transformation law of fields in $\mathcal{F}_I$ under the action of $L_i^\pm \in U_I^p(R)$ follows from the following commutation rule

$$L_i^\pm \bullet \psi_j = \lambda^\pm \cdot (\psi_j \bullet L_i^\pm). \tag{6.46}$$

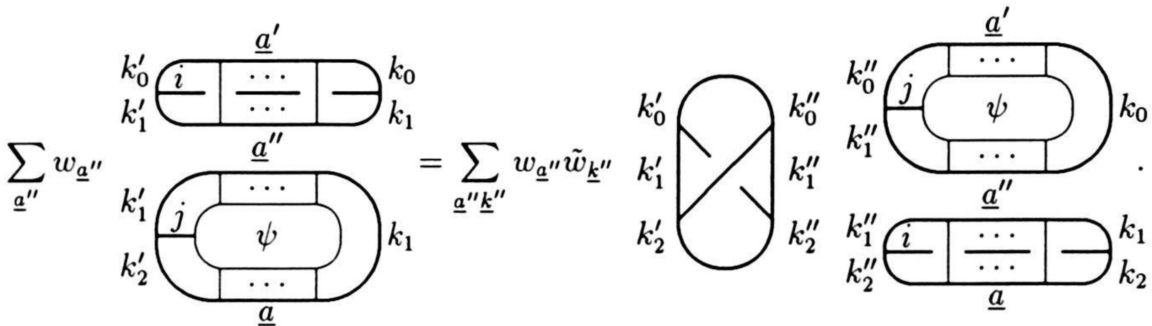
Thus for the field of Example 4.4 λ is equal to the R-matrix, e.g.

$L_i^- \bullet \psi_j = R_{ij} \cdot (\psi_j \bullet L_i^-) \quad :$



(6.47)

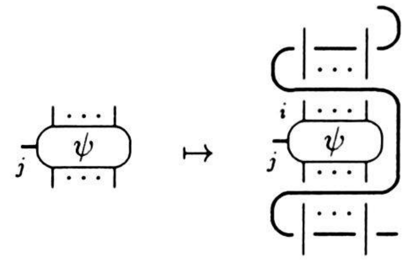
or in terms of matrix elements



(6.48)

Definition 6.7 *The q-transformation law for fields in $\mathcal{F}_I$ under the q-symmetry algebra $\mathcal{L}_I$ is given by*

$\psi_j \mapsto \psi_j^{L_i} = L_i \psi_j \cdot L_i^{-1} \quad :$



(6.49)

for all $L_i \in \mathcal{L}_I$. *A field is invariant if*

$$\psi^{L_i} = \psi \times^{\urcorner} \epsilon_i^1 \tag{6.50}$$

for all L_i.

Theorem 6.8 *a) The fixedpoint algebra* $\mathcal{F}_I^{\mathcal{L}_I}$ *of all fields invariant under the q-symmetry is equal to the algebra of observables* $\hat{\mathcal{A}}_I$.

b) The $\mathcal{L}_I$*-average of a transformed field* ψ^{L_i} *is an observable:*

$$\frac{1}{w^2}\sum_i w_i^4 \mathrm{tr}_h(\psi_j^{L_i}) = \delta_{j0}\psi_j. \tag{6.51}$$

c) Any field may be decomposed into its irreducible components by

$$\psi = \sum_j \psi_j \quad \text{where} \quad \psi_j = \sum_i w_i^2 S_{0j} S_{ji}\, \mathrm{tr}_h(\psi^{L_i}). \tag{6.52}$$

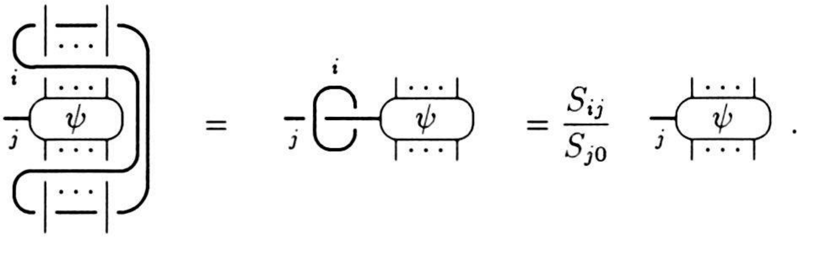
d) The "volume" of the orbit $\psi_j^{\mathcal{L}_I}$ *is the q-dimension* $(2j+1)_q$*:*

$$\sum_{ik} (-1)^{2k} w_k^4 S_{ki} w_i^2\, \mathrm{tr}_h(\psi_j^{L_i}) = (2j+1)_q\, \psi_j. \tag{6.53}$$

Proof: Using eqs. (2.11), (2.17) and (6.47) we find

$$\mathrm{tr}_h(\psi_j^{L_i}) = \mathrm{tr}_h(R_{ji}\cdot R_{ij})\cdot\psi_j = w_i^{-2} S_{ij}/S_{0j}\ \psi_j \tag{6.54}$$

where again $S_{ij} = (-1)^{2i+2j}(\sin\frac{\pi}{r}(2i+1)(2j+1))/(w\sin\frac{\pi}{r})$ is the Verlinde matrix (see e.g. [9]). Part a) of the theorem follows since $S_{ij} = S_{i0} \Leftrightarrow j = 0 \Leftrightarrow \psi_i \in \mathcal{A}_I$. The parts b), c) and d) of the theorem follow from the orthogonality of S_{ij} and $S_{i0} = w_i^2/w$. The graphical interpretation of eq. (6.54) is

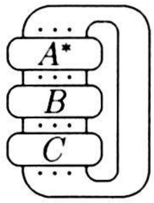


(6.55)

7 The thermodynamic limit

Let $\mathcal{A}$ be the C^*-algebra defined as the norm closed inductive limit of the family $\mathcal{A}_I$ w.r.t. the inclusions $\iota_{I',I}$ $(I \subset I')$ (see eq. (3.31)). By construction viewing the $\mathcal{A}_I$ as C^*-algebras, one also has isometric injective $*$-homomorphisms $\iota_I : \mathcal{A}_I \to \mathcal{A}$ with $\iota_{I'} \circ \iota_{I',I} = \iota_I$ $(I \subseteq I')$ and $\bigcup_I \iota_I(\mathcal{A}_I)$ is dense in $\mathcal{A}$. Furthermore the compatible (see eq. (3.38)) traces tr_I on $\mathcal{A}_I$ extend by continuity to a faithful trace tr on $\mathcal{A}$ such that $\mathrm{tr}_I(A) = \mathrm{tr}(\iota_I(A))$ for all $A \in \mathcal{A}_I$. For simplicity, in the following we will sometimes identify $A \in \mathcal{A}_I$ with its image $A = \iota_I(A) \in \mathcal{A}$.

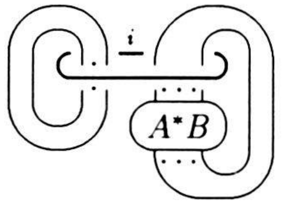
Let $\overline{\mathcal{A}}$ the von Neumann algebra obtained as the weak closure of $\mathcal{A}$ with respect to the representation obtained from the trace tr by the GNS construction. This means that the resulting states $|A\rangle$ with $A \in \bigcup_I \mathcal{A}_I$ form a dense set in the vacuum sector of the Hilbert space and the matrix element of an operator $B \in \bigcup_I \mathcal{A}_I$ is

$$\langle A|B|C\rangle = \mathrm{tr}(A^*BC) =$$  $$(A, B, C \in \mathcal{A}_I). \qquad (7.1)$$

Lemma 7.1 *The other superselection sectors are obtained by application of field operators (see below). The weak limits of the minimal idempotents of the $\mathcal{A}_I$ are proportional to the unit operator, more precisely with $\bar{p}^I_j = 1/2(p^I_j + p_j^{I\cup\{x_{I+1}\}})$*

$$\mathrm{w}\text{-}\lim_{I\to\mathbb{Z}} \bar{p}^I_j = \frac{d_j^2}{w^2}\mathbf{1} \in \overline{\mathcal{A}} \quad (j \in \mathcal{I}),\ w^2 = \sum_j d_j^2\ . \qquad (7.2)$$

Proof: Let $C^{I'}_i \in \mathcal{A}_{I'}$ be the central elements of $\mathcal{A}_{I'}$ (see eq. (6.20)). Then with $A, B \in \mathcal{A}_I$ for $I \subset I'$

$$\langle A|C^{I'}_i|B\rangle = \frac{1}{w_{1/2}^{2|I'|}w_i^2}$$  $$= \left(\frac{(2(2i+1))_q}{2_q(2i+1)_q}\right)^{|I'|-|I|}\langle A|C^I_i|B\rangle. \qquad (7.3)$$

The claim follows since $((2(2i+1))_q)/(2_q(2i+1)_q))^N \approx \delta_{i,0} + (-1)^N\delta_{i,r/2-1}$ for $N \to \infty$ and $\bar{p}^{I'}_j = S_{0j}\sum_i S_{ji}w_i^2\bar{C}^I_j \to S_{0j}^2\mathbf{1}$ with $\bar{C}^I_j$ defined as $\bar{p}^I_j$.

Also ρ_x extends to an isometric injective $*$-homomorphism from $\mathcal{A}$ into $\mathcal{A}$, also denoted by ρ_x and ϕ_x extends to a left inverse of ρ_x on $\mathcal{A}$, also denoted by ϕ_x. Finally the conditional expectation E_x on each $\mathcal{A}_{\bar{I}}$ extends to a conditional expectation on $\mathcal{A}$ with range $\rho_x(\mathcal{A})$.

In analogy to the definition of $\overline{\mathcal{A}}$ let $\overline{\rho_x(\mathcal{A})}$ be the weak closure of $\rho_x(\mathcal{A})$ w.r.t. the representation obtained from the trace tr by the GNS construction.

Theorem 7.2 *The index $[\mathcal{A} : \rho_x(\mathcal{A})]$ of the inclusion $\rho_x(\mathcal{A}) \subset \mathcal{A}$ of C^*-algebras equals $w_{1/2}^2$ as does the index of the inclusion $\overline{\rho_x(\mathcal{A})} \subset \overline{\mathcal{A}}$ of type II_1 von Neumann algebras, both of which are factors. tr is the unique tracial state on both $\overline{\rho_x(\mathcal{A})}$ and $\overline{\mathcal{A}}$.*

Proof: We now fix I with $|I| \geq r-2$ and with $x \in I$ and set (see proof of Theorem 3.11)

$$v_x(b'_{x_f}, \underline{b}) = \iota_{\tilde{I}}\left(v_x^{\tilde{I}}(b'_{x_f}, \underline{b})\right) \in \mathcal{A} . \tag{7.4}$$

These $v_x(b'_{x_f}, \underline{b})$ and their adjoints (modulo a sign) form a quasibasis in $\mathcal{A}$ for the conditional expectation $E_x = \rho_x \phi_x$ satisfying

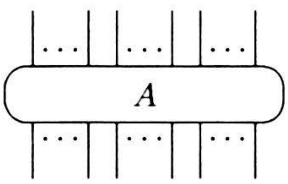
$$\sum_{b'_{x_f}, \underline{b}} (-1)^{2b'_{x_f}+1} (v_x(b'_{x_f}, \underline{b}))^* v_x(b'_{x_f}, \underline{b}) = w_{1/2}^4 \mathbf{1} \tag{7.5}$$

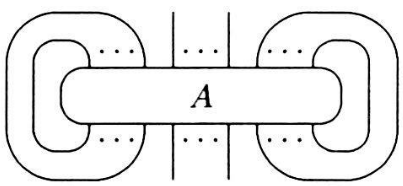
concluding the proof of the first part of the theorem. In fact for any $I' \supseteq I$ it is easy to see that the elements $v_x^{\tilde{I}'}(b'_{x_f}, \underline{b}) = \iota_{\tilde{I}',\tilde{I}}(v_x^{\tilde{I}}(b'_{x_f}, \underline{b})) \in \mathcal{A}_{\tilde{I}'}$ and their adjoints (modulo a sign) form a quasibasis for the conditional expectation $E_x = \rho_x \circ \phi_x : \mathcal{A}_{\tilde{I}'} \to \mathcal{A}_{\tilde{I}'}$ satisfying a relation similar to (3.56). The claim now follows from the continuity of E_x and the fact that by construction the set $\bigcup_{I'} \iota_{I'}(\mathcal{A}_{I'})$ is dense in $\mathcal{A}$.

We turn now to the proof of the second part. By the graphical definitions it follows directly that the maps ρ_x and $\iota_{I',I}$ are compatible in the sense that $\rho_x(\iota_{I',I}(A)) = \iota_{\tilde{I}',\tilde{I}}\rho_x(A))$ for all $I \subseteq I'$ and all $A \in \mathcal{A}_I$. In particular $\iota_{\tilde{I}',\tilde{I}}$ maps $\rho_x(\mathcal{A}_I)$ into $\rho_x(\mathcal{A}_{I'})$. We claim that

$$\begin{array}{ccc} \mathcal{A}_{\tilde{I}} & \xrightarrow{\iota_{\tilde{I}',\tilde{I}}} & \mathcal{A}_{\tilde{I}'} \\ \cup & & \cup \\ \rho_x(\mathcal{A}_I) & \xrightarrow{\iota_{\tilde{I}',\tilde{I}}} & \rho_x(\mathcal{A}_{I'}) \end{array} \tag{7.6}$$

is a commuting square in the sense of Popa [26] (see also e.g. [7]). In fact for all $I \subseteq I'$ the map $\phi_{I,I'} : \mathcal{A}_{I'} \to \mathcal{A}_I$ on graphs $A \in \mathcal{A}_{I'}$

$\phi_{I,\tilde{I}}$: 

$\mapsto \dfrac{1}{(w_{1/2}^2)^{|I'|-|I|}}$ 

(7.7)

is a left inverse of $\iota_{I',I}$. More generally $\phi_{I,I'}(\iota_{I',I}(A)B\iota_{I',I}(A')) = A\phi_{I,I'}(B)A'$ holds for all $A, A' \in \mathcal{A}_I$ and all $B \in \mathcal{A}_{I'}$, Also $\mathrm{tr}_I(\phi_{I,I'}(A)) = \mathrm{tr}_{I'}(A)$ for all $A \in \mathcal{A}_{I'}$. Therefore $E_{I',I} = \iota_{I',I} \circ \phi_{I,I'} : \mathcal{A}_{I'} \to \mathcal{A}_{I'}$ is a conditional expectation with range $\iota_{I',I}(\mathcal{A}_I)$ compatible with the trace in the sense that $\mathrm{tr}_{I'}(E_{I',I}(A)) = \mathrm{tr}_{I'}(A)$ holds for all $A \in \mathcal{A}_{I'}$ and for all $I \subseteq I'$. Also the restriction of $E_{\tilde{I}',\tilde{I}}$ to $\rho_x(\mathcal{A}_{I'})$ is the conditional expectation for the inclusion $\iota_{\tilde{I}',\tilde{I}} : \rho_x(\mathcal{A}_I) \mapsto \rho_x(\mathcal{A}_{I'})$ whenever $x \in I$. Again this follows easily from the graphical definition of ρ_x, proving the claim. Now Theorem 7.2 follows from Theorem 3.11. and a well known result by Wenzl [14] (see also [7]).

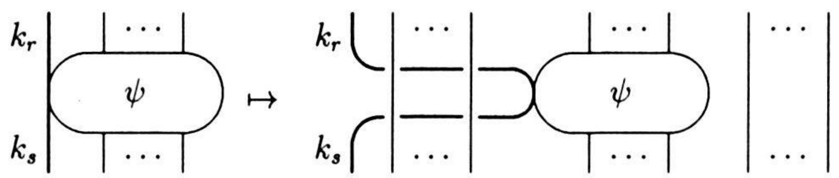
In fact, what remains to be shown is that the inclusion matrices for the inclusions $\iota_{I',I} : \mathcal{A}_I \to \mathcal{A}_{I'}$ and $\iota_{\tilde{I}',\tilde{I}} : \rho_x(\mathcal{A}_I) \to \rho_x(\mathcal{A}_{I'})$ are primitive (see e.g. [7], p. 12) whenever $|I| \geq r-2$ and $|I'| - |I| \geq r-2$. This will in particular prove that $\overline{\mathcal{A}}$ and $\overline{\rho_x(\mathcal{A})}$ both are factors. Now it is easy to see that $p_{j'}^{I'}\iota_{I',I}(p_j^I \mathcal{A}_I p_j^I)p_{j'}^{I'}$ and $p_{j'}^{I'}\iota_{I',I}(p_j^I)\mathcal{A}_I\iota_{I',I}(p_j^I)p_{j'}^{I'}$ are equal and nonvanishing for any j, j' wit $2j+|I|$ and$2j'+|I'|$ even (compare the discussion leading to eq. (3.52)). Hence all entries of the inclusion matrix are equal to one, thus establishing the primitivity in the first case. The second inclusion is treated similarly.

The canonical injective $*$-homomorphisms $\iota_{I',I}$ may be extended to $\mathcal{F}_I$, $\tilde{\mathcal{F}}_I$ and $\mathcal{G}_I$. The map

$$\iota_{I',I} : \mathcal{F}_I \to \mathcal{F}_{I'} \quad (I \subset I'),$$

which restrict to maps from $\mathcal{A}_I$ into $\mathcal{A}_{I'}$, are defined on graphs (and extended linearly) as follows:

$\iota_{I',I}:$



(7.8)

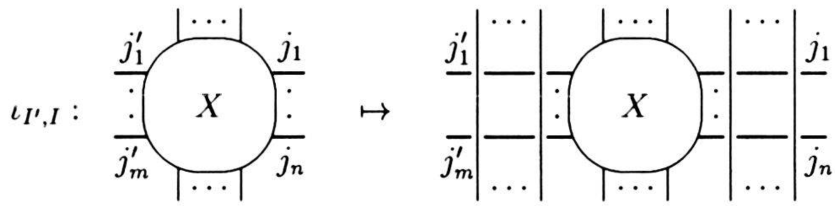
Here $x'_f - x_f$ vertical lines with colour 1/2 have been added on the right and $x_i - x'_i$ lines, again with colour 1/2, on the left. We have local commutativity of the local field and observable algebras in the sense that $\iota_{I,I_1}(\psi)$ and $\iota_{I,I_2}(A)$ commute in $\mathcal{F}_I$ ($I \supseteq I_1 \cup I_2$) for all $\psi \in \mathcal{F}_{I_1}, A \in \mathcal{A}_{I_2}$ whenever $I_1 \cap I_2 = \emptyset$. Obviously we have compatibility with the trace (see eq. (4.14))

$$tr_{I'}\Big(\iota_{I',I}(\psi)\Big) = tr_I(\psi) \tag{7.9}$$

for all $\psi \in \mathcal{F}_I$ and with the conditional expectation E_I (see (4.17)) with range $\mathcal{A}_I$

$$\iota_{I',I} \circ E_I = E_{I'} \circ \iota_{I',I} \quad (I \subseteq I'). \tag{7.10}$$

Analogously we have a canonical injective $*$-homomorphisms $\iota_{I',I} : \tilde{\mathcal{F}}_I \to \tilde{\mathcal{F}}_{I'}$. Finally these maps induce injective $*$-homomorphisms from $\mathcal{G}_I$ into $\mathcal{G}_{I'}$ respecting the filtering, also denoted by the same symbol. They are given on graphs $X \in \mathcal{G}_I^{mn}$ as (compare (7.8))



(7.11)

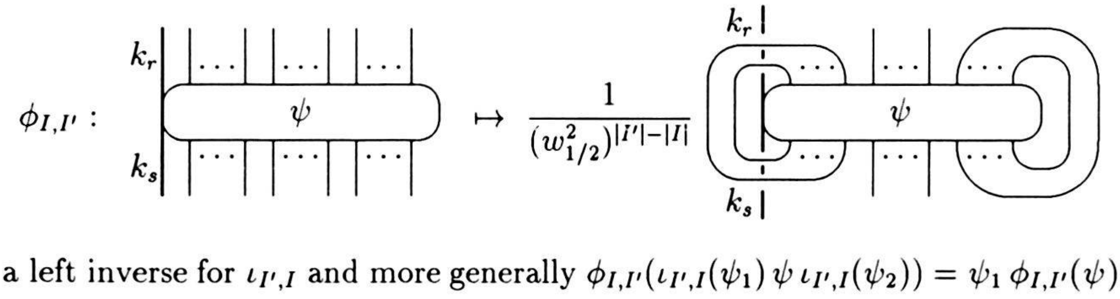
with $x_i - x'_i$ vertical lines added to the left and $x'_f - x_f$ lines added to the right, each with colour 1/2.

Let $\mathcal{F}$, $\hat{\mathcal{A}}(\subset \mathcal{F})$ and $\mathcal{A}(k)$ be the inductive C^*-algebra limits of the families $\mathcal{F}_I$, $\hat{\mathcal{A}}_I$ and $\mathcal{A}_I(k)$ with respect to the inclusion maps $\iota_{I',I}$ respectively. By construction we have injective isometric $*$-homomorphisms $\iota_I : \mathcal{F}_I \to \mathcal{F}$ satisfying $\iota_{I'} \circ \iota_{I',I} = \iota_I$ $(I \subseteq I')$ and mapping $\hat{\mathcal{A}}_I$ into $\hat{\mathcal{A}}$ such that $\bigcup_I \iota_I(\mathcal{F}_I)$ and $\bigcup_I \iota_I(\hat{\mathcal{A}}_I)$ are dense in $\mathcal{F}$ and $\hat{\mathcal{A}}$ respectively. In particular since $\iota_{I',I}(\mathbf{1}_I(k)) = \mathbf{1}_{I'}(k)$, the elements $\mathbf{1}(k) = \iota_I(\mathbf{1}_I(k))$ are well defined and satisfy $\mathbf{1}(k)\mathbf{1}(k') = \delta_{kk'}\mathbf{1}(k)$, $\mathbf{1}^*(k) = \mathbf{1}(k)$ and $\sum \mathbf{1}(k) = \mathbf{1}$. The element $\mathbf{1}(k)$ is the unit in $\mathcal{A}(k)$ and in the center of $\hat{\mathcal{A}}$. Analogously $\bigcup_I \iota_I(\mathcal{A}_I(k))$ is dense in $\mathcal{A}(k)$, $\mathcal{A}(k)\mathcal{A}(k') = 0$ $(k \neq k')$ and $\hat{\mathcal{A}} = \bigoplus_k \mathcal{A}(k)$. By the compatibility property (7.9), the family of traces tr_I defines a faithful trace tr on $\mathcal{F}$. Again via a GNS construction this trace gives rise to a representation of $\mathcal{F}$. Let $\overline{\mathcal{F}}$, $\overline{\hat{\mathcal{A}}}$ and $\overline{\mathcal{A}(k)}$ be the weak closures of $\mathcal{F}$, $\hat{\mathcal{A}}$ and $\mathcal{A}(k)$, respectively in this representation. In particular all $\overline{\mathcal{A}}(k)$ commute and $\overline{\hat{\mathcal{A}}} = \bigoplus_k \overline{\mathcal{A}}(k)$. Also tr extends to a tracial state on $\overline{\mathcal{F}}$.

Now

$$\begin{array}{ccc} \mathcal{F}_I & \stackrel{\iota_{I',I}}{\longrightarrow} & \mathcal{F}_{I'} \\ \cup & & \cup \\ \hat{\mathcal{A}}_I & \stackrel{\iota_{I',I}}{\longrightarrow} & \hat{\mathcal{A}}_{I'} \end{array} \tag{7.12}$$

is a commuting square in the sense of Popa. Indeed, the linear map $\phi_{I,I'} : \mathcal{F}_{I'} \to \mathcal{F}_I$ $(I' \supseteq I)$ given for graphs $\psi \in \mathcal{F}_{I'}$ by (compare (7.7))



(7.13)

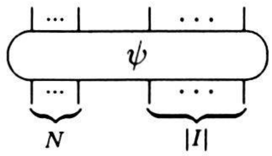
is a left inverse for $\iota_{I',I}$ and more generally $\phi_{I,I'}(\iota_{I',I}(\psi_1)\,\psi\,\iota_{I',I}(\psi_2)) = \psi_1\,\phi_{I,I'}(\psi)\,\psi_2$ holds for all $\psi_1, \psi_2 \in \mathcal{F}_I, \psi \in \mathcal{F}_{I'}$. The resulting conditional expectation $E_{I',I} = \iota_{I',I} \circ \phi_{I,I'}$: $\mathcal{F}_{I'} \to \mathcal{F}_{I'}$ with range $\iota_{I',I}(\mathcal{F}_I)$ is again compatible with the trace in the sense that $\mathrm{tr}_{I'}(E_{I',I}(\psi))$ $=\mathrm{tr}_{I'}(\psi)$ holds for all $\psi \in \mathcal{F}_{I'}$. When restricted to $\mathcal{A}_{I'}, \phi_{I,I'}$ is the left inverse for the bottom inclusion in (7.12). Again by arguments similar to those given in the proof of Theorem 4.5 this leads to

Theorem 7.3 *The index of the inclusion $\mathcal{A} \subset \mathcal{F}$ of C^*-algebras is given by*

$$[\mathcal{F} : \mathcal{A}] = \left(\sum_k d_q(k) \right)^2 = \beta. \tag{7.14}$$

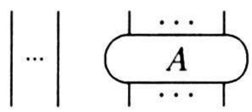
β is also the Jones index for the inclusion $\overline{\mathcal{A}} \subset \overline{\mathcal{F}}$. $\overline{\mathcal{A}}$ and $\overline{\mathcal{F}}$ are type II_1 factors and tr is the unique tracial state on $\overline{\mathcal{F}}$. $\overline{\mathcal{A}}$ is not irreducible in $\overline{\mathcal{F}}$ and $\overline{\hat{\mathcal{A}}}$ is not a factor.

We note that this index also appears in the context of certain subfactors studied by Choda and Ocneanu [27]. In fact, there is an alternative way of introducing a field algebra and which is more closely related to this work. It is obtained by introducing an algebra at infinity in the following way: Replace the path space V_I by the path space V_I^∞ spanned by symbols of the form $|b_1, \cdots b_N, a_{x_i - 1}, \cdots a_{x_j} \rangle$. Here N is chosen to be equal to $r/2 - 1$. The colours a_k are again subject to the usual fusion rules and the colours b_k satisfy similar fusion rules, i.e. $N_{b_N}^{1/2}{}_{a_{x_i - 1}} = 1 = \prod_{j=1}^{N-1} N_{b_i b_{i+1}}^{1/2}$. The field algebra $\mathcal{F}_I^\infty$ is then the linear span of all graphs ψ considered as endomorphisms of V_I^∞ of the form



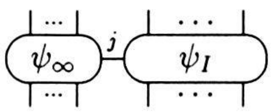
(7.15)

with $N + |I|$ horizontal and vertical legs respectively. Our previous $\mathcal{A}_I$ is obviously isomorphic to the subalgebra of $\mathcal{F}_I^\infty$ spanned by elements of the form



(7.16)

Note that $\mathcal{F}_I^\infty$ is generated by elements of the form



(7.17)

where the color j is arbitrary (due to the choice $N = \frac{r}{2} - 1$). Those elements with $j = 0$ span an algebra $\mathcal{A}_I^\infty \subset \mathcal{F}_I^\infty$ containing $\mathcal{A}_I$. The corresponding part ψ_∞ may be viewed as an observable at infinity. Now the inclusion $\mathcal{A}_I^\infty \subset \mathcal{F}_I^\infty$ in the thermodynamic limit $|I| \to \infty$ is essentially the situation considered by Choda and Ocneanu (apart from the fact that in our case N stays fixed). At the moment, however, we do not see the physical relevance of this construction.

Proof of Theorem 7.3: To prove the first part we introduce the quantities

$$v(\underline{b}, l, \underline{k}, j) = \iota_I(v_I(\underline{b}, l, \underline{k}, j)) \in \mathcal{F}. \tag{7.18}$$

where I is an arbitrary fixed interval with $|I| > r-2$. Then these elements and their adjoints (modulo a sign) form a quasibasis w.r.t. the conditional expectation $E : \mathcal{F} \to \hat{\mathcal{A}}$ given as the inductive limit of the E_I. Indeed, it is easy to see that the elements $v_{I'}(\underline{b}, l, \underline{k}, j) = \iota_{I',I}(v_I(\underline{b}, l, \underline{k}, j))$ (with $v(\underline{b}, l, \underline{k}, j) = \iota_{I'}(v_{I'}(\underline{b}, l, \underline{k}, j)))$ and their adjoints (modulo a sign) form a quasibasis in $\mathcal{F}_{I'}$ w.r.t. the conditional expectation $E_{I'} : \mathcal{F}_{I'} \to \mathcal{A}_{I'}$ for any $I' \supseteq I$. The claim then follows from the continuity of E and the fact that by construction $\bigcup_{I'} \iota_{I'}(\mathcal{F}_{I'})$ and $\bigcup_{I'} \iota_{I'}(\mathcal{A}_{I'})$ are dense in $\mathcal{F}$ and $\mathcal{A}$ respectively. Finally, the $v(\underline{b}, l, \underline{k}, j)$ satisfy a relation similar to (4.28), thus concluding the first part of the theorem. Since the conditional expectation E extends to a conditional expectation for the inclusion $\overline{\mathcal{A}} \subset \overline{\mathcal{F}}$ of von Neumann algebras the above quasi basis turns into a Pimsner-Popa basis thus proving that β is also the index of the inclusion $\overline{\mathcal{A}} \subset \overline{\mathcal{F}}$.

To see that $\overline{\mathcal{F}}$ is a factor and that tr is the unique tracial state we proceed as in the proof of the preceeding theorem. The algebras $P_{l'}^{I'} \iota_{I',I}(P_l^I \mathcal{F}_I P_l^I) p_{l'}^{I'}$ and $P_{l'}^{I'} \iota_{I',I}(P_l^I) \mathcal{F}_I \iota_{I',I}(P_l^I) p_{l'}^{I'}$ are equal and nonvanishing for all $l, l' \in \mathcal{I}$ and $I \subseteq I'$ provided $|I| \geq r-2$ and $|I'| - |I| \geq r-2$. Thus all the entries of the inclusion matrix for the inclusion $\iota_{I',I}(\mathcal{F}_I) \subset \mathcal{F}_{I'}$ are equal to one. By similar arguments $\overline{\mathcal{A}}$ is also a factor. To see that $\overline{\hat{\mathcal{A}}}$ is not a factor, note that the elements $\mathbf{1}(k)$ are also nontrivial central elements of $\overline{\hat{\mathcal{A}}}$. Also since these $\mathbf{1}(k)$ commute with $\mathcal{A}$ and hence with $\overline{\mathcal{A}}$, $\overline{\mathcal{A}}$ is not irreducible in $\mathcal{F}$.

Remark 7.4 *Let $L^2(\mathcal{F})$ be the Hilbert space obtained from the GNS construction. It contains $L^2(\mathcal{A})$ in a natural way, where $L^2(\mathcal{A})$ is the corresponding GNS Hilbert space for $\mathcal{A}$. Let $|\Omega\rangle \in L^2(\mathcal{A})$ be the state corresponding to the unit element. Consider $|\Omega_k\rangle = \mathbf{1}(k)|\Omega\rangle$, which is nonzero since $\langle \Omega_k | \Omega_k \rangle = tr(\mathbf{1}(k)) = d_q(k)/\beta^{1/2}$ see eq. (4.16). These vectors are pairwise orthogonal and $|\Omega\rangle = \sum_k |\Omega_k\rangle$. Let $L^2(\mathcal{A}(k))$ be the closure of $\mathcal{A}(k)|\Omega\rangle$, which is also the closure of $\mathcal{A}|\Omega_k\rangle$. In fact $L^2(\mathcal{A}(k))$ is obtained from the GNS construction w.r.t. $\mathcal{A}(k)$ using $tr_k(\cdot) = tr(\cdot\, \mathbf{1}(k))/tr(\mathbf{1}(k))$. Then each $L^2(\mathcal{A}(k))$ is invariant under $\overline{\mathcal{A}}$ and we have the decomposition*

$$L^2(\hat{\mathcal{A}}) = \bigoplus_{k \in \mathcal{I}} L^2(\mathcal{A}(k))$$

which is analogous to multiplicities w.r.t. magnetic quantum numbers in the tensor picture.

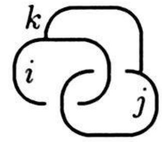
Finally we discuss the q-symmetry algebra in the thermodynamic limit $I \to \mathbb{Z}$. As above let $\tilde{\mathcal{F}}$ be the C^*-algebra limit of the families $\tilde{\mathcal{F}}_I$ and $\iota_I : \tilde{\mathcal{F}}_I \to \tilde{\mathcal{F}}$ the injective isometric $*$-homomorphisms satisfying $\iota_{I'} \circ \iota_{I',I} = \iota_I$ $(I \subseteq I')$ such that $\bigcup_I \iota_I(\tilde{\mathcal{F}}_I)$ is dense in $\tilde{\mathcal{F}}$. Also we have $\iota_I(L) \in \tilde{\mathcal{F}}$ for $L \in \mathcal{L}_I$.

Lemma 7.5 *The q-transformation $\psi \to \psi^L$ for $\psi \in \mathcal{F}$ as an extension of relation (6.49) is well defined.*

Proof: First note that the definition of the q-transformation of fields (6.49) is compatible with the maps $\iota_{I',I}$ since by the definition of $\iota_{I',I}$ for $\psi \in \mathcal{F}_{I_1}$ and $L \in \mathcal{L}_{I_2} \subset \tilde{\mathcal{F}}_{I_2}$ the map $\iota_{I_4,I_1}(\psi) \to \iota_{I_4,I_3}((\iota_{I_2,I_1}(\psi))^{\iota_{I_3,I_2}(L)})$ is independent of I_3 for $I_1 \subset I_2 \subset I_3 \subset I_4$. Therefore by $\iota_{I_1}(\psi) \to \iota_{I_1}(\psi^L)$ it yields a well defined map $\psi \to \psi^L \in \tilde{\mathcal{F}}$ for a dense set of ψ's in $\mathcal{F}$. However, note that ψ^L cannot be written as $L\psi \cdot L^{-1}$ by an $L \in \tilde{\mathcal{F}}$ as for finite lattices in relation (6.49).

Analogously to eq. (6.28) we may define $\mathcal{L}_I^l$ as generated by the $L_i^l = L_i^- \cdot S(L_i^+)$ (as a left version of $\mathcal{L}_I$ obtained by mirroring along a vertical line). For a field operator in $\psi_k \in \mathcal{F}_I$ of spin k we have

$$\langle\Omega|L_i^l \psi_k p_j^l|\Omega\rangle \propto S_{ij}^k = \quad (7.19)$$

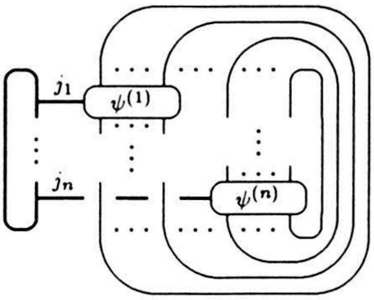


where the S_{ij}^k are generating elements of matrix representations of the mapping class groups of arbitrary genus, which have been discussed in [28] (see also [29]).

Note that the algebra of fields $\mathcal{F}$ is generated by elements of the form $\psi = AL^l$ for $A \in \mathcal{A}$ and $L^l \in \mathcal{L}_I^l$ for any interval I. The transformation $A_2^{L_1^l} = L_1^l A_2 \cdot L_1^{l\,-1}$ is in general nontrivial if the interval I is contained in but smaller than the support of A.

We conclude with the following remark. Usually the internal group symmetry and the external space-time symmetry form a tensor product. Due to the nontrivial R-matrix the situation is different for the q-symmetry. Nevertheless, we have the following cluster property. If the fields $\psi_{j_i}^{(i)}$ are localized in $I^{(i)}$ for $1 \leq i \leq n$ such that pairwise $I^{(i)} \cap I^{(i')} = \emptyset$, then the vacuum expectation value of a product of fields factorizes

$$\langle\Omega|\prod_{i=1}^{n} \psi_{j_i}^{(i)}|\Omega\rangle \propto \quad [\text{diagram}] \quad \propto \prod_{i=1}^{n} \delta_{j_i 0}\langle\Omega|\psi_{j_i}^{(i)}|\Omega\rangle. \quad (7.20)$$

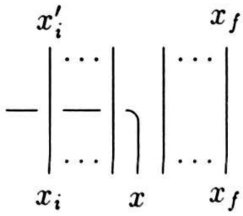
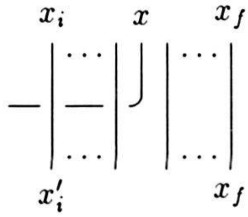


The seemingly unphysical feature, that factorization takes place for all nonoverlapping supports of the fields, is due to the fact that the state given by the trace is an infinite temperature state. For other states like ground states of a dynamical system given by a local Hamiltonian we expect the usual cluster property.

8 Applications and Outlook

In this section we will show how to describe models of statistical mechanics in our path space picture by providing explicit Hamiltonians. In particular we will show how the RSOS model of Baxter [2] fits into this picture.

It is convenient to slightly extend the above concepts and to introduce fields in $\mathcal{F}_{I',I} \subset Hom(V_I, V_{I'})$ which map elements of V_I into $V_{I'}$ for $I \neq I'$. We also introduce the corresponding linear space $\mathcal{G}_{I'I}$, such that $\mathcal{G}_I = \mathcal{G}_{II}$. For example for $I = \{x_i, \dots, x_f\}$ and $I' = \{x_i' = x_i + 1, \dots, x_f\}$

$\psi(x) =$  and $\psi^*(x) =$  . (8.1)

Note that the local endomorphism of eq. (3.41) and its left inverse (3.44) may be written in terms of these fields

$$\begin{aligned} \rho_x(A) &= \sigma \cdot (\psi^*(x) \bullet A \bullet \psi(x)) \\ \phi_x(A) &= \sigma \cdot (\psi(x) \bullet A \bullet \psi^*(x)) \end{aligned} \tag{8.2}$$

where $\sigma = \subset \in \mathcal{G}_\emptyset{}^{02}$ or $\langle a'|\sigma(\underline{k})|a\rangle = w_{\underline{a}'} w_{\underline{a}} \tilde{w}_{\underline{k}} \delta_{a'a} \delta_{ak_0} \delta_{ak_2} N_{ak}^{1/2} = w_{k_1}/w_a \delta_{a'a} \delta_{ak_0} \delta_{ak_2} N_{ak}^{1/2}$.

The fields ψ and ψ^* fulfil "Cuntz-algebra" like relations:

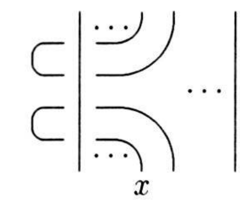
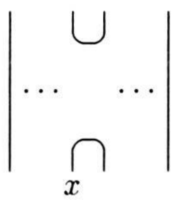
$$\sigma \cdot (\psi^* \bullet \psi) = \mathbf{1} \quad \text{and} \quad \psi \bullet \psi^* = \bar{\sigma} \cdot \mathbf{1} \tag{8.3}$$

where $\bar{\sigma} = \supset \in \mathcal{G}_\emptyset{}^{20}$.

The RSOS-Hamiltonian (for the gap-less case) acting in V_I is

$$H = J(\sigma \bullet \sigma) \cdot \Big(\sum_{x=x_i}^{x_f - 1} \psi^*(x) \bullet \psi^*(x+1) \bullet \psi(x+1) \bullet \psi(x) \Big) \tag{8.4}$$

and can be written as a sum of Temperly-Lieb projectors

$H = J \sum_x$  $= J \sum_x$  (8.5)

The Hamiltonian is obviously quantum group invariant $[H, U_I^p(R)] = 0$. An example of a symmetry breaking contribution to the Hamiltonian is provided by the local field $\psi_j(\underline{k})(x)$

of eq. (4.13) for $j \neq 0$ or by $\phi(\underline{k})(x) = (\psi^*(x) \bullet \psi(x))(\underline{k})$ in the form

$$H_1 = \sum_x J_1(\underline{k}, x)\psi_j(\underline{k})(x) \quad \text{or} \quad H_2 = \sum_x J_2(\underline{k}, x)\phi(\underline{k})(x) \tag{8.6}$$

where $J_i(\underline{k}, x)$ is an external (classical) field. Spontaneous symmetry breaking would be present if the ground state $|0\rangle$ satisfies

$$\langle 0|\psi_j(\underline{k})(x)|0\rangle \neq 0 \quad \text{or} \quad \langle 0|\phi(\underline{k})(x)|0\rangle \neq 0. \tag{8.7}$$

Such a ground state $|0\rangle$ and also that of the Hamiltonian (8.4) which can be obtained by the Bethe ansatz are usually very different from the ground state $|\Omega\rangle$ discussed in Section 7 given by the trace. The latter one is typically an infinite temperature state. It would be interesting to repeat the GNS construction and the discussion of Section 7 for physically more interesting ground states.

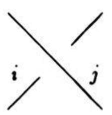
Some further points which we intend to investigate elsewhere are:

- For RSOS-like models order parameters, phase structure and the spontaneous symmetry breaking mentioned above.
- Techniques developed in [30] may be used to transfer the results of this paper to the case of periodic boundary conditions, i.e. to the lattice on a circle.
- In addition to the thermodynamic limit discussed in Section 7 the continuum limit may be analysed using "cabling" techniques.
- Techniques of topological quantum field theory developed in [9] may also be used to apply the ideas and results of this paper to lattices in two space dimensions. Thereby one might obtain a formulation of the q-symmetry for quantum field theories describing particles with braid group statistics in 2+1-dimensions.

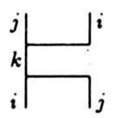
A Appendix

This appendix contains the proof of second part of Lemma 3.6 and will be carried out in several steps. It is a path space formulation of the fact that all representations may be obtained by tensoring the fundamental representation.

Step 1. We first claim that any graph $A \in \mathcal{A}_I^0$ is a sum of graphs in $\mathcal{A}_I^0$ with no 4-vertices. Indeed, we may successively eliminate all 4-vertices by (2.12) and (2.15)



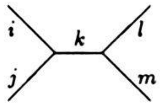
$$= \sum_k w_k^2 \frac{q_k}{q_i q_j}$$



(A.1)

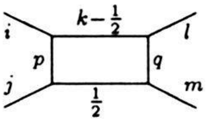
With this identity the proof is complete.

Step 2. We claim that each $A \in \mathcal{A}_I$ can be written as a sum of graphs with no 4-vertices and with lines whose colours are $= 1/2$. We proceed by induction on the largest colour and consider the local case

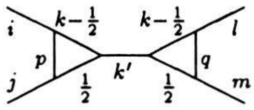


(A.2)

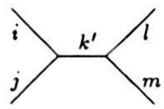
with $k \geq \max(i,j,l,m)$. Without loss of generality, by (2.13) we may assume $i \neq 0, j \neq 0, l \neq 0, m \neq 0$). Using (2.12) and (2.18) we have



$$= \sum_{k'=k-1}^{k} w_{k'}^2$$

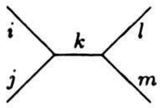


$$= \sum_{k'=k-1}^{k} w_{k'}^2 \begin{vmatrix} i & j & k' \\ \frac{1}{2} & k-\frac{1}{2} & p \end{vmatrix} \begin{vmatrix} l & m & k' \\ \frac{1}{2} & k-\frac{1}{2} & q \end{vmatrix}$$

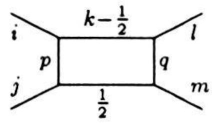


(A.3)

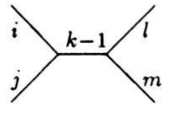
This gives



$$= \frac{1}{w_k^2} \begin{vmatrix} i & j & k \\ \frac{1}{2} & k-\frac{1}{2} & p \end{vmatrix}^{-1} \begin{vmatrix} l & m & k \\ \frac{1}{2} & k-\frac{1}{2} & q \end{vmatrix}^{-1} \Bigg\{$$



$$-w_{k-1}^2 \begin{vmatrix} i & j & k-1 \\ \frac{1}{2} & k-\frac{1}{2} & p \end{vmatrix} \begin{vmatrix} l & m & k-1 \\ \frac{1}{2} & k-\frac{1}{2} & q \end{vmatrix}$$



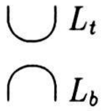
$\Bigg\}$ (A.4)

for any p, q for which

$$\begin{vmatrix} p & k-\frac{1}{2} & \frac{1}{2} \\ k & j & i \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} q & k-\frac{1}{2} & \frac{1}{2} \\ k & m & l \end{vmatrix}$$

are nonvanishing. Now we use that fact that the 6j symbols are nonvanishing whenever all the relevant fusion rules are satisfied. We claim this is the case for the choice $p = \max(i,j) - \frac{1}{2}$, $q = \max(l,m) - \frac{1}{2}$. It suffices to consider the case $i \leq j$ and $l \leq m$ such that $p = j - \frac{1}{2}$, $q = m - \frac{1}{2}$. By assumption $N_{jk}^i = N_{mk}^l = 1$. It is easy to show that $i \leq j \leq k \geq 1$ and $N_{jk}^i = 1$ implies $N_{pk-\frac{1}{2}}^i = 1$ and $N_{p\frac{1}{2}}^j = 1$ with $p = j - \frac{1}{2}$. The other two fusion rules involving $\frac{1}{2}, k-\frac{1}{2}, q, l$ and m are verified similarly. By (A.4) we have (locally) decreased the maximal colour and since by this procedure no 4-vertices are generated the claim is complete.

Step 3. Let $A \in \mathcal{A}_I$ by any graph with no 4-vertices and with lines whose colours are all $1/2$. By the fusion rule it therefore cannot contain any 3-vertices. A is formed of $|I|$ lines starting and ending at the top or the bottom and closed loops which are disconnected from the rest of the graph. Using eqs. (2.2) and (2.11) the loops can be replaced by numbers. We claim that A may be written as a linear combination of graphs with all $|I|$ lines starting at the bottom and ending at the top. This will also conclude the proof of Theorem 3.1. To prove this claim we proceed by induction on $k(A)$, where $k(A) \leq |I|$ is the number of lines starting and ending at the bottom (equal to the number of lines starting and ending at the top). For $k(A) = 0$ there is nothing to prove. Now let $k(A) > 0$. Pick a line L_b starting and ending at the bottom and L_t a line starting and ending at the top. By iterative application of (2.8), we may place a part of L_b close to a part of L_t depicted as



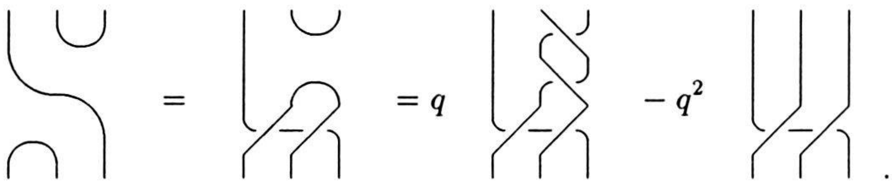
(A.5)

We now use the skein relation (for lines with colour 1/2) in the form

(A.6)

and insert it into (A.5).

Example A.1 *($|I| = 3, k(A) = 1$):*



(A.7)

In the general case we have written A as a linear combination of two elements A_1 and A_2 in $\mathcal{A}_I$ with $k(A_1) = k(A_2) = k(A) - 1$. This concludes the proof of the theorem, since the procedure again does not generate closed loops. The following remark is a reformulation of a well known result (Kaufmann [31])

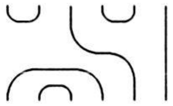
Remark A.2 *After step 2 we have written any graph as a linear combination of graphs A with no 4-vertices and with lines whose colours are all 1/2. Such graphs may be written in terms of generalized Temperly-Lieb operators. Let c_{2n} be the graph*

$c_{2n} =$  (A.8)

consisting of n lines with colour 1/2. Then the linear hull of expressions of the form

$$A = \left((c^*_{m_1} \circ \mathbf{1}_{m_2}) \circ \cdots \circ (c^*_{m_{M-1}} \circ \mathbf{1}_{m_M})\right) \mathbf{1}_K \left((c_{n_1} \circ \mathbf{1}_{n_2}) \circ \cdots \circ (c_{n_{N-1}} \circ \mathbf{1}_{n_N})\right) \quad \text{(A.9)}$$

span $\mathcal{A}_I$ *where* $\sum m_i = \sum n_i = |I|$ *and* $K = \sum_{i=\text{even}} m_i = \sum_{i=\text{even}} n_i$. *As an example we have*



$$= \left(c_2^* \circ \mathbf{1}_1 \circ c_2^* \circ \mathbf{1}_1\right)\mathbf{1}_2\left(c_4 \circ \mathbf{1}_2\right) . \tag{A.10}$$

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