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Atoms and Oscillators in Quasi-Periodic External Fields

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Abstract. We review some results on the spectrum of oscillators and two-level systems in external fields which are quasi-periodic in time.

It is a pleasure to dedicate this paper to Professors K. Hepp and W. Hunziker on the occasion of their sixtieth birthday.

In recent years, there has been a increasing interest in quantum systems subject to periodic or quasi-periodic perturbations. Some of this interest is due to the fact that such systems constitute a paradigm for "quantum chaos" because the dynamics of their classical counterparts is, as a rule, chaotic. Perhaps the most interesting examples of experimental relevance are Rydberg atoms in intense external electric microwave fields [1]. One of the most striking manifestations of quantum mechanics in the latter is the "quantum suppression of classical diffusion", which occurs for large frequencies and leads to *localization*. At least two independent mechanisms of localization occur in Rydberg atoms: a dynamic (pseudorandom) Anderson localization [1] and localization by scars of special (unstable) periodic orbits [2].

What happens in the quasi-periodic case? Several results exist for particles in discrete quasi-periodic potentials, beginning with the pioneering work of W. Craig [3], J. Pöschel [4] and Bellissard, Lima, Scoppola and Testard [5], which relied on KAM methods. More recently, a global (i.e., nonperturbative) result has been obtained by Süto [6] for the Fibonacci Hamiltonian: the spectrum is singular continuous, supported by a Cantor set of zero Lebesgue measure. For systems under perturbations which are quasi-periodic in *time*

— e.g., atoms in bichromatic electric fields —, there are fewer rigorous or exact results. Perturbative KAM results for small coupling were obtained by Blekher, Jauslin and Lebowitz [7] for two-level atoms in quasi-periodic fields:

$$H(t) = \beta\sigma_z + \epsilon f(t)\sigma_x \tag{1}$$

where 2β is the energy difference between the unperturbed levels important and f is quasi-periodic. The results of [7] rely on previous work by Bellissard [8] and M. Combescure [9]. Moreover, nonempty continuous spectrum was proven to exist generically for models of two-level systems [7]. The latter is a nonperturbative result which we shall comment upon later. Finally, in this paper we have generalized the results of [7] to a restricted class of quasi-periodic perturbations in the case of large coupling. It should be mentioned that for two-level Fibonacci Hamiltonians there exists a non-perturbative result [10] which states that the spectrum (as defined in [10]) has no pure-point part. This result provides additional support (a first indication came from the numerical results for the kicked rotor with two or three incommensurate frequencies [11]) to the expectation that localization is weaker (if present at all) in the quasi-periodic case. It seems to be difficult, however, to establish the result of [10] using any of the two equivalent definitions of spectrum used in the present paper.

Since there are so few nonperturbative results for the spectrum of two-level atoms under quasi-periodic perturbation, it is natural to ask what happens for the forced harmonic oscillator, where a solution in closed form exists:

$$H(t) = \omega_0\, a^+ a + \lambda f(t)(a + a^+)\ . \tag{2}$$

Here a and a^+ are standard annihilation and creation operators, and f is quasi-periodic. This has been done by Jauslin and Nerurkar [12] and by us [13]. We shall focus on [13], which proves slightly more than [12] in the resonant case, namely, that the spectrum is transient absolutely continuous.

In order to pose the problems more precisely, it is better to consider the periodic case first [14]. Let us assume we are given a Hamiltonian

$$H(t) = H_0 + V(t)$$

where H_0 is a self-adjoint operator on a Hilbert space $\mathcal{H}$ with discrete spectrum $\{E_n\}_{n=1}^{\infty}$ and V is, for instance, a bounded periodic operator, i.e., $V(t+T) = V(t)$. The quantum analogue of Howland's method in classical mechanics [15], which transforms time-dependent systems into autonomous ones by substituting time by a new dynamical variable, corresponds to introducing the *Floquet operator*

$$K(t) = K_0(t) - V(t) \tag{3a}$$

where

$$K_0 = i\frac{\partial}{\partial t} - H_0 \tag{3b}$$

Both $K_0(t)$, the unperturbed Floquet operator, and $K(t)$ are operators on

$$\mathcal{H} \otimes L^2[0,T] \ . \tag{3c}$$

The spectrum of K_0 is $E_{n,m} = \omega m + E_n \quad m \in \mathbb{Z}, n = 1,2,3,\ldots$ where $\omega = 2\pi/T$, and is, in general, dense pure point, unless there are some commensurability conditions between ω and the E_n. The basic question is: Is the pure point spectrum of K_0 *stable*? The stability of the pure point spectrum of K_0 is usually called *quantum stability* [14]. As we shall see, there is strong evidence for this term because in the stable case time evolution happens essentially in a subspace of finite dimension. The case of *continuous* spectrum is referred to as *unstable*. One reason is that it frequently occurs due to the presence of resonances (as in example (2)), in close analogy to the (unstable) resonant tori in classical mechanics, which lead to chaotic behaviour. What happens more generally is that, while the initial state is localized in "phase space", time-evolution leads to delocalization. As a consequence, one may have unstable behaviour, such as the unbounded growth of the kinetic energy as in the case of the kicked rotor [8]. There may be several degrees of delocalization, with a hierarchy of time decays of certain quantities, such as the autocorrelation function [16].

As in [17] we now consider the general situation described by the Hamiltonian

$$H(t) = H_0(x) + V(x, \underset{\sim}{\theta}(t)) \tag{4a}$$

where x denotes the internal dynamical variables of the system, which act on a Hilbert space $\mathcal{H}$. Moreover,

$$\underset{\sim}{\theta}(t) = g_t \, \underset{\sim}{\theta}$$

where g_t is an invertible flow corresponding to the trajectory of a classical dynamical system on a manifold Ω, having an invariant ergodic measure μ. Thus $\underset{\sim}{\theta}(t)$ is a classical variable whose time-dependence is *independent* of the state of the system evolving according to $H(t)$, corresponding to an "external bath". Let $U(t,s;\underset{\sim}{\theta})$ be the unitary propagator associated to (3), strongly continuous in t and s, and such that

$$U(t+a, \ s+a; \ \underset{\sim}{\theta}) = U(t,s; \ g_a \, \underset{\sim}{\theta}) \ , \qquad a \in \mathbb{R} \ . \tag{5}$$

In analogy to the construction (3), let us define [17] the family of operators on

$$\mathcal{K} = \mathcal{H} \otimes \ L_2(\Omega, d\mu) \tag{6}$$

given by

$$\begin{aligned} [W(t)\psi](\underset{\sim}{\theta}) &= U(0,-t; \ \underset{\sim}{\theta})\tau_{-t} \, \psi(\underset{\sim}{\theta}) \\ &\equiv \tau_{-t} \, U(t,0; \ \underset{\sim}{\theta}) \, \psi(\underset{\sim}{\theta}) \ , \qquad \psi \in \mathcal{K} \end{aligned} \tag{7}$$

where

$$(\tau_{-t}\psi)(\underset{\sim}{\theta}) \equiv (g_{-t}\underset{\sim}{\theta}) \ . \tag{8}$$

Then [17] W is a strongly continuous family of unitary operators with

$$W(t) = e^{-\mathrm{i}Kt} , \tag{9}$$

and

$$(K\psi)(\underset{\sim}{\theta}) \equiv -i\frac{d}{dt}\,\psi(g_t\underset{\sim}{\theta})\Big|_{t=0} + H(\underset{\sim}{\theta})\psi \tag{10}$$

is the *generalized quasienergy operator* [17]. If the flow has a generator $\underset{\sim}{G}$,

$\frac{d\underset{\sim}{\theta}}{dt} = \underset{\sim}{G}(\underset{\sim}{\theta}(t))$, then

$$K = -i\,\underset{\sim}{G}(\underset{\sim}{\theta}) \cdot \frac{\partial}{\partial \underset{\sim}{\theta}} + H(\underset{\sim}{\theta}) . \tag{11}$$

We list a few examples:

1) Periodic force: $H = H_0 + f(x)\cos(\omega t + \theta)$, $\Omega = S^1$, $\theta(t) = \theta + \omega t$, $d\mu = d\theta$, and

$$K = -i\omega\frac{\partial}{\partial\theta} + H(\theta) \quad \text{with} \quad H(\theta) = H_0 + f(x)\cos\theta$$

2) Quasi-periodic force with two frequencies ω_1, ω_2: $H = H_0 + f(x)[\cos(\omega_1 t + \theta_1) + cos(\omega_2 t + \theta_2)]$, $\Omega = S^1 \times S^1$, $g_t(\theta_1, \theta_2) = (\theta_1 + \omega_1 t,\ \theta_2 + \omega_2 t)$, $d\mu = d\theta_1\, d\theta_2$

$$K = -i\omega_1\frac{\partial}{\partial\theta_1} - i\omega_2\frac{\partial}{\partial\theta_2} + H(\theta_1, \theta_2) , \tag{12}$$

$$H(\underset{\sim}{\theta}) = H_0 + f(x)\,(\cos\,\theta_1 + \cos\theta_2) . \tag{13}$$

We shall be specially interested in example 2. We may now describe the relation between stability and the *pure point spectrum of the quasienergy operator*

$$K\,\psi_m = \lambda_m\,\psi_m , \qquad \psi_m \in \mathcal{K} , \tag{14}$$

more precisely. We follow [18]. Since K is defined on an enlarged space $\mathcal{K}$ (6), one must first embed the initial state $\varphi(0) \in \mathcal{H}$ in $\mathcal{K}$, which may be done by the correspondence

$$\varphi(0) \in \mathcal{H} \rightarrow \varphi(0) \otimes \mathbf{1} \in \mathcal{K}$$

where $\mathbf{1}$ is the function on $L_2(\Omega, d\mu)$ which assigns the value 1 to all $\underset{\sim}{\theta}$. Because of (7) and (8) it may then be proven easily [18] that for all $\epsilon > 0$ there exists an M *independent of* t such that

$$\| \varphi(t) - \sum_{m<M} C_m\, e^{-i\lambda_m t}\tau_t\,\psi_m \|_k^2 =$$

$$= \| \sum_{m\geq M} C_m\, e^{-i\lambda_m t}\tau_t\,\psi_m \|_k^2 \leq \sum_{m\geq M} |C_m|^2 < \epsilon^2$$

where

$$\varphi(t) = U(t,0;\theta_1,\theta_2)(\varphi(0)\otimes \mathbf{1})$$

and

$$C_m = \langle \psi_m\,,\; \varphi(0)\otimes \mathbf{1}\rangle_k$$

The above uniformity in time means that the trajectories in Hilbert space are precompact and is a sign of stability.

A second notion of spectrum is what we call the *autocorrelation spectrum* in [13], but is actually due to Birkhoff and von Neumann (see [18] and references given there). Consider the solution of the Schrödinger equation

$$i\,\frac{\partial\psi(t)}{\partial t} = H(t)\,\psi(t) \qquad \psi(0) = \psi$$

where $H(t)$ is given by (4). Define the *autocorrelation function*

$$C_\psi(t) = \lim_{t\to\infty} \frac{1}{2T}\int_{-T}^{T} ds\langle\psi(s),\psi(s+t)\rangle \tag{15}$$

when the limit exists. Under this assumption, C_ψ is positive-definite, hence by Bochner's theorem there exists a Fourier-Stieltjes measure μ_ψ such that

$$C_\psi(t) = \int e^{-it\lambda}\, d\mu_\psi(\lambda)\ . \tag{16}$$

Let $\{\psi_i\}_{i=1}^{\infty}$ be a countable dense set in $\mathcal{H}$ (which we assume to be separable). We define the *autocorrelation spectrum* as the union of the supports of the measures μ_ψ. The sets of ψ such that μ_ψ is absolutely continuous (ac), singular continuous (sc) or pure point define the subspaces $\mathcal{H}_{ac}$, $\mathcal{H}_{sc}$ or $\mathcal{H}_{pp}$, respectively. Equivalence between the above definitions follows from ergodicity of the measure μ: for almost all θ [18]. Furthermore,

$$\begin{aligned}
C_\psi(t) &= \lim_{t\to\infty}\frac{1}{2T}\int_{-T}^{T} ds\langle\psi, U(t+s,s;\underset{\sim}{\theta}\psi\rangle_{\mathcal{H}} = \\
&= \int_\Omega d\mu(\underset{\sim}{\theta})\langle\psi, U(t,0;\underset{\sim}{\theta}\psi)\rangle_{\mathcal{H}} = \\
&= \langle\psi\otimes\mathbf{1}, e^{iKt}\,\psi\otimes\mathbf{1}\rangle_K
\end{aligned}$$

where (5) was used.

We now consider the oscillator model (2), with

$$f(t) = \lambda_1\,\cos\omega_1 t + \lambda_2\,\cos\omega_2 t\ . \tag{17}$$

We have the following result [13].

Theorem 1 In the *resonant* case $\omega_0 = \omega_1$, ω_2 incommensurate with ω_1, the autocorrelation function satisfies the inequality

$$|C_\psi(t)| \leq a\, e^{-\frac{\lambda_1^2}{4}t^2 + bt} \tag{18}$$

with a and b independent of t, and for ψ in a dense set of (coherent) states. In the nonresonant case, and under suitable diophantine conditions,

$$|\underset{\sim}{\omega} \cdot \underset{\sim}{m}| \geq c|m|^{-\alpha} \qquad |\underset{\sim}{m}| \neq 0 \tag{19}$$

for some $\alpha \in \mathbb{R}$ and $C > 0$, where $\underset{\sim}{\omega} \cdot \underset{\sim}{m} \equiv \omega_0 m_0 + \omega_1 m_1 + \omega_2 m_2$ $|\underset{\sim}{m}| \equiv |m_0| + |m_1| + |m_2|$, and m_i, $i = 0, 1, 2$ are arbitrary in $\mathbb{Z}$, $C_\psi(t)$ is a special almost-periodic function, which is not identically zero for ψ in a dense set of (coherence) states.

It follows from well-known theorems [19,20] that

Corollary 1 In the resonant case the autocorrelation spectrum is transient absolutely continuous (ψ is in the transient Hilbert space $\mathcal{H}$ if C_ψ decreases in $|t|$ faster than any power of $|t|$ [21]) and covers the whole line. In the nonresonant case, under the assumption (19), it is pure point.

If $\alpha > 3$ in (19), the Lebesgue measure of the complement of the set of $\underset{\sim}{\omega}$ which satisfies (19) is zero. The structure of $C_\psi(t)$, for $\psi = |0\rangle$, the ground state of the harmonic oscillator (but generalizable to a dense set of coherent states) is

$$C_{|0\rangle}(t) = \lim_{T\to\infty} \frac{1}{2T} \sum_{\substack{m_0,\ldots,m_N=-\infty \\ \underset{\sim}{\omega}\tilde{m}\cdot\neq 0}}^{\infty} \frac{e^{i\underset{\sim}{\omega}\cdot\tilde{m}T} - e^{-i\underset{\sim}{\omega}\cdot\tilde{m}T}}{i\underset{\sim}{\omega}\cdot\tilde{m}} \,.$$

$$\cdot \prod_{i=0}^{N} I_{m_i}(u_i(t)) + \sum_{\substack{m_0,\ldots,m_N \\ \underset{\sim}{\omega}\cdot\tilde{m}=0}} \prod_{i=0}^{N} I_{m_i}(u_i(t)) \tag{20}$$

where N is fixed, $\tilde{m}_0, \tilde{m}_1$ and $\tilde{m}_2$ are linear combinations of the integers $m_0, \ldots, m_N$, the I_m are modified Bessel functions and the $u_i(t)$ are almost-periodic functions of t. Due to the bound

$$|I_{m_i}(u_i(t))| \leq \sqrt{n} \left|\frac{u_i(t)}{2}\right|^{m_i} \frac{e^{|u_i(t)|}}{\Gamma\left(m_i + \frac{1}{2}\right)} \tag{21}$$

and (19), the first summation in (20) tends to zero. The same bound (21) and the fact that the almost-periodic functions form a closed subalgebra of $L^\infty(\mathbb{R})$ [21], show that the last summation defines an almost-periodic function which is easily seen to be not identically zero. If, however, the ratios ω_1/ω_0 are Liouville numbers, the bound (21) does not suffice to show that the first summation in (20) is zero, In fact, it is conjectured that in this case the spectrum is singular-continuous.

We now turn to the much more interesting model (1). The generalized quasienergy operator is given by (12) with $H_0 = \epsilon\,\sigma_z$ and (13), with $f(x) = \epsilon\,\sigma_x$ and $(\cos\theta_1 + \cos\theta_2)$ replaced by a general quasi-periodic function. Or, more generally, by

$$K = K_0 + \epsilon\, V(\theta_1, \theta_2) \tag{22}$$

where

$$K_0 \equiv -\imath\omega_1 \frac{\partial}{\partial\theta_1} - i\omega_2 \frac{\partial}{\partial\theta_2} + \beta\sigma_z \tag{23}$$

on $\mathcal{K} = \mathbf{C}^2 \otimes L^2(\mathbb{T}, d\mu)$, $\mathbb{T}$ is the two dimensional torus $S^1 \times S^1$, $d\mu = \frac{1}{(2\pi)^2} d\theta_1\, d\theta_2$, and $V(\theta_1, \theta_2)$ is a 2×2 matrix. In [7] the following theorem has been proven.

Theorem 2 [7] Let $V(\theta_1, \theta_2)$ be such that each component is an analytic function in the strip $\{\underset{\sim}{\theta} | \mathrm{Im}\theta_j < \tau_0\}$. Assume e.g. $\alpha = \omega_2/\omega_1 > 1|$ and $(2\beta/\omega_1)_{\mathrm{mod1}} > 0$.

Then, for any given $n > 0$ and fixed ω_1, there exists a set of $\alpha's$, $S_\eta \subset (1, \infty)$ of Lebesgue measure $\mathcal{L}(S_\eta) < \eta$ and a value $\epsilon_c(\eta)$ such that, if $\alpha \in (1, \infty) \setminus S_\eta$ and $\epsilon < \epsilon_c$, the spectrum of k is pure point.

The operator KAM method of [7] constructs a unitary operator $R(\alpha, \epsilon)$ such that

$$RKR^{-1} = K_0 + \begin{pmatrix} g_+(\alpha, \epsilon) & 0 \\ 0 & g_-(\alpha, \epsilon) \end{pmatrix} \tag{24}$$

where $g_\pm(\alpha, \epsilon)$ are independent of $\underset{\sim}{\theta} \equiv (\theta_1, \theta_2)$, i.e., R transforms K into an operator which is diagonal in the basis of eigenfunctions of the unperturbed K_0 which are

$$\psi_{\underset{\sim}{n},m} = \begin{cases} e^{i\underset{\sim}{n}\cdot\underset{\sim}{\theta}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } m = +1\,, \qquad \underset{\sim}{n} \equiv (n_1, n_2) \in \mathbb{Z}^2 \\ e^{i\underset{\sim}{n}\cdot\underset{\sim}{\theta}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } m = -1\,, \qquad m \in \{-1, 1\} \end{cases} \tag{25}$$

The transformation R is constructed by iteration $R = \ldots R_k \ldots R_2 R_1$. At each step the order of the $\underset{\sim}{\theta}$-dependent perturbation is reduced from ϵ^j to ϵ^{2j}. The k-th step is defined starting from an operator of the form

$$K_k = D_k + V_k \tag{26}$$

where

$$D_k = K_0 + g_k \tag{27}$$

with g_k diagonal in the basis $\{\psi_{\underset{\sim}{n},m}\}$, and V_k hermitian. g_k is generated by the previous iterations and can depend explicitly on α, but not on $\underset{\sim}{\theta}$. Writing $R_{k+1} = e^{W_{k+1}}$, with

$W_{k+1}^+ = -W_{k+1}$, and assuming W_{k+1} of the same order as V_k which is verified a posteriori, we have

$$K_{k+1} \equiv e^{W_{k+1}} K_k e^{-W_{k+1}} = K_k + [W_{k+1}, K_k] +$$
$$+\frac{1}{2!}[W_{k+1},[W_{k+1},k_k]] + \frac{1}{3!}[W_{k+1},[W_{K+1},[W_{k+1},K_k]]] + \ldots =$$
$$= D_k + V_k + [W_{k+1}, D_k] + V_{k+1} \tag{28}$$

where

$$V_{k+1} = \left[W_{k+1}, \left\{[W_{k+1}, D_k]\frac{1}{2!} + V_k\right\}\right] +$$
$$+\left[W_{k+1}, \left[W_{k+1}, \left\{[W_{k+1}, D_k]\frac{1}{3!} + \frac{V_k}{2!}\right\}\right]\right] + \ldots \tag{29}$$

Define for an operator A

$$A(m, m', \underset{\sim}{n} - \underset{\sim}{n}') \equiv \langle \psi_{\underset{\sim}{n},m}, A\, \psi_{\underset{\sim}{n};m'} \rangle \ . \tag{30}$$

Then, motivated by (28), we try to determine W_{k+1} and a diagonal δg such that

$$e^{W_{k+1}} K_k\, e^{W_{k+1}} = D_k + \delta g + V_{k+1} \tag{31}$$

which follows if

$$[W_{k+1}, D_k] + V_k = \delta g \ . \tag{32}$$

The diagonal terms of (32) yields

$$\delta g(\alpha, m) = V_k(m, m, \underset{\sim}{0})$$

and one may choose $W_{k+1}(m, m, \underset{\sim}{\theta}) = 0$.

The off-diagonal terms yield

$$W_{k+1}(m, m', \underset{\sim}{n} - \underset{\sim}{n}') = \frac{V_k(m, m', \underset{\sim}{n} - \underset{\sim}{n}')}{D_k(m, \underset{\sim}{n}) - D_k(m', \underset{\sim}{n}')} \tag{33}$$

where the denominator in (33) is

$$D_k(m, \underset{\sim}{n}) - D_k(m', \underset{\sim}{n}') = \underset{\sim}{\omega} \cdot (\underset{\sim}{n} - \underset{\sim}{n}') + (m - m')\beta + g_k(\alpha, m) - g_k(\alpha, m') \tag{34a}$$

with

$$g_k(\alpha, m) = \sum_{k'=1}^{k} \delta g_{k'}(\alpha, m) \ . \tag{34b}$$

By (34a), the denominator in (33) may be zero or orbitrarily close to zero for infinitely many m, m', and $\underset{\sim}{n} - \underset{\sim}{n}'$. In order to guarantee the convergence of the series for R, one must restrict α to the set characterized by the Diophantine condition

$$\Omega_{k+1}(\gamma_{k+1}) = \left\{ \alpha \in \Omega_k(\gamma_k) \quad \text{such that for} \quad \forall \underset{\sim}{n} \in \mathbb{Z}^2 \quad \text{and} \right.$$

$$\left. m, m' \in \{-1, 1\}, |\underset{\sim}{\omega} \cdot \underset{\sim}{n} + (m - m')\beta + g_k(\alpha, m) = -g_k(\alpha, m')| \geq \frac{\gamma_{k+1}}{(1 + |\underset{\sim}{n}|)^\sigma} \right\} \tag{35}$$

where $\underset{\sim}{n}$ and $m - m'$ are not simultaneously zero, $\Omega_0 = (1, \infty), \tau > 2$ and γ_{k+1} is a constant chosen for each step. Due to the explicit dependence of g_k upon α, one has to control the size of g_k as well as its variation with α, and this is accomplished by the operator norm [7,9]

$$\| A \|_{r,\Omega} = \sum_{\underset{\sim}{n}, \Delta m} e^{r|\underset{\sim}{n}|} \sup_{\alpha\alpha' \in \Omega} \sup_{m} \Bigg(|A(m, m + \Delta m, \underset{\sim}{n}, \alpha)| +$$

$$+ \frac{|A(m, m + \Delta m, \underset{\sim}{n}, \alpha) - A(m, m + \Delta m, \underset{\sim}{n}, \alpha')|}{|\alpha - \alpha'|} \Bigg) . \tag{36}$$

The space of infinite matrices endoved with the above norm is a Banach algebra $\mathcal{A}(r, \Omega)$. The sets Ω in (36) depend on k and are given by (35), and, correspondingly, the numbers in (36), are related to the width of the strip of analyticity in Theorem 2. Starting from r_0 given there, the width is reduced at the k'th step to r_k with $r_\infty > 0$. The sets Ω_k are by construction such that $\Omega_{k+1} \subset \Omega_k$, and the Lebesgue measure of the complement of the final set $\Omega_\infty = \bigcap_{k=0}^{\infty} (\Omega_k \backslash \Omega_{k+1})$ is proved to be small (see the statement of theorem 2). The convergence of the operators $R_k \ldots R_2 R_1$ is in the Banach algebra $\mathcal{A}(r_\infty, \Omega_\infty)$.

An interesting problem, relevant from the point of view of both experiment [1] and theory, is the nature of the spectrum of K given by (22), (23), for *large* coupling or, alternatively, of

$$K = K_0 + f(\theta_1, \theta_2)\sigma_x \tag{37}$$

where

$$K_0 = -i\omega_1 \frac{\partial}{\partial \theta_1} - i\omega_2 \frac{\partial}{\partial \theta_2} + \epsilon \sigma_z . \tag{38}$$

We have

Theorem 3 Let

$$f(\theta_1, \theta_2) = f_1(\theta_1) + f_2(\theta_2) + \nu \tag{39a}$$

where

$$f_i(\theta_i) = \sum_{\substack{n \in \mathbf{Z} \\ n \neq 0}} C_{i,n} \, e^{in\theta_i} \qquad i = 1, 2 \tag{39b}$$

with

$$|C_{i,n}| \le \delta_i \exp(-\mu_i|n|) \qquad \delta_i, \mu_i > 0, i = 1,2 \tag{39c}$$

$$\overline{C}_{i,n} = C_{i,-n} \tag{39d}$$

and

$$\nu \neq 0 \tag{39e}$$

Then K is unitarily equivalent to an operator

$$\widetilde{K} \equiv -i\omega_1 \frac{\partial}{\partial\theta_1} - i\omega_2 \frac{\partial}{\partial\theta_2} + \nu\sigma_z + V(\alpha,\theta_1,\theta_2) \tag{40}$$

where V is analytic in the strip

$$\left\{\underset{\sim}{\theta} \equiv (\theta_1,\theta_2) \ \ s.t. \ |\mathrm{Im}\,\theta_i| < \mu_i \quad , \qquad i = 1,2\right\}$$

Proof By a unitary rotation through $\pi/2$ about the y axis we map K to the operator K', given by

$$K' \equiv -i\omega_1 \frac{\partial}{\partial\theta_1} - i\omega_2 \frac{\partial}{\partial\theta_2} + \epsilon\,\sigma_x + f(\theta_1,\theta_2)\sigma_z \tag{41}$$

Define, now, the operator U from $\mathcal{K}$ to $\mathcal{K}$ by

$$U \equiv \begin{pmatrix} \exp\left[-\frac{1}{\omega_1}\sum_{\substack{n\neq 0\\ n\in\mathbf{Z}}} \frac{C_{1,n}}{n} e^{in\theta_1} - \frac{1}{\omega_2}\sum_{\substack{n\neq 0\\ n\in\mathbf{Z}}} \frac{C_{2,n}}{n} e^{in\theta_2}\right] & 0 \\ 0 & \exp\left[\frac{1}{\omega_1}\sum_{\substack{n\neq 0\\ n\in\mathbf{Z}}} \frac{C_{1,n}}{n} e^{in\theta_1} + \frac{1}{\omega_2}\sum_{\substack{n\neq 0\\ n\in\mathbf{Z}}} \frac{C_{2,n}}{n} e^{in\theta_2}\right] \end{pmatrix} \tag{42}$$

By (39c), (39d), U is a unitary operator and

$$U^{-1}K'U = \widetilde{K} \tag{43}$$

where $\widetilde{K}$ is given by (40), with

$$V(\alpha,\theta_1,\theta_2) \equiv \epsilon \begin{pmatrix} 0 & h(\alpha,\theta_1,\theta_2) \\ \overline{h}(\alpha,\theta_1,\theta_2) & 0 \end{pmatrix} \tag{44}$$

and

$$h(\alpha,\theta_1,\theta_2) \equiv \exp\left\{-2\sum_{i=1}^{2}\sum_{\substack{n\neq 0\\ n\in\mathbf{Z}}} \frac{1}{n}\left[\frac{C_{i,n}}{\omega_i} e^{in\theta_i}\right]\right\} \tag{45}$$

By (39c) and (45), h is analytic in the strip $|\mathrm{Im}\,\theta_i| < \mu_i$, $i = 1,2$. ■

There is a basic difference between the operator (40) and the operator (23)treated in [7]: in (40) the "potential" V depends on α (see (44), (45)). In order to prove that the spectrum of K, given by (23), is pure point in ref. [7], by an operator KAM technique, one proceeds as described in the discussion of Theorem 2.

The assumption on the potential for the proof in [7] to go through is:

$$\| V \|_{r_0,\Omega_0} < C\,\epsilon \tag{46}$$

where r_0, Ω_0 are the starting values of the parameter r and the set Ω in (36), and C is a constant. In the case (23) of [7] this property followed directly from analyticity on a strip, because the second term in (36) did not contribute, and the $\sup_{\alpha\alpha'\in\Omega_0}$ in (36) was irrelevant, since V (in (23)) was independent of α. From (46) it follows that $||V||_{r_0,\Omega_0}$ is sufficiently small if ϵ is chosen sufficiently small. We collect these remarks in

Proposition 1 Let V in (40) satisfy assumption (46). Then, if $\alpha > 1$ (for definiteness) and $(2\nu/\omega_1)_{\text{mod}1} > 0$, for any given $\eta > 0$ and fixed ω_1, there exists a set $S_\eta \subset \Omega_0$ of Lebesgue measure $\mathcal{L}(S_\eta) < \eta$ and a value $\epsilon_c(\eta)$ such that, if $\alpha \in (1,\infty)\setminus S_\eta$ and $\epsilon < \epsilon_c(\eta)$ the spectrum of K is pure point.

Assumption (46) must be verified in each particular case:

Corollary 2 Let $f_1(\theta_1) = \cos\theta_1$ and $f_2(\theta_2) = \cos\theta_2$ in (39), and $\Omega_0 = (1,\infty)$. Then assumption (46) holds for any $0 < r_0 < \infty$

Proof In this case, by (44), (45)

$$h(\theta_1,\theta_2) = \exp\left(-2i\,\frac{\sin\theta_1}{\omega_1}\right)\exp\left(-2i\,\frac{\sin\theta_2}{\omega_2}\right) \tag{47}$$

and

$$V(1,-1,\underset{\sim}{n},\alpha) = V(-1,1,\underset{\sim}{n},\alpha) = \epsilon\,J_{n_1}(-2/\omega_1)\cdot J_{n_2}\left(-\frac{2}{\omega_1\alpha}\right) \tag{48}$$

where J_n is Bessel's function of order n.

By (36)

$$\begin{aligned}\| V \|_{r_0,\Omega_0} \quad &\leq \quad \text{const.}\,\epsilon \sum_{\underset{\sim}{n}\in\mathbf{Z}^2} e^{r_0|\underset{\sim}{n}|} \sup_{\alpha,\alpha'\in\Omega_0}\Bigg(|V(1,-1,\underset{\sim}{n},\alpha)| \\ &+ \left|\frac{V(1,-1,\underset{\sim}{n},\alpha) - V(1,-1,\underset{\sim}{n},\alpha')}{|\alpha-\alpha'|}\right|\Bigg)\end{aligned} \tag{49}$$

and by (48) and (49):

$$\| V \|_{r_0,\Omega_0} \quad \leq \quad \text{const.}\,\epsilon\left(\sum_{n_1\in\mathbf{Z}} e^{r_0|n_1|}\left|J_{n_1}\left(-\frac{2}{\omega_1}\right)\right|\right)\times\left(\sum_{n_2\in\mathbf{Z}} e^{r_0|n_2|}\,g(n_2)\right) \tag{50}$$

where

$$g(n_2) \equiv \sup_{\alpha\in\Omega_0} \left| J_{n_2}\left(-\frac{2}{\omega_1\alpha}\right)\right| + $$
$$+ \sup_{\alpha\alpha'\in\Omega_0} \left| \frac{J_{n_2}\left(-\frac{2}{\omega_1\alpha}\right) - J_{n_2}\left(-\frac{2}{\omega_1\alpha'}\right)}{|\alpha-\alpha'|} \right| \tag{51}$$

Using now the bounds, for α real

$$|J_n(x)| \le \frac{(x/2)^{|n|}}{\Gamma(|n|+1)}$$

$$|J_n'(x)| \le \frac{1}{2}\left[\frac{(x/2)^{|n|-1}}{\Gamma(|n|)} + \frac{(x/2)^{|n|+1}}{\Gamma(|n|+2)}\right]$$

together with the mean-value theorem and the fact that $\alpha, \alpha' > 1$ in (51), because $\Omega_0 = (1, \infty)$, we get

$$g(n_2) \le \frac{(1/\omega_1)^{|n_2|}}{\Gamma(|n_2|+1)} + \frac{2}{\omega_1}\left[\frac{(1/\omega_1)^{|n_2|-1}}{\Gamma(|n_2|)} + \frac{(1/\omega_1)^{|n_2|+1}}{\Gamma(|n_2|+2)}\right]$$

and hence, by (50):

$$\| V \|_{r_0,\Omega_0} \le \epsilon\, C_1 \exp(r_0\, C_2)$$

where C_1 and C_2 are constants. ■

What happens in the case $\nu = 0$ in (39a)? In this case, the spectrum of

$$\widetilde{K}_0 = -i\omega_1 \frac{\partial}{\partial\theta_1} - i\omega_2 \frac{\partial}{\partial\theta_2} \tag{52}$$

with $\widetilde{K} = \widetilde{K}_0 + \epsilon V(\alpha, \theta_1, \theta_2)$ in (40), is doubly degenerate, and some essential modifications in the structure of the proof are necessary [22]. The same happens for f nonseparable, i.e., not the sum of two functions, each dependent on only one variable, as in (39) [22].

We now come to a harder problem, that of the existence of continuous spectrum [7]. For this purpose, it is interesting to introduce a generalized Floquet operator U_F [17] which acts on $K_1 \equiv \mathcal{H} \otimes L_2(S^1, d\theta_1)$ and whose spectral properties are equivalent [17] to those of the quasi-energy operator:

$$U_F \equiv \tau^1_{-T_2}\, u_1(\theta_1) \tag{53}$$

where $u_1(\theta_1) \equiv U(T_2, 0; \theta_1, 0)$ (the monodromy operator) and

$$(\tau'_{-T_2}\phi)(\theta_1) = \phi(\theta_1 - \omega_1 T_2)\ . \tag{54}$$

In the case of two-level systems, u_1 is an element of $SU(2)$ and may thus be represented as

$$u_1(\theta_1) = \begin{pmatrix} a & b^* \\ -b & a \end{pmatrix} \tag{55}$$

with $|a|^2 + |b|^2 = 1$. It may be proven [7] that for any choice of functions $a(\theta), b(\theta)$ with $|a|^2 + |b|^2 = 1$ there is some quasi-periodic Hamiltonian

$$H = f_0(t)\mathbf{1} + \sum_{j=1}^{3} f_j(t)\sigma_j \ , \tag{56}$$

with f_0, f_1, f_2 and f_3 real quasi-periodic functions,

$$f_j(t) = \tilde{f}_j\left(\omega_1 t + \theta_1, \omega_2 t + \theta_2\right) \ , \quad j = 0,1,2,3$$

with $\tilde{f}_j(\theta_1, \theta_2)$ continuous and 2π-periodic in $(\theta_1, \theta_2) \in S^1 \times S^1$ and τ such that u_1, given by (46), is the corresponding monodromy operator. This construction [7] relies on the fact that $SU(2)$ is simply connected and is not applicable to the scalar case ($f_j = 0$ in (56) for two of the indices $j = 1,2,3$) because $U(1)$ is not simply connected. A simple example due to Rychlik, reported in [7], is

$$u_1(\theta_1) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} \tag{57}$$

The corresponding Floquet operator has no pure point spectrum (no eigenvalues). Since u_1 is diagonal any eigenvector candidate is $(y(\theta_1), 0)$ or $(0, z(\theta_1))$. Consider the first type. The eigenvalue equation is, by (53)

$$\tau^1_{-T_2} e^{i\theta_1} y(\theta_1) = e^{-i\lambda T_2} y(\theta_1)$$

or

$$e^{i\theta_1}\, y(\theta_1) = e^{-i\lambda T_2}\, y(\theta_1 + \omega_1\, T_2) = e^{-i\lambda T_2}\, y(\theta_1 + 2\pi\alpha) \tag{58}$$

with $\alpha = \omega_1/\omega_2$. An eigenfunction is $y \in L_2(S^1, d\theta_1)$.

Hence

$$y(\theta_1) = \sum_{n=-\infty}^{\infty} y_n\, e^{i\eta\theta_1} \tag{59}$$

with

$$\sum_{n=-\infty}^{\infty} |y_n|^2 < \infty \tag{60}$$

By (58) and (59)

$$y_{n+1} = e^{-i\lambda T_2}\, e^{-2\pi i\alpha n} y_n$$

whence $|y_{n+1}| = |y_n|$, which is incompatible with (60).

Now $||(y(\theta_1), 0)||^2_{\mathcal{H}} = |y(\theta_1)|^2 \in L_1(S^1, d\theta_1)$ is invariant under the map $\theta_1 \to \theta_1 + 2\pi\alpha$ on S^1 because of (49). Assuming α irrational, this map is ergodic and hence $|y(\theta_1)|$ is constant for almost all θ_1, and we may write

$$y(\theta_1) = \exp[i\varphi(\theta_1)] \tag{61}$$

with φ real. Substituting (61) into (58), we see that the *index* (i,e., the number of times the image wraps around the circle when θ_1 goes from 0 to 2π) of the left-hand side is larger by one than that of the right-hand side. Therefore [7] the equation, for topological reasons, cannot have a solution. Actually much more is proven in [7]: the spectrum is absolutely continuous for all irrational α.

An interesting and important open problem is to find out whether K, as given by (22) and (23), has some continuous spectrum for intermediate values of ϵ.

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