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Diffeomorphism Invariant Integrable Field Theories

By Jens Hoppe [1]

Institut für Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich

Abstract. In a Hamiltonian formulation of hypersurface motions in Riemannian manifolds, diffeomorphism invariant field theories of arbitrary dimension are presented, for which infinitely many Poisson-commuting, diffeomorphism invariant, conserved charges exist.

In this talk [2], I would like to consider field theories described by a Hamiltonian of the form

$$H[x,p] = \int_\Sigma d^M\varphi \sqrt{g}\, h\,(p/\sqrt{g}) \tag{1}$$

with Σ being some M-dimensional Riemannian manifold, fields $x : \Sigma \to \mathcal{N}$ (with components x^i, $i = 1, 2, ..., N = M+1$) describing an embedding of Σ into a $M+1$ dimensional Riemannian manifold $\mathcal{N}$ (with metric $\zeta_{ij}(x)$), g being the determinant of the metric $g_{rs} := \frac{\partial x^i}{\partial \varphi^r}\frac{\partial x^j}{\partial \varphi^s}\zeta_{ij}(x)$ induced on $x(\Sigma)$, $p := \sqrt{p_i p_j \zeta^{ij}(x)}$ (with p_i being the momentum canonically conjugated to x^i), and h a real function of one variable ($p/\sqrt{g} =: u$) [1].

The classical equations of motion derived from (1),

$$\begin{aligned}
\dot{x}^i = \frac{\delta H}{\delta p_i} = &\quad h'(u)\tfrac{p^i}{p} \\
\dot{p}_i = -\frac{\delta H}{\delta x^i} = &\quad \zeta_{ij}\partial_r\left((h - h'u)\sqrt{g}g^{rs}\partial_s x^j\right) + \\
&\quad \text{(terms containing derivatives of} \\
&\quad \text{the embedding metric}\,\zeta_{ij})
\end{aligned} \tag{2}$$

[1] Heisenberg Fellow
On leave of absence from Karlsruhe University
[2] based on work done in collaboration with M. Bordemann

by construction (cp. H; both p and $\sqrt{g}$ transform as densities under diffeomorphisms) imply that the generators of diffeomorphisms,

$$C_r := p_i \frac{\partial x^i}{\partial \varphi^r},\ r = 1, ..., M \tag{3}$$

are constants of motion: $\dot{C}_r = 0$.

In the following let us restrict to solutions of (2) for which

$$C_r \equiv 0,\ r = 1, ..., M. \tag{4}$$

Using (2) and(4), one may easily show, that for such motions the Hamiltonian density, $\mathcal{H} := \sqrt{g}\, h$, will also be time-independent:

$$\frac{\partial}{\partial t} \underbrace{(\sqrt{g} h\,(p/\sqrt{g}))}_{=:\rho(\varphi^1,...,\varphi^M)} = 0. \tag{5}$$

This allows one to express u (hence also $h'(u)$) as a function, α, of $\sqrt{g}/\rho$ - implying that the first equation in (2) can be written in the form

$$\dot{x}^i = \alpha\,(\sqrt{g}/\rho)\, n^i,\ i = 1, ..., N, \tag{6}$$

as (4) implies that the unit vector $\frac{p^i}{p}$ must be normal to the hypersurface $\Sigma_t := x(\Sigma)$. Thus, if one is only interested in motions through the Riemannian manifold $\mathcal{N}$, for which (4) is satisfied, one may forget about the (complicated) second equation in (2), and merely consider (6). Unlike usual geometric first order equations (6) is 'secretly second order' (for non-constant α): its solution requires either a *parametrized* initial hypersurface $\Sigma_{t=0}$ (with parametrisations that differ by more than an area preserving one leading to geometrically inequivalent motions) *or*: an initial hypersurface Σ_0 *and* an initial velocity distribution (for which a parametrisation must be found such that (6) is satisfied at $t = 0$).

Before solving (6) for one particular, nontrivial, choice of α, let me mention that constant α (resp. $h(u) = u$, $H = \int \sqrt{p_i p_j \zeta^{ij}(x)}$) yields 'free' motion, $\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$, while solutions for $\alpha(v) = \sqrt{\mp v^2 \pm \epsilon^2}$, resp.

$$h(u) = \sqrt{\frac{1}{\epsilon^2} \pm \epsilon^2 (u - u_0)^2} \tag{7}$$

($\zeta_{ij} = \delta_{ij}$), lead to extremal hypersurface motions in $R^{1,M+1}(R^{M+2})$ [1, 2].

By choosing the constant u_0 equal to $\frac{1}{\epsilon^2}$

$$H = \int_\Sigma d^M\varphi (4p^2 g)^{1/4}, \tag{8}$$

corresponding to $\alpha(v) = v$, i.e.

$$\dot{x}^i = \sqrt{g}/\rho n^i,\ i = 1, ..., N, \tag{9}$$

can be obtained as a ($\epsilon \to 0$) limit of the 'maximal hypersurface motion' also in the Hamiltonian formulation (The following geometric property may be noted at this point: Consider, $M = 1$ for visual simplicity, the motion of a closed curve in the plane, from Σ_0 at $t = t_0$ to Σ_1 at $t = t_1$; calculate the area covered in the plane, as well as the surface area obtained by drawing the motion in R^3, with the vertical axis being ϵ^2 times t (the time); the $\epsilon \to 0$ limit of $\frac{1}{\epsilon^2}$ times the difference of the two surface areas is minimal if the closed curve moves according to (9)).

In any case, one can easily show (cp. [1]) that

$$\mathcal{Q} := \int_\Sigma d^M\varphi (4p^2 g)^{1/4} Q\,(x(t,\varphi)) \tag{10}$$

will Poisson-commute (weakly; i.e. modulo terms containing C_r) with (8), and with each other, provided Q, as a function of the x^i ($i = 1, ..., N$) satisfies

$$\nabla^i \nabla_i Q = 0; \tag{11}$$

this is related to the fact [1, 3] that the time at which the hypersurface Σ_t reaches a point x in the Riemannian manifold $\mathcal{N}$ is a harmonic function of x: $\nabla^i\nabla_i t = 0$ (for non-linear α, t will satisfy a *non-linear* second-order equation [1]). As (10) may be viewed as a Hamiltonian $\tilde{H} = \int_\Sigma d^M\varphi \tilde{\mathcal{H}}$ of the form (8), corresponding to an embedding metric

$$\tilde{\zeta}_{ij} = Q^{\frac{4}{N-2}} \zeta_{ij}, \tag{12}$$

the hierarchy of integrable systems related by (12) (with Q harmonic), or rather: the hypersurface motions in the corresponding manifolds, resp. the corresponding time-harmonic functions, should be related by Bäcklund-transformations.

In [4], a multilinear form for (9), which automatically implies the conservation of (10), was presented (for $\mathcal{N} = R^N$); these results were extended to certain conformally flat manifolds in [5], and (for $N = 3$) to 'quantized' time-harmonic flows in [6].

References

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