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Representation Theory of Deformed Oscillator Algebras

By Christiane Quesne [1] and Nicolas Vansteenkiste [2]

Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles, Campus de la Plaine CP229, Boulevard du Triomphe, B-1050 Brussels, Belgium

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Abstract. The representation theory of deformed oscillator algebras, defined in terms of an arbitrary function of the number operator N, is developed in terms of the eigenvalues of a Casimir operator C. It is shown that according to the nature of the N spectrum, their unitary irreducible representations may fall into one out of four classes, some of which contain bosonic, fermionic or parafermionic Fock-space representations as special cases. The general theory is illustrated by classifying the unitary irreducible representations of the Arik-Coon, Chaturvedi-Srinivasan, and Tamm-Dancoff oscillator algebras, which may be derived from the boson one by the recursive minimal-deformation procedure of Katriel and Quesne. The effects on non-Fock-space representations of the minimal deformation and of the quommutator-commutator transformation, considered in such a procedure, are studied in detail.

1 Introduction

Since the pioneering works of Arik and Coon [1], Kuryshkin [2], Biedenharn [3], and Macfarlane [4], many forms of deformed oscillator algebras have been considered and played an important role in the construction of q-deformed Lie algebras (see e.g. [5, 6]). They have found various applications to physical problems, such as the description of systems with non-standard statistics [7, 8], the construction of integrable lattice models [9], the algebraic

[1]Directeur de recherches FNRS; E-mail: cquesne@ulb.ac.be
[2]E-mail: nvsteen@ulb.ac.be

treatment of quantum mechanical exactly solvable systems [10, 11], of pairing correlations in nuclear physics [12], and of vibrational spectra of diatomic and polyatomic molecules [13], as well as the search for nonlinearities due to high-intensity electromagnetic fields [14].

The necessity to introduce some order in the rich and varied choice of deformed commutation relations did however appear soon and various classification schemes were therefore proposed [15, 16, 17, 18]. More recently, a unifying recursive procedure was introduced, generating at appropriate steps all the familiar deformed oscillators, along with some multiparametric generalizations [19]. Each iteration consists in a minimal deformation of a commutator into a quommutator, followed by a transformation of the latter into a new commutator, to which it is equivalent within the corresponding Fock space.

As it was already observed in Ref. [19], the equivalence between quommutators and corresponding commutators, which is a central ingredient of the recursive minimal deformation procedure, is not valid any more in the additional non-Fock-space representations, which are known to exist in general for deformed oscillator algebras. Although various works have been devoted to determining such representations for some particular algebras [20, 21, 22, 23], there still remains a need for a general theory.

The purpose of the present paper is twofold: first to fill in this gap by discussing the representation theory of general deformed oscillator algebras; then to illustrate both the effects of minimal deformation and of the quommutator-commutator transformation on the non-Fock-space representations by studying some selected examples.

The recursive minimal deformation procedure is briefly reviewed in Sec. 2. The representation theory of general deformed oscillator algebras is then developed in Sec. 3, and illustrated on some examples in Sec. 4. Finally, Sec. 5 contains the conclusion.

2 Recursively Minimally-Deformed Oscillators

Let us consider a given oscillator algebra $\mathcal{A}_0$, generated by the operators $N = N^\dagger$, $a^\dagger$, $a = \left(a^\dagger\right)^\dagger$, satisfying the commutation relations

$$\begin{aligned} \left[N, a^\dagger\right] &= a^\dagger, \qquad [N, a] = -a, && (2.1)\\ \left[a, a^\dagger\right] &= f_0(N), && (2.2) \end{aligned}$$

for some function $f_0(N) = (f_0(N))^\dagger$. The algebra $\mathcal{A}_0$ will serve as a starting point for a recursive procedure, wherein other oscillator algebras will be generated [19].

Let us assume that at the kth step, we have obtained an algebra $\mathcal{A}_k$, still generated by N, $a^\dagger$, a, and satisfying Eq. (2.1), but with Eq. (2.2) replaced by

$$\left[a, a^\dagger\right] = f_k(N), \tag{2.3}$$

where $f_k(N) = (f_k(N))^\dagger$. Then the next minimal deformation $\tilde{\mathcal{A}}_k$ of this algebra is defined by Eq. (2.1) and

$$\left[a, a^\dagger\right]_{q_{k+1}} = f_k(N), \tag{2.4}$$

where q_{k+1} is some real parameter, and the left-hand side of Eq. (2.4) is a quommutator, defined by $\left[a, a^\dagger\right]_{q_{k+1}} \equiv aa^\dagger - q_{k+1}a^\dagger a$.

The minimally-deformed relation (2.4) implies that in the bosonic Fock-space representation, i.e., with respect to the eigenvectors $|n\rangle$ of the number operator N, corresponding to the eigenvalues $n = 0, 1, 2, \ldots$, the operators $a^\dagger$ and a satisfy the relations

$$a^\dagger|n\rangle = \sqrt{F_{k+1}(n+1)}\,|n+1\rangle, \qquad a|n\rangle = \sqrt{F_{k+1}(n)}\,|n-1\rangle, \tag{2.5}$$

where the vacuum state $|0\rangle$ is assumed to fulfil the condition

$$a|0\rangle = 0, \tag{2.6}$$

and the function $F_{k+1}(n)$ is defined by

$$F_{k+1}(n) = \sum_{i=0}^{n-1} q_{k+1}^i f_k(n-1-i). \tag{2.7}$$

In Eq. (2.7), $\sum_{i=0}^{-1} \equiv 0$ so that $F_{k+1}(0) = 0$ in accordance with Eqs. (2.5) and (2.6). The corresponding function of the number operator $F_{k+1}(N)$, which satisfies the equation

$$F_{k+1}(N+1) - q_{k+1}F_{k+1}(N) = f_k(N), \tag{2.8}$$

is referred to as the structure function of the algebra $\tilde{\mathcal{A}}_k$.

It follows that the algebra $\mathcal{A}_{k+1}$, defined by Eq. (2.1) and

$$\left[a, a^\dagger\right] = f_{k+1}(N), \tag{2.9}$$

where

$$f_{k+1}(N) = (f_{k+1}(N))^\dagger \equiv F_{k+1}(N+1) - F_{k+1}(N), \tag{2.10}$$

is equivalent to $\tilde{\mathcal{A}}_k$ in such a Fock-space representation. In other words, both algebras $\tilde{\mathcal{A}}_k$ and $\mathcal{A}_{k+1}$ have the same structure function $F_{k+1}(N)$.

By iterating the transformations $\mathcal{A}_k \to \tilde{\mathcal{A}}_k \to \mathcal{A}_{k+1}$, one gets a sequence of deformed oscillator algebras depending upon an increasing number of parameters.

Both $\mathcal{A}_k$ and $\tilde{\mathcal{A}}_k$ have a nonvanishing central element or Casimir operator, defined by [19, 24]

$$C_k = F_k(N) - a^\dagger a, \tag{2.11}$$

and

$$\tilde{C}_k = q_{k+1}^{-N}\left(F_{k+1}(N) - a^\dagger a\right) = q_{k+1}^{-N} C_{k+1}, \tag{2.12}$$

respectively, which may be used to characterize their irreducible representations. As in the Fock-space representation, $F(N)|0\rangle = a|0\rangle = 0$, it follows from (2.11) and (2.12) that C_k and $\tilde{C}_k$ have a vanishing eigenvalue in such a representation.

In Sec. 4, we shall consider as examples the first few iterations obtained by starting from the standard boson oscillator algebra, for which

$$f_0(N) = 1, \qquad F_0(N) = N, \qquad C_0 = N - a^\dagger a. \tag{2.13}$$

In such a case, the first minimal deformation $\tilde{\mathcal{A}}_0$ is the Arik-Coon oscillator algebra [1, 2], for which

$$\left[a, a^\dagger\right]_{q_1} = 1. \tag{2.14}$$

Its structure function and Casimir operator are given by

$$F_1(N) = [N]_{q_1} \equiv \frac{q_1^N - 1}{q_1 - 1}, \qquad \tilde{C}_0 = \frac{1 - q_1^{-N}}{q_1 - 1} - q_1^{-N} a^\dagger a, \tag{2.15}$$

respectively. The Fock-space equivalent algebra $\mathcal{A}_1$ corresponds to the Chaturvedi-Srinivasan oscillator [25], for which

$$\left[a, a^\dagger\right] = q_1^N, \qquad F_1(N) = [N]_{q_1}, \qquad C_1 = [N]_{q_1} - a^\dagger a. \tag{2.16}$$

The second minimal deformation $\tilde{\mathcal{A}}_1$ is the Chakrabarti-Jagannathan two-parameter oscillator algebra [26], for which $\left[a, a^\dagger\right]_{q_2} = q_1^N$. It has two important special cases: the Biedenharn [3] and Macfarlane [4] oscillator algebra, corresponding to $q_2 = q_1^{-1}$, and the Tamm-Dancoff oscillator algebra [27], for which $q_2 = q_1$, and

$$\left[a, a^\dagger\right]_{q_1} = q_1^N, \qquad F_2(N) = q_1^{N-1} N, \qquad \tilde{C}_1 = q_1^{-1} N - q_1^{-N} a^\dagger a. \tag{2.17}$$

Many more examples can be found in Ref. [19].

3 Representation Theory of Deformed Oscillator Algebras

In the present section, we shall review some general properties of the unitary irreducible representations (unirreps) of the deformed oscillator algebras considered in the previous one. For such purpose, it is enough to consider the case of $\tilde{\mathcal{A}}_k$, as that of $\mathcal{A}_k$ can be obtained from it by restricting the q_{k+1} values to $q_{k+1} = 1$. For simplicity's sake, we shall define the algebra commutation relations by Eq. (2.1) and

$$\left[a, a^\dagger\right]_q = f(N) = (f(N))^\dagger, \tag{3.1}$$

and denote the corresponding structure function and Casimir operator by $F(N)$ and C, respectively. From (2.7) and (2.12), the latter satisfy the relations

$$F(N+1) - qF(N) = f(N), \qquad C = q^{-N}\left(F(N) - a^{\dagger}a\right), \tag{3.2}$$

from which it follows that

$$a^{\dagger}a = F(N) - q^{N}C, \qquad aa^{\dagger} = F(N+1) - q^{N+1}C. \tag{3.3}$$

We shall classify the unirreps of this algebra under the assumption that the spectrum of N is discrete and nondegenerate.

Let us start with a normalized simultaneous eigenvector $|c, \nu_0\rangle$ of the Casimir operator C, defined in (3.2), and of N, corresponding to the eigenvalues c and ν_0 respectively,

$$C|c,\nu_0\rangle = c|c,\nu_0\rangle, \qquad N|c,\nu_0\rangle = \nu_0|c,\nu_0\rangle, \qquad \langle c,\nu_0|c,\nu_0\rangle = 1. \tag{3.4}$$

>From (2.1), it results that the vectors

$$|c,\nu_0+n) = \begin{cases} \left(a^{\dagger}\right)^n |c,\nu_0\rangle & \text{if } n = 1,2,\ldots, \\ a^{-n}|c,\nu_0\rangle & \text{if } n = -1,-2,\ldots, \end{cases} \tag{3.5}$$

are also simultaneous eigenvectors of C and N,

$$C|c,\nu_0+n) = c|c,\nu_0+n), \qquad N|c,\nu_0+n) = (\nu_0+n)|c,\nu_0+n), \tag{3.6}$$

as long as they are nonvanishing. In (3.5), we use a round bracket instead of an angular one to denote unnormalized states. Definition (3.5) can be extended to $n = 0$ by setting $|c,\nu_0) = |c,\nu_0\rangle$.

Eq. (3.3) implies that

$$a^{\dagger}a|c,\nu_0+n) = \lambda_n|c,\nu_0+n), \qquad aa^{\dagger}|c,\nu_0+n) = \mu_n|c,\nu_0+n), \tag{3.7}$$

where

$$\lambda_n = F(\nu_0+n) - q^{\nu_0+n}c, \qquad \mu_n = F(\nu_0+n+1) - q^{\nu_0+n+1}c = \lambda_{n+1}. \tag{3.8}$$

As eigenvalues of a positive operator, only those λ_n that are positive or null are admissible in a unitary representation. In particular, the condition

$$\lambda_0 = F(\nu_0) - q^{\nu_0}c \geq 0 \tag{3.9}$$

restricts the possible values of c and ν_0.

So it is straightforward to derive the following unitarity conditions:

Proposition 1 *If there exists some $m_1 \in \{-1,-2,-3,\ldots\}$ such that $\lambda_{m_1} < 0$, and $\lambda_n \geq 0$ for $n = 0, -1, \ldots, m_1+1$, then an irreducible representation of a deformed oscillator algebra can be unitary only if $\lambda_{n_1} = 0$ for some $n_1 \in \{0,-1,\ldots,m_1+1\}$. If there exists some $m_2 \in \{2,3,4,\ldots,\}$ such that $\lambda_{m_2} < 0$, and $\lambda_n \geq 0$ for $n = 0, 1, \ldots, m_2-1$, then it can be unitary only if $\lambda_{n_2} = 0$ for some $n_2 \in \{1,2,\ldots,m_2-1\}$.*

Proof. In the first part of the proposition, we must have $|c, \nu_0 + m_1) = 0$ as otherwise $a^\dagger a$ would have a negative eigenvalue. This implies that $a|c, \nu_0 + m_1 + 1) = 0$, which can be achieved in two ways, either $|c, \nu_0 + m_1 + 1) = 0$, or $|c, \nu_0 + m_1 + 1) \neq 0$ and $\lambda_{m_1+1} = 0$. In the former case, we can proceed in the same way and find that at least one of the conditions $\lambda_{m_1+2} = 0$, $\lambda_{m_1+3} = 0$, ..., $\lambda_{-1} = 0$, or $|c, \nu_0 - 1) = 0$ must be satisfied. But the last one is equivalent to $\lambda_0 = 0$, since $|c, \nu_0) \neq 0$ by hypothesis. This concludes the proof of the first part of the proposition. The second part can be demonstrated in a similar way by using $aa^\dagger$, and $\mu_n = \lambda_{n+1}$ instead of $a^\dagger a$, and λ_n.

According to Proposition 1, the unirreps may belong to one out of four classes. If there exists some $n_1 \in \{0, -1, -2, \ldots\}$ such that $\lambda_{n_1} = 0$, and $\lambda_n > 0$ for $n = n_1 + 1$, $n_1 + 2$, ..., 0, 1, 2, ..., then $|c, \nu_0 + n_1)$ satisfies the relation $a|c, \nu_0 + n_1) = 0$, and is an eigenvector of $a^\dagger a$ and N with eigenvalues $\lambda_{n_1} = 0$ and $\tilde{\nu}_0 = \nu_0 + n_1$. By repeating construction (3.5) with $|c, \nu_0\rangle$, ν_0, λ_0 replaced by $|c, \tilde{\nu}_0\rangle$, $\tilde{\nu}_0$, $\tilde{\lambda}_0 = 0$, respectively , we obtain for the corresponding normalized states [3]

$$|c, \tilde{\nu}_0 + n\rangle = \left(\prod_{i=1}^{n} \tilde{\lambda}_i\right)^{-1/2} \left(a^\dagger\right)^n |c, \tilde{\nu}_0\rangle, \qquad n = 0, 1, 2, \ldots, \tag{3.10}$$

where $\tilde{\lambda}_n = \lambda_{n+n_1} = F(\tilde{\nu}_0 + n) - q^n F(\tilde{\nu}_0)$, and the Casimir operator eigenvalue c is entirely determined by $\tilde{\nu}_0$ through the relation $c = q^{-\tilde{\nu}_0} F(\tilde{\nu}_0)$. The states (3.10) carry an infinite-dimensional unirrep, characterized by a lower bound $\tilde{\nu}_0$ (bounded from below or BFB unirrep). In basis (3.10), the generators are represented by

$$\begin{aligned} a|c, \tilde{\nu}_0 + n\rangle &= \sqrt{\tilde{\lambda}_n}\, |c, \tilde{\nu}_0 + n - 1\rangle, \qquad a^\dagger |c, \tilde{\nu}_0 + n\rangle = \sqrt{\tilde{\lambda}_{n+1}}\, |c, \tilde{\nu}_0 + n + 1\rangle, \\ N|c, \tilde{\nu}_0 + n\rangle &= (\tilde{\nu}_0 + n)\, |c, \tilde{\nu}_0 + n\rangle. \end{aligned} \tag{3.11}$$

In the special case where $\tilde{\nu}_0 = 0$, we obtain a bosonic Fock-space representation of type (2.5), wherein the spectrum of N is $\{0, 1, 2, \ldots\}$, and $c = F(0) = 0$.

If, on the contrary, there exists some $n_2 \in \{1, 2, 3, \ldots\}$ such that $\lambda_{n_2} = 0$, and $\lambda_n > 0$ for $n = n_2 - 1$, $n_2 - 2$, ..., 0, -1, -2, ..., then $|c, \nu_0 + n_2 - 1)$ satisfies the relation $a^\dagger|c, \nu_0 + n_2 - 1) = 0$, and is an eigenvector of $aa^\dagger$ and N with eigenvalues $\mu_{n_2-1} = \lambda_{n_2} = 0$ and $\tilde{\nu}_0 = \nu_0 + n_2 - 1$. If we repeat construction (3.5) by starting from $|c, \tilde{\nu}_0\rangle$, $\tilde{\nu}_0$, $\tilde{\mu}_0 = \tilde{\lambda}_1 = 0$, instead of $|c, \nu_0\rangle$, ν_0, $\mu_0 = \lambda_1$, we obtain for the corresponding normalized states

$$|c, \tilde{\nu}_0 + n\rangle = \left(\prod_{i=0}^{|n|-1} \tilde{\lambda}_{-i}\right)^{-1/2} a^{-n} |c, \tilde{\nu}_0\rangle, \qquad n = 0, -1, -2, \ldots, \tag{3.12}$$

where $\tilde{\lambda}_n = \lambda_{n+n_2-1} = F(\tilde{\nu}_0 + n) - q^{n-1} F(\tilde{\nu}_0 + 1)$, and c is again determined by $\tilde{\nu}_0$ through the relation $c = q^{-\tilde{\nu}_0 - 1} F(\tilde{\nu}_0 + 1)$. Such states now carry an infinite-dimensional unirrep, characterized by an upper bound $\tilde{\nu}_0$ (bounded from above or BFA unirrep). The representation of the generators in basis (3.12) is still given by (3.11), but where n now takes the values indicated in (3.12), instead of those shown in (3.10).

[3]In Eq. (3.10), we assume that $\prod_{i=1}^{0} \equiv 1$. A similar convention is used in subsequent formulae too.

It may also happen that there exist both $n_1 \in \{0, -1, -2, \ldots\}$, and $n_2 \in \{1, 2, 3, \ldots\}$ such that $\lambda_{n_1} = \lambda_{n_2} = 0$, and $\lambda_n > 0$ for $n = n_1 + 1,\ n_1 + 2,\ \ldots,\ -1,\ 0,\ 1,\ \ldots,\ n_2 - 2,\ n_2 - 1$. The corresponding unirrep is then finite-dimensional (FD unirrep), and may be characterized by its lower and upper bounds, $\tilde{\nu}_0 = \nu_0 + n_1$ and $\tilde{\nu}_0 + n_2 - n_1 - 1 = \nu_0 + n_2 - 1$, or alternatively by $\tilde{\nu}_0$ and $p = n_2 - n_1 - 1$. It is spanned by the $d = p + 1$ normalized states

$$|c, \tilde{\nu}_0 + n\rangle = \left(\prod_{i=1}^{n} \tilde{\lambda}_i\right)^{-1/2} \left(a^\dagger\right)^n |c, \tilde{\nu}_0\rangle, \qquad n = 0, 1, \ldots, p, \tag{3.13}$$

where $\tilde{\lambda}_n = \lambda_{n+n_1} = F(\tilde{\nu}_0 + n) - q^n F(\tilde{\nu}_0)$, and $c = q^{-\tilde{\nu}_0} F(\tilde{\nu}_0) = q^{-\tilde{\nu}_0 - p - 1} F(\tilde{\nu}_0 + p + 1)$. They still satisfy Eq. (3.11), but we note that now

$$a^\dagger |c, \tilde{\nu}_0 + p\rangle = 0. \tag{3.14}$$

The unirrep is an order-p parafermionic Fock-space representation if $c = \tilde{\nu}_0 = 0$. It is fermionic in the special case where $p = 1$.

Finally, if $\lambda_n > 0$ for $n \in \mathbb{Z}$, we get an unbounded unirrep (UB unirrep), which may be characterized by c and by the fractional part $\tilde{\nu}_0$ of ν_0 (i.e., $\nu_0 = [\nu_0] + \tilde{\nu}_0$, where $0 \leq \tilde{\nu}_0 < 1$, and $[\nu_0]$ denotes the largest integer contained in ν_0), as different values of $[\nu_0]$ lead to equivalent unirreps. Its representation space is spanned by the states

$$\begin{aligned} |c, \tilde{\nu}_0 + n\rangle &= \left(\prod_{i=1}^{n} \tilde{\lambda}_i\right)^{-1/2} \left(a^\dagger\right)^n |c, \tilde{\nu}_0\rangle, \qquad n = 0, 1, 2, \ldots, \\ |c, \tilde{\nu}_0 + n\rangle &= \left(\prod_{i=0}^{|n|-1} \tilde{\lambda}_{-i}\right)^{-1/2} a^{-n} |c, \tilde{\nu}_0\rangle, \qquad n = -1, -2, \ldots, \end{aligned} \tag{3.15}$$

where $\tilde{\lambda}_n = \lambda_{n-[\nu_0]}$, and the generators are still represented by Eq. (3.11).

4 Some Selected Examples

In the present section, we shall apply the theory developed in the previous one to some of the deformed oscillator algebras considered in Sec. 2. The results are summarized in Tables 1, 2, and 3.

As explained in Sec. 2, the starting algebra $\mathcal{A}_0$ of the recursive procedure considered here is the boson oscillator algebra, defined by Eqs. (2.1), (2.2), and (2.13). It is worth emphasizing that contrary to the Heisenberg algebra for which the number operator is defined as $N \equiv a^\dagger a$, the boson oscillator algebra has some non-Fock-space representations. From (3.8) and (3.9), we indeed obtain that

$$\lambda_n = \nu_0 + n - c, \qquad \text{where } \nu_0 \geq c, \tag{4.1}$$

may become negative for $n < c - \nu_0$. Hence unitarity imposes that there exists some $n_1 \in \{0, -1, -2, \ldots\}$ such that $\lambda_{n_1} = \nu_0 + n_1 - c = 0$. The algebra has therefore BFB unirreps, characterized by

$$\tilde{\nu}_0 = \nu_0 + n_1 = c, \qquad \tilde{\lambda}_n = n, \tag{4.2}$$

where $\tilde{\nu}_0$ may take any real value. For $\tilde{\nu}_0 = c \neq 0$, such representations are non-Fock-space unirreps.

We shall successively review the cases where $0 < q \neq 1$, and $q < 0$. The latter is omitted in most studies, because the corresponding algebras are considered as deformations of the fermion oscillator algebra, instead of the boson one. It is worth noting however that in some definitions of deformed oscillator algebras [16], both a commutation and an anticommutation relations are assumed. We chose here to keep only one of them. As explained in Refs. [24, 28], such a modified definition leads to the existence of a Casimir operator. Considering negative q values for the minimally-deformed oscillator algebras is therefore in some way equivalent to selecting anticommutation relations instead of the commutation ones associated with positive q values.

Note that for $q < 0$, except when otherwise stated, we shall restrict ν_0 to integer values so that q^{ν_0} is well defined.

4.1 The Arik-Coon Oscillator Algebra

4.1.1 Positive Values of the Deforming Parameter

For the Arik-Coon oscillator algebra $\tilde{\mathcal{A}}_0$ [1, 2], defined by Eqs. (2.1) and (2.14), we find from (2.15), (3.8), and (3.9) that for $q > 0$

$$\lambda_n = \left(\frac{1}{q-1} - c\right) q^{\nu_0+n} - \frac{1}{q-1}, \qquad \text{where } c \leq \frac{1-q^{-\nu_0}}{q-1}, \tag{4.3}$$

may be an increasing, constant or decreasing function of n according to the values taken by q and c. To classify its unirreps, we have to distinguish between the cases where $0 < q < 1$ and $q > 1$.

Whenever $0 < q < 1$, we note from Eq. (4.3) that the Casimir operator eigenvalue may satisfy either of the conditions $c \leq (q-1)^{-1}$, or $(q-1)^{-1} < c \leq (1 - q^{-\nu_0})/(q-1)$. In the former case, $\lambda_n > 0$ for any $n \in \mathbb{Z}$, so we obtain UB unirreps, whereas in the latter case, λ_n may become negative for some negative n values, so we get BFB unirreps, characterized by

$$\tilde{\nu}_0 = \nu_0 + n_1, \qquad c = \frac{1-q^{-\tilde{\nu}_0}}{q-1}, \qquad \tilde{\lambda}_n = [n]_q. \tag{4.4}$$

Here $[n]_q$ is defined as in Eq. (2.15), and $n_1 \in \{0, -1, -2, \ldots\}$, so that $\tilde{\nu}_0$ may take any real value. Note that for $c = (q-1)^{-1}$, the UB unirrep degenerates into a unirrep for which

$$a^\dagger a = a a^\dagger = (1-q)^{-1}. \tag{4.5}$$

Whenever $q > 1$, we always have $c \leq (1 - q^{-\nu_0})/(q-1) < (q-1)^{-1}$. Since λ_n may again become negative for some negative n values, we obtain BFB unirreps, similar to those defined in (4.4).

In the limit where $q \to 1^-$ or 1^+, the only surviving unirreps are the BFB ones, which go over into those of $\mathcal{A}_0$, defined in (4.2). The UB unirreps, which diverge for $q \to 1^-$, are referred to as classically singular representations [29].

Our results do agree with those previously derived by Kulish [20] by a similar type of approach. The method used here, as well as in Ref. [20], contrasts with that of Chaichian *et al.* [22]. Indeed the latter do not postulate the existence of a number operator, hence of Eq. (2.1). Their unirrep classification is therefore not performed in terms of a Casimir operator C, but in terms of some noncentral element $K \equiv aa^\dagger - a^\dagger a$, whose sign cannot change in a given unirrep. Whenever $K \neq 0$, they set $|K| = q^M$, where the operators M, $a^\dagger$, and a satisfy some relations similar to Eq. (2.1). The connection between their approach [4] and ours is easily established by noting that K can be rewritten in terms of our operators N and C as $K = q^N(1+(1-q)C)$. Hence $K > 0$, $K = 0$, and $K < 0$ correspond to $c > (q-1)^{-1}$ if $0 < q < 1$, or $c < (q-1)^{-1}$ if $q > 1$, $c = (q-1)^{-1}$, and $c < (q-1)^{-1}$ if $0 < q < 1$, respectively, and for $K \neq 0$, one may set $M = N + \log_q |1 + (1-q)c|$.

4.1.2 Negative Values of the Deforming Parameter

>From Eq. (4.3), it is obvious that for any negative q value, and $c = (q-1)^{-1}$, there exists a degenerate UB unirrep, for which Eq. (4.5) is valid, and which may be characterized by any $\tilde{\nu}_0$ such that $0 \leq \tilde{\nu}_0 < 1$.

Assuming now $c \neq (q-1)^{-1}$ and $\nu_0 \in \mathbb{Z}$, Eq. (4.3) becomes

$$\lambda_n = (-1)^{\nu_0+n+1}\left(\frac{1}{1+|q|} + c\right)|q|^{\nu_0+n} + \frac{1}{1+|q|}, \tag{4.6}$$

where

$$\begin{aligned} c &\leq \frac{|q|^{-\nu_0} - 1}{1+|q|} \qquad \text{if } \nu_0 \in 2\mathbb{Z}, \\ c &\geq -\frac{|q|^{-\nu_0} + 1}{1+|q|} \qquad \text{if } \nu_0 \in 2\mathbb{Z}+1. \end{aligned} \tag{4.7}$$

For successive n values, λ_n oscillates around the positive constant $(1+|q|)^{-1}$. To classify the unirreps, we have to distinguish between the cases where $0 < |q| < 1$, $|q| > 1$, and $|q| = 1$.

Whenever $0 < |q| < 1$, $|\lambda_n|$ decreases from $+\infty$ to $(1+|q|)^{-1}$. Hence if $(-1)^{\nu_0}((1+|q|)^{-1} + c) > 0$, λ_n may become negative for some negative even n values. Unitarity then

[4] It is worth noting that Chaichian *et al.* call any BFB unirrep a Fock-space representation, whereas we do reserve this name for a very specific BFB unirrep.

Table 1: Unirrep classification for the Arik-Coon oscillator algebra. The cases where $q = 1$ and $q = -1$ correspond to the boson and fermion oscillator algebras, respectively.

q	Type	Characterization
$q > 1$	BFB	$\tilde{\nu}_0 \in \mathbb{R}$, $c = q^{-\tilde{\nu}_0}[\tilde{\nu}_0]_q$, $\tilde{\lambda}_n = [n]_q$
$q = 1$	BFB	$\tilde{\nu}_0 \in \mathbb{R}$, $c = \tilde{\nu}_0$, $\tilde{\lambda}_n = n$
$0 < q < 1$	BFB	$\tilde{\nu}_0 \in \mathbb{R}$, $c = q^{-\tilde{\nu}_0}[\tilde{\nu}_0]_q$, $\tilde{\lambda}_n = [n]_q$
	UB	$0 \le \tilde{\nu}_0 < 1$, $c \le (q-1)^{-1}$, $\tilde{\lambda}_n = [\tilde{\nu}_0 + n]_q - cq^{\tilde{\nu}_0+n}$
$-1 < q < 0$	BFB	$\tilde{\nu}_0 \in \mathbb{Z}$, $c = q^{-\tilde{\nu}_0}[\tilde{\nu}_0]_q$, $\tilde{\lambda}_n = [n]_q$
	UB	$0 \le \tilde{\nu}_0 < 1$, $c = (q-1)^{-1}$, $\tilde{\lambda}_n = (1-q)^{-1}$
$q = -1$	FD	$\tilde{\nu}_0 \in 2\mathbb{Z}$, $p = 1$, $c = 0$, $\tilde{\lambda}_n = (1-(-1)^n)/2$
	FD	$\tilde{\nu}_0 \in 2\mathbb{Z}+1$, $p = 1$, $c = -1$, $\tilde{\lambda}_n = (1-(-1)^n)/2$
	UB	$\tilde{\nu}_0 = 0$, $-1 < c < -1/2$ or $-1/2 < c < 0$, $\tilde{\lambda}_n = (-1)^{n+1}c + (1-(-1)^n)/2$
	UB	$0 \le \tilde{\nu}_0 < 1$, $c = -1/2$, $\tilde{\lambda}_n = 1/2$
$q < -1$	BFA	$\tilde{\nu}_0 \in \mathbb{Z}$, $c = q^{-\tilde{\nu}_0-1}[\tilde{\nu}_0+1]_q$, $\tilde{\lambda}_n = [n-1]_q$
	UB	$0 \le \tilde{\nu}_0 < 1$, $c = (q-1)^{-1}$, $\tilde{\lambda}_n = (1-q)^{-1}$

imposes that there exists some $n_1 \in \{0, -2, -4, \ldots\}$ such that

$$\lambda_{n_1-2} < 0, \qquad \lambda_{n_1} = 0, \qquad \lambda_{n_1-1}, \lambda_{n_1+1}, \lambda_{n_1+2}, \ldots > 0, \tag{4.8}$$

corresponding to $m_1 = n_1 - 2$ in Proposition 1. We therefore obtain BFB unirreps, characterized by

$$\tilde{\nu}_0 = \nu_0 + n_1 \in 2\mathbb{Z}, \qquad c = \frac{|q|^{-\tilde{\nu}_0} - 1}{1+|q|}, \qquad \tilde{\lambda}_n = \frac{1 + (-1)^{n+1}|q|^n}{1+|q|}, \tag{4.9}$$

if $\nu_0 \in 2\mathbb{Z}$, or

$$\tilde{\nu}_0 = \nu_0 + n_1 \in 2\mathbb{Z}+1, \qquad c = -\frac{|q|^{-\tilde{\nu}_0} + 1}{1+|q|}, \qquad \tilde{\lambda}_n = \frac{1 + (-1)^{n+1}|q|^n}{1+|q|}, \tag{4.10}$$

if $\nu_0 \in 2\mathbb{Z}+1$. If, on the contrary, $(-1)^{\nu_0}((1+|q|)^{-1} + c) < 0$, λ_n may become negative for some negative odd n values. This shows that Eq. (4.8) must be satisfied for some $n_1 \in \{-1, -3, -5, \ldots\}$. Hence, we again obtain BFB unirreps, but this time the $\tilde{\nu}_0$, c, and $\tilde{\lambda}_n$ values that characterize them are given by Eq. (4.9) or (4.10) according to whether

$\nu_0 \in 2\mathbb{Z}+1$ or $\nu_0 \in 2\mathbb{Z}$. The results obtained in the various cases can be put together by writing that for $0 < |q| < 1$, there exist BFB unirreps specified by

$$\tilde{\nu}_0 \in \mathbb{Z}, \qquad c = \frac{q^{-\tilde{\nu}_0} - 1}{1-q}, \qquad \tilde{\lambda}_n = [n]_q, \tag{4.11}$$

where the $\tilde{\nu}_0$ value is arbitrary.

Whenever $|q| > 1$, $|\lambda_n|$ increases from $(1+|q|)^{-1}$ to $+\infty$, so that λ_n may become negative for some positive even or odd values, according to whether $(-1)^{\nu_0}((1+|q|)^{-1}+c)$ is positive or negative. Unitarity now imposes that there exists some $n_2 \in \{2,4,6,\ldots\}$ or $n_2 \in \{1,3,5,\ldots\}$ respectively, such that

$$\lambda_{n_2+2} < 0, \qquad \lambda_{n_2} = 0, \qquad \lambda_{n_2+1}, \lambda_{n_2-1}, \lambda_{n_2-2}, \ldots > 0, \tag{4.12}$$

corresponding to $m_2 = n_2 + 2$ in Proposition 1. By proceeding as in the case where $0 < |q| < 1$, we conclude that there exist BFA unirreps, characterized by

$$\tilde{\nu}_0 \in \mathbb{Z}, \qquad c = \frac{q^{-\tilde{\nu}_0-1} - 1}{1-q}, \qquad \tilde{\lambda}_n = [n-1]_q, \tag{4.13}$$

for any $\tilde{\nu}_0$ value.

Finally, for $|q| = 1$, corresponding to the fermion oscillator algebra, Eq. (4.6) becomes

$$\lambda_n = (-1)^{\nu_0+n+1}\left(c + \frac{1}{2}\right) + \frac{1}{2}, \tag{4.14}$$

where $c \leq 0$ or $c \geq -1$ according to whether $\nu_0 \in 2\mathbb{Z}$ or $\nu_0 \in 2\mathbb{Z}+1$. In the former case, we obtain that for $c = 0$, $\lambda_n = 0$ for any $n \in 2\mathbb{Z}$, while $\lambda_n = 1$ for any $n \in 2\mathbb{Z}+1$, thereby showing that there exist FD unirreps, characterized by $p = 1$ and any $\tilde{\nu}_0 \in 2\mathbb{Z}$. For $\tilde{\nu}_0 = 0$, this is the standard fermionic Fock-space representation. For $c = -1$, $\lambda_n = 1$ for any $n \in 2\mathbb{Z}$, while $\lambda_n = 0$ for any $n \in 2\mathbb{Z}+1$. So we again get FD unirreps characterized by $p = 1$, but this time $\tilde{\nu}_0 \in 2\mathbb{Z}+1$. They can be derived from the previous ones by interchanging the roles of $a^\dagger$ and a. For $-1 < c < -1/2$ or $-1/2 < c < 0$, λ_n is always positive, hence the corresponding unirreps are UB ones. Similar results are obtained by starting from any $\nu_0 \in 2\mathbb{Z}+1$.

4.2 The Chaturvedi-Srinivasan Oscillator Algebra

4.2.1 Positive Values of the Deforming Parameter

For the Chaturvedi-Srinivasan oscillator algebra $\mathcal{A}_1$ [25], defined by Eqs. (2.1) and (2.16), we find from (3.8) and (3.9) that

$$\lambda_n = \frac{q^{\nu_0+n} - 1}{q-1} - c, \qquad \text{where } c \leq \frac{q^{\nu_0} - 1}{q-1}, \tag{4.15}$$

is an increasing function of n for any positive q.

Whenever $0 < q < 1$, λ_n increases from $-\infty$ to $(1-q)^{-1} - c$, which is a positive constant as it follows from Eq. (4.15). Hence, we only get BFB unirreps, characterized by

$$\tilde{\nu}_0 = \nu_0 + n_1, \qquad c = [\tilde{\nu}_0]_q, \qquad \tilde{\lambda}_n = q^{\tilde{\nu}_0}[n]_q, \tag{4.16}$$

where $n_1 \in \{0, -1, -2, \ldots\}$, showing that $\tilde{\nu}_0$ may take any real value.

Whenever $q > 1$, λ_n increases from $-(q-1)^{-1} - c$ to $+\infty$, and we have to distinguish between the cases where $-(q-1)^{-1} < c \leq [\nu_0]_q$, and $c \leq -(q-1)^{-1}$. In the former, λ_n may become negative, so that we obtain BFB unirreps similar to those defined in (4.16). In the latter, on the contrary, λ_n is always positive, hence we get UB unirreps.

In the limit where $q \rightarrow 1^-$ or 1^+, the UB unirreps again diverge so that we are only left with the BFB ones, which go over into those of $\mathcal{A}_0$, as it was the case for the Arik-Coon algebra.

Comparing now the $\mathcal{A}_1$ unirreps with those of $\tilde{\mathcal{A}}_0$, we note that only the Fock-space representations of these algebras do coincide since they are both characterized by $\tilde{\nu}_0 = c = 0$, and $\tilde{\lambda}_n = [n]_q$. The remaining BFB unirreps are however different and, more strikingly, the classically singular representations appear for different q values, namely $0 < q < 1$ for $\tilde{\mathcal{A}}_0$ and $q > 1$ for $\mathcal{A}_1$.

Table 2: Unirrep classification for the Chaturvedi-Srinivasan oscillator algebra.

q	Type	Characterization
$q > 1$	BFB	$\tilde{\nu}_0 \in \mathbb{R}$, $c = [\tilde{\nu}_0]_q$, $\tilde{\lambda}_n = q^{\tilde{\nu}_0}[n]_q$
	UB	$0 \leq \tilde{\nu}_0 < 1$, $c \leq -(q-1)^{-1}$, $\tilde{\lambda}_n = [\tilde{\nu}_0 + n]_q - c$
$0 < q < 1$	BFB	$\tilde{\nu}_0 \in \mathbb{R}$, $c = [\tilde{\nu}_0]_q$, $\tilde{\lambda}_n = q^{\tilde{\nu}_0}[n]_q$
$-1 < q < 0$	BFB	$\tilde{\nu}_0 \in 2\mathbb{Z}$, $c = [\tilde{\nu}_0]_q$, $\tilde{\lambda}_n = q^{\tilde{\nu}_0}[n]_q$
$q = -1$	FD	$\tilde{\nu}_0 \in 2\mathbb{Z}$, $p = 1$, $c = 0$, $\tilde{\lambda}_n = (1 - (-1)^n)/2$
	UB	$\tilde{\nu}_0 = 0$, $c < 0$, $\tilde{\lambda}_n = -c + (1 - (-1)^n)/2$
$q < -1$	BFA	$\tilde{\nu}_0 \in 2\mathbb{Z} + 1$, $c = [\tilde{\nu}_0 + 1]_q$, $\tilde{\lambda}_n = q^{\tilde{\nu}_0+1}[n-1]_q$

4.2.2 Negative Values of the Deforming Parameter

For negative q values and $\nu_0 \in \mathbb{Z}$, Eq. (4.15) becomes

$$\lambda_n = (-1)^{\nu_0+n+1}\frac{|q|^{\nu_0+n}}{1+|q|} + \frac{1}{1+|q|} - c, \qquad \text{where } c \le \frac{1-(-1)^{\nu_0}|q|^{\nu_0}}{1+|q|}. \tag{4.17}$$

For successive n values, λ_n oscillates around the constant $(1+|q|)^{-1} - c$, which is positive for $\nu_0 \in 2\mathbb{Z}$, but may be positive, null, or negative for $\nu_0 \in 2\mathbb{Z}+1$. To classify the unirreps, we have to distinguish between the cases where $0 < |q| < 1$, $|q| > 1$, and $|q| = 1$.

Whenever $0 < |q| < 1$, $|\lambda_n|$ decreases from $+\infty$ to $|(1+|q|)^{-1} - c|$. If $\nu_0 \in 2\mathbb{Z}$, λ_n may become negative for some negative even n values. Unitarity then imposes that there exists some $n_1 \in \{0, -2, -4, \ldots\}$ such that Eq. (4.8) be satisfied. We therefore obtain BFB unirreps, characterized by

$$\tilde{\nu}_0 = \nu_0 + n_1 \in 2\mathbb{Z}, \qquad c = \frac{1-|q|^{\tilde{\nu}_0}}{1+|q|}, \qquad \tilde{\lambda}_n = |q|^{\tilde{\nu}_0}\frac{1-(-1)^n|q|^n}{1+|q|}. \tag{4.18}$$

If $\nu_0 \in 2\mathbb{Z}+1$, c must satisfy the stronger condition $c \le (1+|q|)^{-1}$. Then λ_n may become negative for some negative odd n values. Hence, Eq. (4.8) must be fulfilled for some $n_1 \in \{-1, -3, -5, \ldots\}$, so that we get BFB unirreps specified by (4.18) again.

Whenever $|q| > 1$, $|\lambda_n|$ increases from $|(1+|q|)^{-1} - c|$ to $+\infty$. Similar arguments show that Eq. (4.12) must be fulfilled for some $n_2 \in \{2, 4, 6, \ldots\}$ or $n_2 \in \{1, 3, 5, \ldots\}$ according to whether $\nu_0 \in 2\mathbb{Z}$ or $\nu_0 \in 2\mathbb{Z}+1$. We therefore obtain BFA unirreps, characterized by

$$\tilde{\nu}_0 = \nu_0 + n_2 - 1 \in 2\mathbb{Z}+1, \qquad c = \frac{1-|q|^{\tilde{\nu}_0+1}}{1+|q|}, \qquad \tilde{\lambda}_n = |q|^{\tilde{\nu}_0+1}\frac{1+(-1)^n|q|^{n-1}}{1+|q|}. \tag{4.19}$$

Finally, for $|q| = 1$, hence for the oscillator algebra defined by (2.1) and $\left[a, a^\dagger\right] = (-1)^N$, Eq. (4.17) simply becomes

$$\lambda_n = \frac{1}{2}\left(1 - (-1)^{\nu_0+n}\right) - c, \tag{4.20}$$

where $c \le 0$, or $c \le 1$, according to whether $\nu_0 \in 2\mathbb{Z}$, or $\nu_0 \in 2\mathbb{Z}+1$. By reasoning as in Sec. 4.1.2, we find two types of unirreps, namely FD unirreps characterized by $c = 0$, $p = 1$, and any $\tilde{\nu}_0 \in 2\mathbb{Z}$, and UB unirreps specified by $\tilde{\nu}_0 = 0$, and any negative c value.

4.3 The Tamm-Dancoff Oscillator Algebra

4.3.1 Positive Values of the Deforming Parameter

For the Tamm-Dancoff oscillator algebra $\tilde{\mathcal{A}}_1$ [27], defined by Eqs. (2.1) and (2.17), we find from (3.8) and (3.9) that

$$\lambda_n = q^{\nu_0+n-1}(\nu_0 + n - qc), \qquad \text{where } c \le q^{-1}\nu_0, \tag{4.21}$$

may become negative for $n < qc - \nu_0$, and any positive q value. Hence, in the present case, we only get BFB unirreps, characterized by

$$\tilde{\nu}_0 = \nu_0 + n_1, \qquad c = q^{-1}\tilde{\nu}_0, \qquad \tilde{\lambda}_n = q^{\tilde{\nu}_0+n-1}n, \tag{4.22}$$

where $n_1 \in \{0, -1, -2, \ldots\}$, and therefore $\tilde{\nu}_0 \in \mathbb{R}$.

Such unirreps were already found before [27]. The fact that $\lim_{n\to+\infty} \tilde{\lambda}_n = 0$, for $0 < q < 1$, explains the name given to the algebra, and referring to the idea of a high-energy cutoff proposed in the context of field theory [30].

Table 3: Unirrep classification for the Tamm-Dancoff oscillator algebra.

q	Type	Characterization
$0 < q \neq 1$	BFB	$\tilde{\nu}_0 \in \mathbb{R}$, $c = q^{-1}\tilde{\nu}_0$, $\tilde{\lambda}_n = q^{\tilde{\nu}_0+n-1}n$

4.3.2 Negative Values of the Deforming Parameter

For negative q values and $\nu_0 \in \mathbb{Z}$, Eq. (4.21) becomes

$$\lambda_n = (-1)^{\nu_0+n+1}|q|^{\nu_0+n+1}(\nu_0 + n + |q|c), \tag{4.23}$$

where $c \leq -\nu_0|q|^{-1}$ or $c \geq -\nu_0|q|^{-1}$ according to whether $\nu_0 \in 2\mathbb{Z}$ or $\nu_0 \in 2\mathbb{Z}+1$. Since λ_n can vanish for at most one integer n value, and it oscillates around zero in the intervals $(-\infty, -\nu_0 - |q|c)$ and $(-\nu_0 - |q|c, +\infty)$, it is obvious that the conditions of Proposition 1 cannot be fulfilled so that no unirrep can exist for negative q values.

We have therefore established that contrary to the remaining deformed oscillator algebras considered in the present paper, the Tamm-Dancoff oscillator algebra has a single class of unirreps.

5 Conclusion

In the present paper, we developed the representation theory of deformed oscillator algebras, defined in terms of an arbitrary function of the number operator N. We showed that the classification of their unirreps can be most easily performed in terms of the eigenvalues of a Casimir operator C. Under the assumption that the spectrum of N is discrete and nondegenerate, we proved that the unirreps may fall into one out of four classes (BFB, BFA, FD, UB) according to the nature of that spectrum, and that bosonic, and fermionic or parafermionic Fock-space representations may occur as special cases of BFB and FD unirreps, respectively.

We did also carry out the unirrep classification in detail for some deformed oscillator algebras, which can be derived from the boson one by the recursive minimal deformation procedure of Katriel and Quesne [19], namely the Arik-Coon [1, 2], Chaturvedi-Srinivasan [25], and Tamm-Dancoff [27] oscillator algebras. For all of them, we considered both positive and negative values of the deforming parameter, which constitutes a distinctive feature of the present study as compared with some previous ones [20, 21, 22, 23].

We showed that all the known unirreps, in particular the bosonic Fock-space representations, can be recovered in our classification scheme, and that in addition, many new unirreps make their appearance. We actually provided some examples for each of the four unirrep classes, although in the FD case, only two-dimensional unirreps were encountered. Higher-dimensional FD unirreps do however arise for some known deformed oscillator algebras [11].

We also illustrated both the effects of minimal deformation and of the quommutator-commutator transformation of the recursive procedure on non-Fock-space representations.

Applications of deformed oscillator algebras have been restricted up to now to their Fock-space representations. Whether non-Fock-space representations, such as those constructed in this paper, may have some useful applications remains an interesting open question.

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