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Analyticity and independence on the classical boundary conditions of the infinite volume thermal KMS states for a class of continuous systems. I. The Maxwell-Boltzmann statistics case.

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Abstract

The method of dual pair of Banach spaces is used to analyze the Kirkwood-Salsburg equations for the reduced density matrices describing continuous systems of particles obeying Maxwell-Boltzmann statistics. The existence, analyticity and equality in the thermodynamic limit of the conditioned (by classical boundary conditions) KMS states is proved for any value of the chemical activity z such that z^{-1} does not belongs to the spectrum of the corresponding Kirkwood-Salsburg operator.

Key words:
KMS state, continuous systems, grand canonical Gibbs ensemble, Kirkwood-Salsburg operator, classical boundary condition, Wiener integral, repulsive interaction, superstable interaction.

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1 Introduction

One of the fundamental problem of the quantum statistical mechanics is the problem of existence of the KMS states corresponding to a (locally) given dynamics [1]. More interesting and important is the description of the whole set of KMS states provided they exist.

Our knowledge about the content of the set of Equilibrium Gibbs states describing continuous systems of quantum-mechanical systems of particles in high density (low temperatures) regime of parameters is still very incomplete. (The same is true for classical particles as well). The rigorous results which have been obtained up to now for such systems are confined essentially to low density (high temperature) regime. The basic results of Ginibre [2] combined with the methods of [3-7] give the existence of Dirichlet infinite volume KMS states in the above domain of parameters. It is worthwhile to quote the papers [8- 12] where the existence problem has been solved for a special class of systems in the wider regime of couplings, see also [45].

The main aim of the present paper is to extend the Ginibre results using different method of analysis of the corresponding Kirkwood-Salsburg equations. The method of the dual pair of Banach spaces (invented in [13] and then improved and applied in similar situations in [14-18]) will be used. This method enables us to generalize slightly the Ginibre results. In particular the existence of the corresponding Dirichlet KMS states can be proved for much wider domain of parameters. Additionally the method used gives the possibility to discuss the eventual dependence of the limiting KMS states on the classical boundary conditions that have been used to construct them. These results (partially) solve the problem posed by Bratteli and Robinson in [1]. It should be stressed that the influence of the particular choice of the boundary conditions on the phase transitions has been demonstrated explicitly for some toy models [19,20]. This is one of our motivations to study the influence of boundary conditions on the limiting KMS states. The second motivation comes from the question concerning the uniqueness of the limiting KMS states. Results of that kind for continuous systems seem to be very exceptional [1]. So far only the possible influence of the classical boundary conditions on the infinite volume free energy density was studied before [21-23].

In the first part of this work we will concentrate ourselves on the exposition of the method of dual pair of Banach spaces and its applications. The restriction to the Maxwell-Boltzmann statistics enables us to obtain existence results (modulo the difficult part of the proof of Lemma 3-4 below) in a rather economic way. The case of quantum statistics is more complicated and the corresponding results will be presented in the second part of this work [24], see also [45]. In Section 2 of the present paper the Kirkwood-Salsburg equations are formulated and some introductory discussion of the classical boundary conditions is included also. The main result is formulated as *Theorem 4-1.* It gives the existence and independence on all classical boundary conditions of the limiting KMS states on the whole resolvent set of the corresponding Kirkwood-Salsburg operator. The complicated proof of the difficult part of Lemma 3-4 is referred to another publication [24].

2 Some preparations

2.1 Admissible boundary conditions

Let $-\tilde{\Delta}_\Lambda$ denote the Laplace operator $-\nabla^2$ defined on twice-continuously differentiable functions in $L_2(\Lambda)$, where Λ is assumed to have a piecewise C^1 boundary $\partial\Lambda$. The class of self-adjoint extensions of $-\tilde{\Delta}_\Lambda$ can be described by the condition that ϕ belongs to the domain of such extension iff $\partial_n\phi = \sigma\phi$, where $\sigma \in C^1(\partial\Lambda)$ and ∂_n means the inward normal derivative. This corresponds to the so called classical boundary conditions and corresponding extensions will be denoted by $-\Delta_\Lambda^\sigma$. The case of Dirichlet extension corresponds formally to $\sigma = +\infty$ on $\partial\Lambda$ and the case of Neumann extension to $\sigma = 0$ on $\partial\Lambda$. The corresponding extensions will be denoted by $-\Delta_\Lambda^D$, resp. $-\Delta_\Lambda^N$. The infinite volume (Friedrichs) Laplacian will be denoted by $-\Delta$.

Let $\Omega_\beta(\Lambda) = \underset{0\le t\le\beta}{\times} \Lambda$ where Λ is compact region in R^d with the boundary $\partial\Lambda$ being piecewise C^1. Similarly we define $\Omega_\beta = \underset{0\le t\le\beta}{\times} \dot{R}^d$ (where the dot means one point compactification of R^d) to be the space of paths.

Lemma 2-1.

For any classical boundary condition (σ, Λ), any $x, y \in \Lambda$ there exist uniquely defined measure $\mu^{\sigma,\beta}_{\Lambda,\, x|y}$ (resp. $\mu^{\beta}_{x|y}$) defined on the Borel σ-algebra of $\Omega_\beta(\Lambda)$ (resp. Ω_β) such that for any cylindric function $\phi(\omega) = \phi(\omega(t_1), \omega(t_2), \ldots, \omega(t_n))$ with $0 \le t_1 < t_2 < \ldots\ t_n \le \beta$ we have

$$\begin{aligned} \mu^{\sigma,\beta}_{\Lambda,\, x|y}(\phi) &\equiv \int\limits_{\Omega_\beta(\Lambda)} \mu^{\sigma,\beta}_{\Lambda,\, x|y}(d\omega)\phi(\omega) \\ &= \int\limits_{\Lambda} dx_1 \ldots dx_n \phi(x_1, \ldots, x_n) p^\sigma_\Lambda(x, x_1|t_1) \ldots p^\sigma_\Lambda(x_n, y|\beta - t_m) \end{aligned} \tag{2-1}$$

where

$$\begin{aligned} p^\sigma_\Lambda(x, y|t) &= (exp - t\Delta^\sigma_\Lambda)(x, y), \\ p(x|t) &= (exp - t\Delta)(x) \end{aligned} \tag{2-2}$$

The corresponding conditioned by Dirichlet (resp. Neumann) boundary condition Wiener measure will be denoted by $\mu^D_{\Lambda,x|y}$ (resp. $\mu^N_{\Lambda,x|y}$).

To some extent, the deviation of $\mu^{\sigma,\beta}_{\Lambda,\, x|y}$ from $\mu^{\beta}_{x|y}$ is measured by the compensating Green function $\Delta p^\sigma_\Lambda(x, y|t) = p(x - y|t) - p^\sigma_\Lambda(x, y|t)$. Certain fundamental properties of Δp^σ have been established in [1,21,22]. For the applications to the present exposition we need

Lemma 2-2 [1,22]

For each $\beta > 0$ there exist constants $C, c', c'' > 0$ such that (uniformly in σ and $t \leq \beta$)

$$\Delta p^\sigma_\Lambda(x,y|t) \leq C \cdot e^{c' \cdot t} \cdot t^{-d/2} \cdot exp - \left\{ \frac{c'' \cdot (dist(x,\partial\Lambda)^2 + dist(y,\partial\Lambda)^2)}{4t} \right\} \qquad (2-3)$$

for every bounded convex domain $\Lambda \subset R^d$, whose boundary is C^3-surface.

It is important to note that the constants in (2-3) depend only on the mean curvature of $\partial\Lambda$.

Another, well known lemma expresses the expected fact that the measures $\mu^{\sigma,\beta}_{\Lambda,\, x|y}$ and $\mu^{\beta}_{x|y}$ differ on the boundary σ-algebra $\sigma(\partial\Lambda) \equiv \sigma\{\omega | \exists 0 \leq t \leq \beta\ : \omega(t) \in \partial\Lambda\}$ only.

Lemma 2-3 [1,22]

For any Borel set $B \in \Omega_\beta(\mathring{\Lambda})$, where $\mathring{\Lambda}$ means the interior of Λ, the identity

$$\mu^{\sigma,\beta}_{\Lambda,\, x|y}(\alpha_\Lambda \cdot B) = \mu^{\beta}_{x|y}(\alpha_\Lambda \cdot B) \qquad (2-4)$$

holds, where

$$\alpha_\Lambda(\omega) \equiv \begin{cases} 1 & \text{if} \quad \omega(t) \in \mathring{\Lambda} \quad \text{for any} \quad t \in [0,\beta] \\ 0 & \text{otherwise} \end{cases}$$

A (σ, Λ) boundary condition is compatible iff there exists $\mu^{\beta}_{x|y}$-measurable function $\epsilon^\sigma_{\Lambda,x|y} : \Omega_\beta \to \Omega_\beta(\Lambda)$ that

$$\mu^{\sigma,\beta}_{\Lambda,\, x|y}(B) = \mu^{\beta}_{x|y}\left(\epsilon^\sigma_{\Lambda,x|y}\right)^{-1}(B). \qquad (2-5)$$

From Lemma 2-3 it follows that if the boundary condition is compatible then $\left(\epsilon^\sigma_{\Lambda,x|y}\right)^{-1} = id$ on $\Omega(\mathring{\Lambda})$. Therefore, the measure $\mu^{\beta}_{x|y}\left(\epsilon^\sigma_{\Lambda,x|y}\right)^{-1}$ differs from $\mu^{\beta}_{x|y}$ on $\sigma(\partial\Lambda)$ only. Let us decompose

$$\mu^{\beta}_{x|y}\left(\left(\epsilon^\sigma_{\Lambda,x|y}\right)^{-1}(B)\right) = r^{\sigma,\beta}_{\Lambda,x|y}(B) + s^{\sigma,\beta}_{\Lambda,x|y}(B) \qquad (2-6)$$

where $r^{\sigma,\beta}_{\Lambda,x|y}$ denotes the corresponding Radon-Nikodym derivative and $s^{\sigma,\beta}_{\Lambda,x|y}$ the singular part of the general decomposition of the measure $\mu^{\sigma,\beta}_{\Lambda,\, x|y}$, with respect to $\mu^{\beta}_{x|y}|\sigma(\partial\Lambda)$. From the inequality $s^{\sigma,\beta}_{\Lambda,x|y}(1 - \alpha_\Lambda) \leq \Delta p^\sigma_\Lambda(x,y|\beta)$ it follows that locally $\mu^{\sigma,\beta}_{\Lambda,\, x|y} \to \mu^{\beta}_{x|y}$ in the weak topology, for $(\Lambda_n)_{n=1,\dots}$ being any monotonic sequence of convex bounded regions whose boundaries are C^3-surfaces with mean curvatures uniformly bounded. Taking into account this observation we support conventional wisdom that only in the strongly correlated state of the system the boundary condition could influence the thermodynamic limit. The results presented below support this picture.

Finally it seems worthwhile to point out that there are many mathematical papers [25-28] in which great portion of a more detailed information concerning the conditioned Wiener measure is still to be encoded.

2.2 Interactions

Two classes of interactions will be considered in this paper.
(SSR) superstable and strongly regular interactions defined by $V \in SSR$ iff

(SS)

$$\exists_{\substack{A>0\\B>0}} : U(x^n) = \sum_{1\leq i<j\leq n} V(x_i - x_j) \geq \sum_{r\in Z^d} \left(An^p(r,x^n) - Bn(r,x^n)\right) \qquad (2-8)$$

with

$$\begin{cases} p-1 \geq \frac{dp+(2-d)}{2(p-1)} \quad , \quad p \geq 2 \quad , & \text{if } d > 3, \\ p = 2 \quad , & \text{if } d \leq 3, \end{cases} \qquad (2-9)$$

where $n(r, x^n)$ denotes the number of particles belonging to the configuration $x^n = (x_1, \ldots, x_n)$ that are located in the unit cubes $\square_r = \left\{x \in R^d | \, r - \frac{1}{2} \leq x_i < r + \frac{1}{2}\right\}$, $z \in \mathbf{Z}^d$; and

(SR) there exists a positive decreasing monotonously function Φ on $(0, \infty)$ such that $\Phi(x) \simeq x^{-(d+\epsilon)}$ for some $\epsilon > 2$ as $x \uparrow \infty$ and moreover

$$|\mathcal{E}(x^n|y^m)| \leq \frac{1}{2} \sum_{r,s\in\mathbf{Z}^d} \Phi(|r-s|) \cdot \left(n^2(r,x^n) + n^2(s,y^m)\right) \qquad (2-10)$$

where

$$\mathcal{E}(x^n|y^m) = \sum_{i=1}^{n}\sum_{j=1}^{m} V(x_i - y_j).$$

The second class is

RR - purely repulsive, strongly regular interactions defined by: $V \in RR$ iff

$(R)_1$ V is nonnegative measurable function on R^d and

$(R)_2$ $\int_{R^d} V(x)dx < \infty$.

We impose also that for both of these cases there exists a closed set $F \subset R^d$ of (Newton) capacity 0 such that V is continous outside F.

We can hardly quote a reference in which (but see Reed & Simon [44]) complete treatment of the self-adjointness preserving perturbation theory for the operators $-\Delta_\Lambda^\sigma$

is given. Several partial results can be obtained suitably by adopted standard technique (presented originally for the operator $-\Delta_\Lambda^D$ only). For the following analysis we need to impose also that the interactions V leads to the self-adjointness preserving perturbation of $-\Delta_\Lambda^\sigma$ for any classical boundary condition σ. Then we have

Lemma 2-4 (Feynman-Kac formula) [30]
Let V belongs to $SSR \cup RR$ and such that $-\Delta_\Lambda^\sigma + V$ is self-adjoint in $L_2(\Lambda)$ for any classical b.c. (σ, Λ). Then the following formula holds

$$(exp - \beta(\Delta_\Lambda^\sigma + V))(x,y) = \int_{\Omega_\beta(\Lambda)} \mu_{\Lambda,\, x|y}^{\sigma,\beta}(d\omega) \cdot exp - \int_0^\beta dtV(\omega(t)). \qquad (2-12)$$

Notations and Abbreviations

$\Lambda \uparrow R^d$ means always that we have a countable generated filter $(\Lambda_\alpha)_\alpha$ of convex bounded regions that tends to R^d and such that for any α the boundary $\partial\Lambda_\alpha$ is of class C^3 and moreover the mean curvatures of the family $(\partial\Lambda_\alpha)_\alpha$ are uniformly bounded (in α).

$$\omega^n = (\omega_1, \ldots, \omega_n) \in \Omega_\beta(\cdot)^{\otimes n}.$$
$$x^n = (x_1, \ldots, x_n) \in R^{dn}$$
$$\omega'^{n-1} = (\omega_2, \ldots, \omega_n)$$

etc.

$$U_\beta(\omega^n) = U(\omega^n) = \sum_{1 \le i < j \le n} \int_0^\beta dtV(\omega_i(t) - \omega_j(t)), \qquad (2-13)$$

$$U_\beta(\omega^n|\tilde{\omega}^m) = U(\omega^n|\tilde{\omega}^m) = \sum_{i=1}^n \sum_{j=1}^m \int_0^\beta dtV(\omega_i(t) - \tilde{\omega}_j(t)). \qquad (2-14)$$

2.3 Reduced density matrices and the Kirkwood-Salsburg equations

The reduced, m particles, σ-conditioned density matrices $\rho_{m,\Lambda}^\sigma$ are given by

$$\rho_{m,\Lambda}^\sigma(x^m|y^m) = \int_{\Omega_\beta(\Lambda)^{\otimes m}} d\mu_{\Lambda,x^m|y^m}^{\sigma,\beta}(\omega^m)\rho_\Lambda^\sigma(\omega^m) \qquad (2-15)$$

where

$$\rho_\Lambda^\sigma(\omega^m) = (Z_\Lambda^\sigma)^{-1} \cdot \sum_{n \ge 0} \frac{z^{m+n}}{n!} \int_{\Omega_\beta(\Lambda)^{\otimes m}} d_\Lambda^\sigma \tilde{\omega}^m exp\left(-U_\beta(\omega^n) - U_\beta(\tilde{\omega}^m) - U_\beta(\omega^n|\tilde{\omega}^m)\right) \qquad (2-16)$$

($z = exp\beta\mu$ is the chemical activity) and

$$d^{\sigma}_{\Lambda}\tilde{\omega}^m = \bigotimes_{i=1}^{m}\left(\int_{\Lambda} dx_i \mu^{\sigma,\beta}_{\Lambda,x_i|x_i}(d\tilde{\omega}_i)\right) \quad \text{in the weak sense} \tag{2-17}$$

Here

$$Z^{\sigma}_{\Lambda} = \sum_{n\geq 0}\frac{z^n}{n!}\int_{\Omega_\beta(\Lambda)^{\otimes n}} d^{\sigma}_{\Lambda}\omega^n exp - U_\beta(\omega^n) \tag{2-18}$$

is the σ-conditioned, grand canonical ensemble partition function in the finite volume Λ.

Let B_0 be the space of all sequences $\phi = (\phi_n(\omega^n))$ of $(dx \otimes \mu^{\beta}_{x|x})^{\otimes n}$ measurable functionals ϕ_n defined on ${\Omega_\beta}^{\otimes n}$. Below we define the following linear operators in the space B_0:

$$(\Pi(\Lambda)\phi)_n(\omega^n) \equiv \prod_{i=1}^{n}\Pi(\Lambda)(\omega_i)\phi_n(\omega^n), \tag{2-19}$$

where

$$\Pi(\Lambda)(\omega) = \begin{cases} 1 & \text{if for any} \quad t\in[0,\beta] : \omega(t)\in\Lambda \\ 0 & \text{otherwise} \end{cases},$$

K (the Kirkwood-Salsburg operator):

$$(K\phi)_n(\omega^n) = exp\left\{-U^1(\omega^n)\right\}\cdot$$
$$\cdot\sum_{m\geq 0}\frac{1}{m!}\int_{{\Omega_\beta}^{\otimes m}} d_\infty\tilde{\omega}^m\; k(\omega_1|\tilde{\omega}^m)\phi(\omega'^{n-1},\tilde{\omega}^m), \quad \text{for } n>1, \tag{2-20a}$$

and

$$(K\phi)_1(\omega_1) = \sum_{n\geq 1}\frac{1}{n!}\int_{{\Omega_\beta}^{\otimes n}} d_\infty\tilde{\omega}^n k(\omega_1|\tilde{\omega}^n)\phi_n(\tilde{\omega}^n)), \quad \text{for } n=1, \tag{2-20b}$$

where

$$d_\infty\omega \equiv \int_{R^d} dx\mu^{\beta}_{x|x}(d\omega); \tag{2-21}$$

$$U^i(\omega^n) = \sum_{j\neq i}^{n}\int_0^\beta dt V(\omega_i(t)-\omega_j(t)); \tag{2-22}$$

$$k(\omega|\tilde{\omega}^m) = \prod_{i=1}^{m}\left[e^{-\int_0^\beta dt V(\omega(t)-\tilde{\omega}_i(t))} - 1\right]. \tag{2-23}$$

K^{σ}_{Λ} - the finite volume σ-conditioned Kirkwood-Salsburg operator:

$$(K^{\sigma}_{\Lambda}\phi)_n(\omega^n) = \begin{cases} exp\{-U^1(\omega^n)\}\cdot \\ \cdot \sum_{m\geq 0} \frac{1}{m!} \int_{\Omega_\beta(\Lambda)^{\otimes m}} d^{\sigma}_{\Lambda}\tilde{\omega}^m k(\omega_1|\tilde{\omega}^m)\phi(\omega'^{n-1},\tilde{\omega}^m), & \text{for } n>1; \quad (2-24a) \\ \sum_{m\geq 1} \frac{1}{m!} \int_{\Omega_\beta(\Lambda)^{\otimes m}} d^{\sigma}_{\Lambda}\tilde{\omega}^n k(\omega_1|\tilde{\omega}^n)\phi(\tilde{\omega}^n), & \text{for } n=1. \quad (2-24b) \end{cases}$$

K^{D}_{Λ} - the finite volume, Dirichlet conditioned Kirkwood-Salsburg operator:

$$(K^{D}_{\Lambda}\phi)_n(\omega^n) = \begin{cases} \alpha_\Lambda(\omega^n) exp\left\{-U^1(\omega^n)\right\}\cdot \\ \left[\sum_{m\geq 0} \frac{1}{m!} \int_{\Omega_\beta(\Lambda)^{\otimes m}} d^{D}_{\Lambda}\tilde{\omega}^m k(\omega_1|\tilde{\omega}^m)\phi(\omega'^{n-1},\tilde{\omega}^m)\right], & \text{for } n>1, \quad (2-25a) \\ \alpha_\Lambda(\omega_1) \sum_{m\geq 1} \frac{1}{m!} \int_{\Omega_\beta(\Lambda)^{\otimes m}} d^{D}_{\Lambda}\tilde{\omega}^m k(\omega_1|\tilde{\omega}^m)\phi(\tilde{\omega}^m), & \text{for } n=1, \quad (2-25b) \end{cases}$$

where now

$$d^{D}_{\Lambda}\,\omega = \int_{\Lambda} dx \alpha_\Lambda(\omega)\mu^{\beta}_{x|x}(d\omega). \qquad (2-26)$$

J - the index juggling operator of Ruelle :
From the stability assumption it follows that $\Omega_\beta^{\otimes m} = \bigcup_{j=1}^{m} \sum_j^m$, where $\sum_j^m = \{\omega^m \in \Omega_\beta{}^{\otimes m} \,|U^j(\omega^m) \geq -2\beta B\}$. Let then η_j denotes the characteristic function of Σ_j^m and let $\Theta_j^m = \eta_j \Big/ \sum_{j=1}^{m} \eta_j$. Let S_k be the operator defined on functionals of m-trajectories $f(\omega^m)$ as the cyclic permutation of k steps on the arguments of these functionals. Then the index juggling operator J is defined as:

$$(J\phi)(\omega^m) = \sum_{j=1}^{m} S_j \left[\Theta_j^m(\omega^m)\phi(\omega^m)\right]. \qquad (2-27)$$

If all components of ϕ are symmetric then the operator J reduces to the identity.

From the above definitions we have the following relations, cf. (2-19), (2-25),

$$K^{D}_{\Lambda} = \Pi(\Lambda)K\Pi(\Lambda), \qquad (2-28)$$

$$K^{\sigma}_{\Lambda} = K^{D}_{\Lambda} + \delta K^{\sigma}_{\Lambda} \qquad (2-29)$$

where the operator $\delta K^{\sigma}_{\Lambda}$ is defined as (on the symmetric functionals)

$$(\delta K^{\sigma}_{\Lambda})(\phi)(\omega^m) = exp\left\{-U^1(\omega^m)\right\}\sum_{n\geq 0}\frac{1}{n!}\sum_{l=1}^{n}\binom{n}{l}\int_{\Omega_\beta(\Lambda)^{\otimes l}} d^{\sigma}_{\Lambda}\tilde{\omega}^l \cdot \int_{\Omega_\beta{}^{\otimes(n-l)}} d_\infty\tilde{\omega}^{n-l}k(\omega^1|\tilde{\omega}^n)\Pi(\partial\Lambda)(\tilde{\omega}^l)\Pi(\Lambda)(\tilde{\omega}^{n-l}) \cdot\phi(\omega'^{m-1},\tilde{\omega}^n), \quad (2-30)$$

for $m > 1$, and

$$(\delta K^{\sigma}_{\Lambda}\phi)(\omega_1) = \sum_{n\geq 1}\frac{1}{n!}\sum_{l=1}^{n}\binom{n}{l}\int_{\Omega_\beta(\Lambda)^{\otimes l}} d\mu^{\sigma}_{\Lambda}(\tilde{\omega}^l)\int_{\Omega_\beta{}^{\otimes(n-l)}} d_\infty\tilde{\omega}^{n-l} \cdot k(\omega^1|\tilde{\omega}^n)\Pi(\partial\Lambda)(\tilde{\omega}^l)\Pi(\Lambda)(\tilde{\omega}^{n-l}) \cdot\phi(\tilde{\omega}^n), \quad (2-31)$$

here we have introduced the operator $\Pi(\partial\Lambda)$ by

$$\Pi(\partial\Lambda)(\phi)(\omega^n) = \prod_{l=1}^{n}\Pi(\partial\Lambda)(\omega_i)\phi(\omega^n), \quad (2-32)$$

where

$$\Pi(\partial\Lambda)(\omega) = \begin{cases} 1 & \text{if } \exists t\in[0,\beta] : \omega(t)\in\partial\Lambda \\ 0 & \text{otherwise.}\end{cases}$$

The following decomposition has been also introduced in (2-30): $\omega^n \equiv (\omega^k, \omega^{n-k})$ for $k = 1,\ldots,n$.

If we now proceed with well-known arguments (see [2]) the following identities between the correlation functionals can be obtained;

$$\begin{aligned}\rho^{\sigma}_{\Lambda} &= z\Pi(\Lambda)JK^{\sigma}_{\Lambda}\Pi(\Lambda)\rho^{\sigma}_{\Lambda} + z\mathcal{A}_\Lambda \\ &= \left[z\Pi(\Lambda)JK^{D}_{\Lambda}\Pi(\Lambda) + z\Pi(\Lambda)J\delta K^{\sigma}_{\Lambda}\Pi(\Lambda)\right]\rho^{\sigma}_{\Lambda} + z\mathcal{A}_\Lambda,\end{aligned} \quad (2-34)$$

where

$$\mathcal{A}_\Lambda = \Pi(\Lambda)\alpha \text{ with } \alpha = (1,0,\ldots,\ldots). \quad (2-35)$$

We call these identities the Kirkwood-Salsburg equations for the finite volume. They have to be compared with the following ones

$$\rho = zJK\rho + z\alpha \quad (2-36)$$

that we call the infinite volume Kirkwood-Salsburg equations. In the next section we provide a rigorous comparison analysis of (2-34) and (2-36).

The generic Kirkwood-Salsburg operator K can be decomposed in the following useful for us way (see [13,43]):

$$K = exp\mathcal{E}^1 \circ k \tag{2-37}$$

where the operator k is defined by

$$(k\phi)(\omega^n) = \phi(\omega'^n) + \sum_{m\geq 1}^{\infty} \frac{1}{m!} \int_{\Omega_\beta{}^{\otimes m}} d_\infty \tilde{\omega}^m k(\omega_1|\tilde{\omega}^m)\phi(\omega'^n, \tilde{\omega}^m), \tag{2-38a}$$

for $n > 1$,

$$(k\phi)(\omega_1) = \sum_{m\geq 1} \frac{1}{m!} \int_{\Omega_\beta{}^{\otimes m}} d_\infty \tilde{\omega}^m k(\omega_1|\tilde{\omega}^m)\phi(\tilde{\omega}^m), \tag{2-38b}$$

for $n = 1$.

The operator $exp\mathcal{E}^1$ is given by

$$(exp\mathcal{E}^1)(\phi)(\omega^n) = \begin{cases} \phi(\omega_1), & \text{for } n = 1 \\ (exp - U^1(\omega^n))\phi(\omega^n), & \text{for } n > 1. \end{cases} \tag{2-39}$$

3 Analysis of the Kirkwood-Salsburg equations.

Here the method of the dual pair of the Banach spaces proposed in [13] will be improved and applied to the analysis of the Kirkwood-Salsburg equations for the Maxwell-Boltzmann statistics. The Kirkwood-Salsburg identities with the Dirichlet boundary condition $\sigma = D$ has been analyzed previously by Ginibre [2]. The work of Ginibre [2,3] is entirely based on the contraction map principle. Our method reproduces his results as simple corollaries.

Let B_ξ be the Banach space defined as in [2], i.e.,B_ξ consists of sequence of essentially bounded (with respect to the measure $\int dx \mu^\beta_{x|x}$) functionals ϕ_m of m trajectories ω^m equipped with the norm

$$\|\phi\|_\xi = \sup_m \xi^{-m} ess \sup_{\omega^m \in \Omega_\beta^{\otimes m}} |\phi_m(\omega^m)|, \tag{3-1}$$

where $\xi > 0$ will be chosen later. The space $B_\xi = ({}^*B_\xi)^*$ is the dual to the Banach space ${}^*B_\xi$ which is defined as follows. It consists of sequences of $L_1(\int dx \mu^\beta_{x|x})$ -integrable functionals equipped with the norm

$${}^*\|\Psi\|_\xi \equiv \sum_{m\geq 1} \xi^m \int_{\Omega_\beta{}^{\otimes m}} d^\beta \omega^m |\psi_m(\omega^m)|. \tag{3-2}$$

From the definition of B_ξ and ρ_Λ^σ we get

$$|\rho_\Lambda^\sigma(\omega^m)| \leq \frac{1}{Z_\Lambda^\sigma} \cdot \sum_{s\geq 0}^{\infty} \frac{|z|^{m+s}}{s!} \; [exp(m+s)\beta \cdot B] \left(\int\limits_{\Omega_\beta(\Lambda)} d_\Lambda^\sigma \omega \right)^s \tag{3-3}$$
$$= \frac{1}{Z_\Lambda^\sigma} \left[|z|e^{\beta B}\right]^m exp\left(e^{\beta B} \cdot |z| \cdot C \cdot |\Lambda|\right)$$

uniformly in σ. Here the monotonicity of the kernels p_Λ^σ (2-2) have been used. The best possible value for the constant C is given by

$$C \cdot |\Lambda| = \int\limits_\Lambda du \int\limits_{\Omega_\beta(\Lambda)} d\mu_{\Lambda,u|u}^N(d\omega) \tag{3-4}$$

as it follows from the mean value theorem.

Similarly one can show that Z_Λ^σ is an entire analytic function (of order at most 1) of z and is uniformly in σ bounded:

$$|Z_\Lambda^\sigma| \leq exp\left(ze^{\beta \cdot B} C \cdot |\Lambda|\right) \tag{3-5}$$

It follows from estimate (3-3) that $\rho_\Lambda^\sigma \in B_\xi$ for any ξ such that $|z| \cdot exp\beta B < \xi$ with the norm

$$\|\rho_\Lambda^\sigma\|_\xi \leq |Z_\Lambda^\sigma|^{-1} exp\left(|z|e^{\beta \cdot B} \cdot |\Lambda| \cdot C\right) \tag{3-6}$$

Now we show that the operators JK, J (as defined in §2c) are bounded operators in the space B_ξ.
From the stability of V and definitions (2-24) and (2-27) we get

$$\begin{aligned} |(JK_\Lambda^\sigma\phi)_m(\omega^m)| &\leq \|\phi\|_\xi \, e^{2\beta B} \cdot \xi^{m-1} \\ &\times \sum_{n\geq 0}^{\infty} \frac{\xi^n}{n!} [C_\Lambda^\sigma(\beta)]^n, \end{aligned} \tag{3-7}$$

where

$$\begin{aligned} C_\Lambda^\sigma(\beta) &= \sup_\omega \int\limits_\Lambda dx \int\limits_{\Omega_\beta(\Lambda)} d\mu_{\Lambda,x|x}^{\sigma,\beta}(d\tilde{\omega}) \left|\left[exp\left[-\int\limits_0^\beta V(\omega(t)-\tilde{\omega}(t))\right]-1\right]\right| \\ &\leq \beta e^{2\beta B} \cdot \|V\|_1 \cdot \sup_x |p_\Lambda^\sigma(x,x|\beta)| \\ &\leq \beta e^{2\beta \cdot B} \cdot \|V\|_1 \sup_{\Lambda,x} \left|p_\Lambda^N(x,x|\beta)\right|. \end{aligned}$$

The quantity

$$\sup_{\Lambda} \sup_{x} p_{\Lambda}^{N}(x,x|\beta) \equiv C(\beta) < \infty. \tag{3-9}$$

Therefore, we have

$$\|JK_{\Lambda}^{\sigma}\|_{\xi} \leq \xi^{-1} exp(2\beta B + \xi \cdot \beta e^{2\beta B} \cdot \|V\|_1 C(\beta)) \tag{3-10}$$

uniformly in the boundary data σ and the constant $C(\beta)$ may be chosen to be equal to $(\pi \cdot \beta)^{1/2}$.

We shall concentrate on the Dirichlet-conditioned Kirkwood-Salsburg equations (2-34) which will be compared to the infinite volume one (2-36). The following theorem generalizes some of Ginibre's results [2].

Theorem 3-1

Let us denote the spectrum of the operator JK in the space B_{ξ} as $\sigma_{\xi}(JK)$. There exists $\xi > 0$ such that for any $z^{-1} \notin \sigma_{\xi}(JK)$ there exists a unique solution ρ_{∞} of (2-36) and moreover ρ_{Λ} tends to ρ_{∞} component-wise and locally uniformly as $\Lambda \uparrow R^d$.

Comparing with Ginibre's results included in [2] the novelty of Theorem 3-1 is in the absence of an *a priori* restriction to the small values of the chemical activity z, cf. [13]. The proof of Theorem 3-1 is based on the following sequence of lemmas.

Lemma 3-2.

There exist bounded linear operator ${}^*J$, ${}^*K$ in the space ${}^*B_{\xi}$ the duals of which are equal to J, resp. K i.e. $({}^*J)^* = J$, resp. $({}^*K)^* = K$ in the dual pair $({}^*B_{\xi}, B_{\xi})$.

Lemma 3-3.

Let ${}^*\Pi(\Lambda)$ denotes the corresponding predual of the operator $\Pi(\Lambda)$. Then the strong convergence in ${}^*B_{\xi}$

$${}^*\Pi(\Lambda)\, {}^*K {}^*J\, {}^*\Pi(\Lambda) \rightarrow\ {}^*K\, {}^*J \tag{3-11}$$

takes place as $\Lambda \uparrow R^d$.

Lemma 3-4.

Let $V \in SSR \cup RR$. There exists a number $\bar{\xi} > 0$ such that, uniformly in σ, the net $(\rho_{\Lambda\alpha}^{\sigma})_{\alpha} \subset B_{\bar{\xi}}$ (where all Λ_{α} are as in Lemma 2-3) is pre-compact in the weak-* topology of the space B_{ξ}.

Note, that for $V \in RR$ we immediately get the estimate $sup|\rho_{\Lambda\alpha}^{\sigma}(\omega^n)| \leq |z|^n$, uniformly in σ and α, to our disposal. As a consequence one gets $\|\rho_{\Lambda\alpha}^{\sigma}\|_{\xi} < 1$ providing $|z| < \xi$. Then application of the Banach-Alaoglu theorem gives the proof of Lemma 3-4. For a general $V \in SSR$ the proof is more complicated [24], cf. [13] and [12,45].

Before formulating the next lemma we need some preparations. It is well-known that the iteration of the Kirkwood-Salsburg equations leads to the Mayer-Montroll equations [31]. Therefore, if ρ_{Λ}^{σ} (resp. ρ_{∞}) fulfills (2-34) (resp. (2-36)) it fulfills also the following identities

$$\rho_{\Lambda}^{\sigma} = \Pi(\Lambda)\mathcal{M}(z)\Xi_{\Lambda}^{\sigma}\Pi(\Lambda)\rho_{\Lambda}^{\sigma} + A_{\Lambda}^{\sigma}(z) \tag{3-12}$$

and

$$\rho_\infty = \mathcal{M}(z)\rho_\infty + z\alpha. \tag{3-13}$$

Here the Mayer-Montroll operator $\mathcal{M}(z)$ is defined by the formulae

$$\begin{aligned}(\mathcal{M}(z)\phi)_m(\omega^m) &= z^m exp - U_\beta(\omega^m)\\ &\times\left[1+\sum_{n\geq 1}\frac{1}{n!}\int\limits_{\Omega_\beta(R^d)^{\otimes n}} d_\infty\tilde{\omega}^n M(\omega^m|\tilde{\omega}^n)\phi_{n+m}(\omega^m,\tilde{\omega}^n)\right],\end{aligned} \tag{3-14}$$

where the Mayer-Montroll kernels are given by

$$M(\omega^n|\tilde{\omega}^m) = \prod_{j=1}^{m}\left(\prod_{i=1}^{n} e^{-\int\limits_0^\beta dtV(\omega_l(t)-(\tilde{\omega})_j(t))} - 1\right) \tag{3-15}$$

and $\mathcal{A}_\Lambda = (\Pi(\Lambda)z, 0, \ldots, 0, \ldots)$.
The operator $\mathcal{M}(z)\Xi^\sigma_\Lambda$ is defined by

$$(\mathcal{M}(z)\Xi^\sigma_\Lambda)(\phi)_m(\omega^m) = z^m exp - U_\beta(\omega^m)$$

$$\times\left[1+\sum_{n\geq 1}\frac{1}{n!}\int\limits_{\Omega_\beta(\Lambda)^{\otimes n}} d^\sigma_\Lambda\tilde{\omega}^n M(\omega^m|\tilde{\omega}^n)\phi_{n+m}(\omega^m|\tilde{\omega}^n)\right] \tag{3-16}$$

Standard application [3] of the Mayer-Montroll equations gives.

Lemma 3-5.

Let assume that $\rho^D_\Lambda \to \rho_\infty$, as $\Lambda \uparrow R^d$, in the weak-* topology of the space B_ξ. Then $\rho^D_\Lambda \to \rho_\infty$ component-wise and locally uniformly.

Having postponed proofs of the listed sequence of lemmas we outline the proof of Theorem 3-1.

Proof of Theorem 3-1.

Let $(\Lambda_\alpha)_\alpha$ be an arbitrary net as described above. Then from Lemma 3-4 it follows that the set of accumulation points of $(\rho^D_{\Lambda\alpha})_\alpha$ is non-empty. Let ρ^D_∞ be any of them. It follows from Lemma 3-3 that ρ^D_∞ fulfills the equality (2-36). By the assumption about z it follows that equation (2-36) has a unique solution ρ_∞. Therefore, it must be $\rho^D_\infty = \rho_\infty$. Additionally from the very definition of ρ^D_∞ it follows that $\rho^D_{\Lambda_\alpha} \to \rho_\infty$ in the weak-* topology on B_ξ. Application of Lemma 3-5 concludes the proof.

Now, we show the validity of those lemmas. The proof of Lemma 3-4 for the case $V \in SSSR$ as the most technical, lenghtly and complicated will be presented in [24]. It is based on the adaptation of the probability estimates of Ruelle [31, 32] (see also [33]) with the fluctuation estimates of Park [12]. For $d \leq 3$ and $\sigma = D$ the proof of the Lemma

3-4 is contained in [45].

Proof of Lemma 3-2.

The explicite calculation done in [13] gives the following expression for the predual operator $^*k$ in $^*B_\xi$:

$$(^*k\psi)_m(\omega^m) = \sum_{l=0}^{m} \frac{1}{l!} \int_{R^d} dx \int_{\Omega_\beta} d\mu^\beta_{x|x}(\tilde{\omega}) k(\omega^l|\tilde{\omega}) \psi_{1+m-l}(\tilde{\omega} \vee (\omega^m - \omega^l)) \qquad (3-17)$$

which is bounded by

$$\|(^*k\psi)_m(\omega^m)\|_{L^1} \leq \sum_{l=0}^{m} \frac{1}{l!} C^l(\beta) \|\psi_{m+1-l}\|_{L^1},$$

$$C(\beta) \equiv \sup_\omega \int d^\beta_\infty \tilde{\omega} \left| exp\left\{ - \int_0^\beta V(\omega(t) - \tilde{\omega}(t)) \right\} - 1 \right|.$$

Therefore,

$$^*\|^*k\|_\xi \leq exp \xi C(\beta). \qquad (3-18).$$

The existence, boundedness and the explicit form of $^*J$ follows from the definition of J and the dualisation in $(^*B_\xi, B_\xi)$. The chain rule for pre-dualisation, cf. [13], and formula (2-27) proves the existence of $^*J^*K \in L(^*B_\xi)$ providing V is stable and integrable.

Proof of Lemma 3-3.

It is application of $2-\epsilon$ argument. The finite-length sequences forms a dense subset in $^*B_\xi$ and moreover for any n we have $L_1(\Omega_\beta^{\otimes n}, d^\beta_\infty \omega^n) = \overline{\lim_{\Lambda \uparrow R^d}} L_1(\Omega_\beta^{\otimes n}(\Lambda), d^\beta_\Lambda \omega^n)$ as it follows from Lemma 2-2 and Lemma 2-3. Therefore, for any $\psi \in {^*B_\xi}$ and any $\epsilon > 0$ there exists a bounded $\Lambda_\epsilon \subset R^d$ such that $^*\|\psi - \Pi(\Lambda_\epsilon)\Psi\|_\xi < \epsilon$. Then we get the result by estimate, cf. [13],

$$\begin{aligned}
&^*\|((^*JK_{\Lambda_\epsilon}) - {^*(JK_\infty)})\psi\|_\xi \\
&= {^*\|}(\Pi(\Lambda_\epsilon)\ (^*k\ \exp \mathcal{E}^1\ {^*J})\Pi(\Lambda_\epsilon) - {^*k}\ exp\ \mathcal{E}^1\ {^*J})\psi\|_\xi \\
&\leq \|k\|_\xi\ {^*\|}(\exp \mathcal{E}^1\ {^*J})\Pi(\Lambda_\epsilon) - \exp \mathcal{E}^1\ {^*J})\psi\|_\xi + \|(1 - \Pi(\Lambda_\epsilon))\ {^*k}\ \exp \mathcal{E}^1\ {^*J}\psi\|_\xi \\
&\leq 2\|k\|_\xi\ \|J \exp \mathcal{E}^1\|_\xi\ {^*\|}(1 - \Pi(\Lambda_\epsilon)\psi\|_\xi
\end{aligned} \qquad (3-19)$$

because by definitions of $^*\|\ \|_\xi$ and the operator $\Pi(\Lambda)$ we have: $\Pi(\Lambda) \to 1$ strongly.

Proof of Lemma 3-5.

For each fixed $\omega^n \in {\Omega_\beta}^{\otimes n}$ the map

$$\omega^n \to \left[\frac{1}{k!} M(\omega^n|\tilde{\omega}^k)\right]_{k=1,2,\ldots} \qquad (3-20)$$

takes values in the space ${}^*B_\xi$. It is consequence of the previous estimates. From

$$\|M(\omega^n|\tilde{\omega}^k)| \leq e^{2\beta\cdot B}\prod_{i=1}^{n}\prod_{j=1}^{k}\left|\left(\int_0^\beta dtV(\omega_l(t)-\tilde{\omega}_j(t)\right)\right| \tag{3-21}$$

it follows

$$\left|\int_{\Omega_\beta{}^{\otimes k}} d^\beta_\infty\tilde{\omega}^k M\left(\omega^n|\tilde{\omega}^k\right)\right| \leq e^{2\beta\cdot B}\beta^{nk}\|V\|_{L_1}\cdot p(0|\beta)^k. \tag{3-22}$$

Therefore,

$$^*\left\|1/k!M\left(\omega^n|\tilde{\omega}^k\right)\right\|_\xi \leq e^{2\beta\cdot B\cdot n}exp\left(\beta^n\|V\|_{L^1}p(0|\beta)\cdot\xi\right). \tag{3-23}$$

uniformly in $\omega^n \in \Omega_\beta{}^{\otimes n}$.

Straithforward comparison of the both sides of Mayer-Montroll equations (3-12) and (3-13) gives the final argument.

We proceed to the general case of an arbitrary classical boundary condition. The result is formulated as follows.

<u>**Theorem 3-6**</u>.

Let σ be an arbitrary classical boundary condition and we take $\xi > max\{|z|exp\beta B, \bar{\xi}\}$ where $\bar{\xi}$ is given by Lemma 3-4. Then, for any value of z^{-1} which do not belongs to the spectrum (in the space B_ξ) of the operator JK we have $\rho^\sigma_\Lambda \to \rho_\infty$ as $\Lambda \uparrow R^d$ where the convergence is component-wise and locally uniform.

The new additional arguments necessary to supply the proof in the spirit of the proof of Theorem 3-1 are listed now.

<u>**Lemma 3-7.**</u>

Let $\Lambda \uparrow R^d$. Then for any classical boundary condition σ and any $\xi > 0$

$$\lim_{\Lambda\uparrow R^d}\|J\delta K^\sigma_\Lambda\|_\xi = 0 \tag{3-24}$$

<u>**Lemma 3-8**</u>.

For any $\xi > 0$ and any classical boundary condition the operators $J\delta K^\sigma_\Lambda$ are weakly continuous on the space B_ξ.

<u>**Proof of Theorem 3-6**</u>.

According to Bourbaki [34] the weak continuity of δK^σ_Λ on B_ξ, given by Lemma 3-8, is a sufficient condition to assure the existence of the pre-dual bounded operator ${}^*(\delta K^\sigma_\Lambda)$ $\in L({}^*B_\xi)$. Then by Lemma 3-7 it follows that ${}^*(\delta K^\sigma_\Lambda) \to 0$ in the uniform topology and, thus, strongly on ${}^{*}B_\xi$ as $\Lambda \uparrow R^d$. The remaining arguments for supplying the complete proof can be given as before.

<u>**Proof of Lemma 3-7**</u>.

From the stability assumption it follows that

$$|exp\left\{-\int_0^\beta dt V(\omega(t)-\tilde{\omega}(t))\right\}-1| \le e^{2B\cdot\beta}\cdot\int_0^\beta |V(\omega(t)-\tilde{\omega}(t))|dt. \qquad (3-25)$$

Using the shift transformation, together with Fubini theorem, we get

$$\left|\int_{\Omega_\beta(\Lambda)} d^\sigma_\Lambda\tilde{\omega}\Pi(\partial\Lambda)K(\omega|\tilde{\omega})\right|_W \le \sup_\omega e^{2\beta\cdot B}\int_{\Omega_\beta(\Lambda)} d^\sigma_\Lambda\tilde{\omega}\Pi(\partial\Lambda)(\tilde{\omega})\int_0^\beta |V(\omega(t)-\tilde{\omega}(t))|dt \qquad (3-26)$$
$$\le \beta e^{2\beta\cdot B}\cdot\|V\|_1\cdot\int d\mu^\sigma_{\Lambda,0|0}(\tilde{\omega})\Pi(\partial\Lambda)(\tilde{\omega})$$
$$\le O_\beta(1)\Delta p^\sigma_\Lambda(0,0|\beta).$$

Therefore,

$$|(J\delta K^\sigma_\Lambda)(\phi)_m(\omega^m)| \le \|\phi\|_\xi\cdot\xi^{m-1}\cdot e^{2\beta\cdot B} \times\sum_{n\ge 0}\frac{\xi^n}{n!}\sum_{k=1}^n\binom{n}{k}O_\beta(1)^k\cdot\mathcal{D}^\sigma_\Lambda\cdot\beta^{n-k}\cdot e^{2\beta\cdot Bk} \times\|V\|_1^{n-k}\left(\frac{\pi}{\beta}\right)^{n-k}\le\|\phi\|_\xi\cdot\xi^{m-1}e^{2\beta\cdot B}\cdot\mathcal{D}^\sigma_\Lambda exp\xi O'_\beta(1), \qquad (3-27)$$

where

$$\mathcal{D}^\sigma_\Lambda = \Delta p^\sigma_\Lambda(0,0|\beta). \qquad (3-28)$$

Thus

$$\|J\delta K^\sigma_\Lambda\|_\xi \le O''_\beta(1)\cdot\mathcal{D}^\sigma_\Lambda \qquad (3-29)$$

Application of Lemma 2-2 gives $\lim_{\Lambda\uparrow R^d}\mathcal{D}^\sigma_\Lambda = 0$ uniformly in the boundary data.

<u>**Proof of Lemma 3-8**</u>.

Let $\mathcal{M}_n$ be the space of all countable additive set functions on ${\Omega_\beta}^{\otimes n}$ which have bounded variation and vanish on the zero sets of the measure $(\int dx\mu^\beta_{x|x})^{\otimes n}$. Then the space $\oplus_\xi \mathcal{M}_n \equiv B^*_\xi$ equipped with the norm

$$\|\mu\|^*_\xi = \sum_{n\leq 0} \xi^n Var(\mu_n) \tag{3-30}$$

is the dual space of the space B_ξ (see [35]). Let ϕ^N be an arbitrary sequence in B_ξ converging weakly to ϕ. We have to show that for any $\mu \in B^*_\xi$, $\mu(\delta K^\sigma_\Lambda(\phi^N))$ converges to $\mu(\delta K^\sigma_\Lambda(\phi))$.

To this end, let $\tilde{\mathcal{M}}_n$ be the space of all countable additive set functions on $\Omega_\beta^{\otimes n}$ which have bounded variation. Then the space $\oplus_\xi \tilde{\mathcal{M}}_n = \tilde{B}^*_\xi$ is the Banach space in the norm $\|\mu\|^*_\xi = \sum_{n\leq 0} \xi^n Var(\mu_n)$ and moreover B^*_ξ is the closed subspace of $\tilde{B}^*_\xi$. For a given $\mu = (\mu_n)$ let us define

$$\begin{aligned}(\Theta^\sigma_\Lambda(\mu))_{n+m}(\tilde{\omega}^n, \tilde{\omega}^m) \;&\equiv \mu_n(d\omega^n)\left(\frac{1}{m!}K(\omega_1|\tilde{\omega}^m\right)\\ &\times exp\left\{-U^1(\omega^n)\right\}\left(\Sigma\binom{m}{k}\Pi^k(\partial\Lambda)\right)\\ &\times \Pi^{m-k}(\mathring{\Lambda})\left(\int_\Lambda dx\mu^\sigma_{\Lambda,x|x}(d\tilde{\omega}^m)\right).\end{aligned} \tag{3-31}$$

From the stability assumption it follows that $\Theta^\sigma_\Lambda(\mu) \in \tilde{B}^*_\xi$ for any $\mu \in \tilde{B}^*_\xi$ and, moreover, we have the following estimate

$$\|\Theta^\sigma_\Lambda(\mu)\|_{\tilde{B}^*_\xi} \;\leq\; \|\mu\|_{\tilde{B}^*_\xi} \cdot exp\xi e^{2\beta\cdot B} O_\Lambda(\beta), \tag{3-32}$$

where now

$$O_\Lambda(\beta) = \Delta p^\sigma_\Lambda(0,0|\beta)(1 + O_\beta(1)). \tag{3-33}$$

Finite length sequences form a dense subset in $\tilde{B}^*_\xi$. Taking $\mu \in \tilde{B}^*_\xi$ of the form $\mu = (\delta_{nm} \cdot \mu_n)$ and noting that

$$\begin{aligned}&\left|\mu((J\delta^\sigma_\Lambda K)(\phi)^N - (J\delta K^\sigma_\Lambda)(\phi))\right|\\ &= \left|\Theta^\sigma_\Lambda(\mu_n)(\phi^N) - \Theta^\sigma_\Lambda(\mu_n)(\phi)\right| \to 0,\end{aligned}$$

as $N \uparrow \infty$, we finish the proof.

4 Existence and properties of the KMS states

Let $L_2(\Lambda)$ be the building space for the associated Fock space $\mathcal{F}(\Lambda)$ describing states of the system in a bounded region $\Lambda \subset R^d$. The total hamiltonian $H^\sigma_\Lambda = \bigoplus_{n\geq 0} H^{n,\sigma}_\Lambda$

given in terms of n-particle hamiltonians $H_\Lambda^{n,\sigma} = -\frac{1}{2}\sum_{i=1}^{n}\Delta_i^\sigma + U(x^n)$ defines then a local dynamics on the local C^* algebra $\mathcal{R}(\Lambda)$. For defining $\mathcal{R}(\Lambda)$ let $a(f)$ and $a^+(f)$ be the annihilation and respectively creation operators defined on $\mathcal{F}(\Lambda)$. Then operator $\Phi(f) = \frac{1}{\sqrt{2}}(a(f) + a^+(f))$ for real f has a self-adjoint extension which is $\bar{\Phi}(f)$. The Weyl operators $W(f) = expi\bar{\Phi}(f)$, $f \in L_2(\Lambda)$ then generate C^*-algebra and let $\mathcal{R} = \overline{\bigcup_{\Lambda\subset R^d}\mathcal{R}(\Lambda)}$ be the corresponding quasi-local algebra of observables. The finite volume Gibbs state ω_Λ^σ is given by

$$\omega_\Lambda^\sigma(-) = (Z_\Lambda^\sigma)^{-1}\, Tr_{\mathcal{F}(\Lambda)}(-)e^{-\beta H_\Lambda^\sigma - \mu N_\Lambda}, \tag{4-1}$$

$$Z_\Lambda^\sigma = Tr_{\mathcal{F}(\Lambda)}e^{-\beta H_\Lambda^\sigma - \mu N_\Lambda}, \tag{4-2}$$

where N_Λ is the particles number operator.

The finite-volume Gibbs state ω_Λ^σ on the local algebra $\mathcal{R}(\Lambda)$ is fully determined by the corresponding finite-volume σ-conditioned Green functions G_Λ^σ:

$$G_\Lambda^\sigma(A_0, A_1, \ldots, A_n; t_1, \ldots, t_n) = \omega_\Lambda^\sigma(A_0\alpha_{t_1}^{\Lambda,\sigma}(A_1)\ldots\alpha_{t_n}^{\Lambda,\sigma}(A_n)), \tag{4-3}$$

where $A_0, A_1, \ldots, A_n \in \mathcal{R}(\Lambda)$ and the local dynamics is given by

$$\alpha_t^{\Lambda,\sigma}(A) = \exp[it(H_\Lambda^\sigma - \mu N_\Lambda)]A\exp[-it(H_\Lambda^\sigma - \mu N_\Lambda)]. \tag{4-4}$$

Using the arguments and the methods of refs. [4-6] and the result of the Theorem 3.6 one gets the existence and the independence of the classical boundary condition σ of the limiting Green functions $G_\infty = \lim_{\Lambda\uparrow R^d} G_\Lambda^\sigma$. The limiting Green functions determine the infinite-volume state ω_∞ on the quasilocal algebra $\mathcal{R}$. Using the GNS construction we define finally the physical Hilbert space $\mathcal{H}_\infty$, the limiting unitary dynamics U_t acting on $\mathcal{H}_\infty$ and the cyclic vector $\Omega_\infty \in \mathcal{H}_\infty$ which defines a vector state $\tilde{\omega}_\infty$ that appears to be KMS state with respect of the dynamics $\tilde{\alpha}_t$ implemented by U_t as a group of autormorphisms of $(\pi_{\tilde{\omega}_\infty}(\mathcal{R}))''$. (For an instructive discussion of the problem of the existence of the infinite-volume dynamics (in the weak sense) on the quasilocal algebra $\mathcal{R}$ see [46] and references therein).

Let us denote by $\Delta^c(z,\beta)$ the set of all possible limiting KMS states that can be obtained from $\{\omega_\Lambda^\sigma\}_\Lambda$, as σ varies over all possible classical boundary conditions and $\Lambda \uparrow R^d$, via construction outlined above. The set of all limiting KMS states describing the systems under consideration will be denoted by $\Delta(z,\beta)$. The results obtained in this paper can be summarized as the following statement

Theorem 4-1.

Consider the interacting particles for which $V \in SSR\bigcup RR$. There exists $\xi > 0$ such that $\#\Delta^C(z,\beta) = 1$ for any value of $z = exp(-\beta\mu)$ such that $z^{-1} \notin \sigma_\xi(JK)$. Moreover this unique KMS state, denoted as $\omega_\infty(z)$ is (weakly) analytic in z, entire analytic and locally normal state over the algebra $\mathcal{R}$.

5 Concluding remarks.

To apply effectively the results obtained in the previous sections the spectral analysis of the (Ruelle)-Kirkwood-Salsburg operator is necessary. In the classical statistical mechanics some spectral properties of the operator $\Pi(\Lambda)JK\Pi(\Lambda)$ are known [36, 37]. It is rather straightforward to extend the arguments of those papers to the case with Maxwell-Boltzmann statistics. However, the actual problems are: the extension of these results to the case of particles obeying Bose-Einstein or Fermi-Dirac statistics and the control of the flow of the finite volume spectral set as we pass to the thermodynamic limit. These problems are now under consideration [38].

The next intriguing question concerns the uniqueness of the corresponding infinite volume KMS state i.e. the question whether $\Delta^C(z,\beta) = \Delta(z,\beta)$. In the classical statistical mechanics a constructive description of the set of all equilibrium states is given by the celebrated DLR equation [39, 40]. From the general theory [39, 40] then follows that any solution of the corresponding DLR equation can be obtained manipulating suitably (allowed) boundary conditions. This fact enables us to prove certain uniqueness theorems known as Dobrushin uniqueness theorems. Only for a class of lattice systems similar constructive description of the limiting KMS states has been given [41, 42]. Uniqueness of the KMS states is known sometimes for the lattice systems [1] but the corresponding proofs are based entirely on the intrinsic operator algebras methods.

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