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BEHAVIOUR OF WAVE FUNCTIONS OF THE UNIVERSE AND COSMOLOGICAL CONSTANT

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ABSTRACT

Our purpose is to study the asymptotic behaviour of cosmological wave functions in the region of Superspace where det(metric)=∞. We are considering curves σ_y in Superspace, with $\lim_y(\det\sigma_y)=\infty$, and by means of a family of these curves an asymptotic region in det(metric)=∞ is defined. The behaviour of solutions to the Einstein-Hamilton-Jacobi equation along the curves σ_y is analysed. From this analysis we deduce for the wave functions of the universe the following behaviour in that asymptotic region: dissipative (dispersive) if $\Lambda>0$ (resp. $\Lambda<0$).

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INTRODUCTION

The behaviours of cosmological wave functions have been analysed by several authors using minisuperspace models of gravity coupled to a scalar field [1 - 8], or cosmological models with a Higgs field [9]. In these models the infinite number of degrees of freedom is reduced to a finite number, by restricting the metric and the field to be homogeneous and isotropic. Thus the Wheeler - De Witt (W-DW) equation is reduced to a partial differential equation in the remaining degrees of freedom; and by means of an asymptotic analysis the controlling factors [10] and the prefactors of solutions are obtained.

In this paper we are considering curves in Superspace [11], i.e. one parameter families of "points" of Superspace. Each point is a pair metric-field, in general with an infinite number of degrees of freedom. Our purpose is to study the asymptotic behaviours of wave functions of the universe along those curves. The controlling factors of the asymptotic behaviours obtained in [1-4,6-9] using a minisuperspace model are reproduced by applying our method to suitable curves (Section 6).

The asymptotic analysis of solutions to the W-DW equation suggests that the solutions of the Einstein-Hamilton-Jacobi (EHJ) equation [12] [13] be studied. The framework in which the EHJ operator will be considered and a definition of distance in Superspace are introduced in Sect. 1.

In Sections 2 and 3 we define a class E of curves $(\sigma_y)_{y\in(b,\infty)}$ in Superspace, with $\lim_{y\to\infty} \det(\sigma_y)=\infty$. E is constructed so that on a curve $\sigma_y \in E$ the EHJ equation takes a suitable form with regard to the asymptotic analysis.

In Section 3 the asymptotic behaviour, along σ_y, of solutions V to the EHJ equation is studied. In Theorem 2 the results are collected. There are two types of behaviour: either $V\sim(\det\sigma_y)^{1/2}$ or $V\sim F((\det\sigma_y)^{1/3})$, being F a function such that $|F(u)|\gg u^{3/2}$, as $u\to\infty$.

Theorem 2 is applied to pure quantum gravity in Sect. 4. In this case, as $y\to\infty$, $V[\sigma_y]$ is real (purely imaginary) if the cosmological constant Λ is negative (resp. $\Lambda>0$). From this result, properties for cosmological wave functions ψ of gravity coupled to a scalar field are deduced: There is a neighbourhood U of field zero such that if $\phi\in U$, on (σ_y,ϕ) the behaviour of semiclassical approximation for ψ is dissipative [10] (dispersive) if $\Lambda>0$ (resp. $\Lambda<0$).

The dispersive behaviour of ψ, if $\Lambda<0$, is reasonably conjectured in an "extensive" asymptotic region of Superspace. As a consequence of this supposition new difficulties arise in the probability interpretation of ψ. So it suggests that the possibility $\Lambda<0$ in pure quantum gravity be rejected. (Similar suggestions based on the analysis of a minisuperspace model can be found in [3] and [7]). These questions are considered in Section 5.

In Section 6 two examples of curves which do not belong to E are considered. In the first example the curve consists of homogeneous, isotropic metrics; the second curve consists of homogeneous metrics. By applying our method to first example we reproduce the types of asymptotic behaviour which appear in the study of minisuperspace models.

1 THE W-DW AND THE EHJ EQUATIONS

We consider gravity coupled to a scalar field; and for the massive scalar field ϕ we take the usual Lagrangian

$$L = -(1/2)\phi_{,\mu}\phi_{,\nu}g^{\mu\nu} - (1/2)m^2c^2\hbar^{-2}\phi^2$$

In absolute units ($16\pi G = c = \hbar = 1$) and with a simple choice of a factor ordering, the respective W-DW equation is [11], [14]

$$\Big\{N_{abcd}(\sigma(x))\frac{\delta^2}{\delta\sigma_{ab}(x)\delta\sigma_{cd}(x)} + \det\big(\sigma_{ij}(x)\big)W(\sigma,\phi,x) + + (1/2)\frac{\delta^2}{\delta\phi(x)\delta\phi(x)}\Big\}\psi = 0 \tag{1.1}$$

$$N_{abcd}(\sigma(x)) \equiv (1/2)\big(\sigma_{ac}(x)\sigma_{bd}(x)+\sigma_{ad}(x)\sigma_{bc}(x)-\sigma_{ab}(x)\sigma_{cd}(x)\big)$$

$$W(\sigma,\phi,x) \equiv {}^3R_\sigma(x) - 2\Lambda - (1/2)\sigma^{ab}(x)\phi_{,a}(x)\phi_{,b}(x) - (1/2)m^2\phi^2(x)$$

σ is a riemannian metric on the 3-dimensional manifold S. And we suppose that S is compact. To simplify we suppose also that S is covered with only one chart ζ, and therefore $\zeta(S)$ is a compact of $\mathbb{R}^3$. The second supposition is not restrictive, because there are finite atlas and compact coverings for S (see Remark 2 to

Theorem 1).

$$\mathcal{M} \equiv \{\sigma \,|\, \sigma \text{ is a } C^2 \text{ riemannian metric on S}\}$$

$$\mathcal{F} \equiv \{q \,|\, q \text{ is a } C^1 \text{ scalar field on S}\}$$

Metrics and fields will be expressed in the chart ζ. The coordinates defined by the chart ζ are denoted by x^1, x^2, x^3; hence $\sigma = \sigma_{ij} dx^i dx^j$, where $\sigma_{ij} : \zeta(S) \longrightarrow \mathbb{R}$.

We define $\quad {}_\zeta\mathcal{M} \equiv \{\{\sigma_{ij}\} \,|\, \sigma_{ij}$ coordinates of $\sigma \in \mathcal{M}$ in the chart $\zeta\}$.

$${}_\zeta\mathcal{F} \equiv \{\phi \equiv q \circ \zeta^{-1} \,|\, q \in \mathcal{F}\}.$$

The solutions of (1.1) may be viewed as maps $\psi : {}_\zeta\mathcal{M} \times {}_\zeta\mathcal{F} \longrightarrow \mathbb{C}$. If, moreover, we demand ψ to satisfy the supermomentum equations

$$-2\left(\frac{\delta\psi}{\delta\sigma_{ac}}\right)\Big|_c + \sigma^{ab}\phi_{,b}\left(\frac{\delta\psi}{\delta\phi}\right) = 0 \tag{1.2}$$

then ψ may be regarded as a map $\psi : \mathcal{M} \times \mathcal{F} \longrightarrow \mathbb{C}$; that is, ψ does not depend on the coordinates in terms of which σ and q are expressed.

The second functional derivatives in the W-DW equation, are not necessarily regular distributions, (p. 1123, [11]). This difficulty can be seen already in the expression for the ground state of linearized gravitational field in the field representation [14]. However the first step in deriving the asymptotic behaviour of a solution to an ordinary differential equation, near an irregular singular point, is to express the solution in the form exp(z), and next to suppose that $z'' \ll (z')^2$ [10]. If we make $\psi[\sigma_{ab}, \phi] = \exp\left(iV[\sigma_{ab}, \phi]\right)$ in the W-DW equation, and remove the second derivatives, the EHJ equation [12] is obtained

$$-N_{abcd}\left(\sigma(x)\right) \frac{\delta V}{\delta\sigma_{ab}(x)} \cdot \frac{\delta V}{\delta\sigma_{cd}(x)} + \det\left(\sigma_{ij}(x)\right) W(\sigma, \phi, x) - \\ -(1/2)\left(\frac{\delta V}{\delta\phi(x)}\right)^2 = 0 \tag{1.3}$$

If we admit the analogy with quantum mechanics, then the semiclassical approximation for ψ is $D[\sigma_{ab}, \phi]\exp\left(iV[\sigma_{ab}, \phi]\right)$, being D a slowly varying functional.

A functional derivative may be considered as a distribution; but the product of two distributions is not necessarily defined [16]. Since the first and third terms of (1.3) are products of distibutions, in order to deduce rigorously the behaviour of V, solution of the EHJ equation, we consider only "regular" solutions.

Definition. Let $V: {}_{\zeta}M \times {}_{\zeta}F \longrightarrow \mathbb{C}$ be a map such that the derivatives $\frac{\delta V}{\delta\phi(x)}$ and $\frac{\delta V}{\delta\sigma_{ab}(x)}$ (for all a,b) are regular distributions; that is, integrable functions on $\zeta(S)$ with respect to the Lebesgue measure. Then we say that V is a regular functional.

The lorentzian metric G_{abcd} [11] on Superspace generates a distance which is not adequate to our asymptotic analysis: the distance between $\{\sigma_{ij}\}$ and $\{\gamma_{ij}\}$ can be zero and ${}^3R_{\sigma} \neq {}^3R_{\gamma}$, therefore the coefficients of the EHJ equation are not continuous with relation to this distance. On the other hand each σ_{ab} is a C^2 function defined on the compact $\zeta(S)$, so it can be viewed as an element of $D^2_{\zeta(S)}$ (space of the C^2 functions with support contained in $\zeta(S)$; see p.24 [16]). Likewise ϕ can be regarded as an element of $D^1_{\zeta(S)}$. In $D^m_{\zeta(S)}$ we can consider the seminorms $N_s(f)=\sup|D^s(f)|$, where $s=(s_1,s_2,s_3)$, $0\le|s|\equiv s_1+s_2+s_3\le m$, and

$$D^s \equiv \frac{\partial^{s_1+s_2+s_3}}{(\partial x^1)^{s_1}(\partial x^2)^{s_2}(\partial x^3)^{s_3}} .$$

The convergence of $\{f_n\} \longrightarrow f$ in D^m (endowed with the topology defined by $\{N_s\}$) is the uniform convergence of $\{D^s f_n\} \longrightarrow D^s f$ for all $|s|\le m$. $D^m_{\zeta(S)}$ is also a metric space (p.27, [17]). Let d_1 and d_2 be the respectives distances in D^1 and in D^2. Starting from d_2 we can define on $\prod_{i\le j} D^2_{\zeta(S)}$, and therefore on ${}_{\zeta}M$, the distance $d'(\{\sigma_{ij}\},\{\gamma_{ij}\})=\max_{a,b}\{d_2(\sigma_{ab},\gamma_{ab})\}$. And finally a distance on ${}_{\zeta}M\times{}_{\zeta}F$

$$d\Big((\{\sigma_{ij}\},\phi),(\{\gamma_{ij}\},\bar{\phi})\Big)=\max\Big(d'(\{\sigma_{ij}\},\{\gamma_{ij}\}),d_1(\phi,\bar{\phi})\Big).$$

The topology defined by d is the adequate to regard the functional derivatives as distributions (p.24, [16] and p.47, [17]). Moreover bearing in mind the expression of ${}^3R$ in terms of the metric, we have :

Proposition 1. With relation to d, the coefficients of the EHJ equation are continuous.□

2 CURVES IN THE SPACE OF METRICS

Given V, solution of the EHJ equation; V depends on the functional variables σ_{ab} and ϕ. We want to reduce the six variables σ_{ab} to one functional variable. For this purpose we consider metrics which depend on a fixed C^2 function $z:\zeta(S)\longrightarrow \mathbb{R}$. Let $\hat{\sigma}:D^2_{\zeta(S)}\longrightarrow \prod_{i\le j} D^2_{\zeta(S)}$ be a continuous function such that $\{\sigma_{ij}(x)\equiv\hat{\sigma}_{ij}(z)(x)\}\in{}_{\zeta}M$. Then V may be considered as a functional of z.

In order to have a curve in ${}_{\zeta}M$ we need an one parameter family of functions z. Let $z:(b,\infty)\times\zeta(S)\longrightarrow\mathbb{R}^+$ be a continuous function such that:

i) For each $y\in(b,\infty)$ $z \equiv z(y,\ ):\zeta(S)\longrightarrow \mathbb{R}^+$ is C^2.

ii) $\lim_{y\to\infty} z(y,x) = \infty$, for each $x\in\zeta(S)$.

If we choose $\hat{\sigma}_{ab}$ so that $\{\hat{\sigma}_{ij}(z_y)\}\in{}_{\zeta}M$, we have a curve $y\in(b,\infty) \longrightarrow \{\sigma_{yij}\equiv\hat{\sigma}_{ij}(z_y)\}\in{}_{\zeta}M$, and the subset $I(\hat{\sigma},z)$ of ${}_{\zeta}M \times {}_{\zeta}F$

$$I(\hat{\sigma},z) \equiv \{(\{\sigma_{yij}\},\phi)\,|\,y\in(b,\ \infty),\phi\in{}_{\zeta}F\}\ .$$

This subset has not isolated points with relation to the distance d. On $I(\hat{\sigma},z)$ the EHJ equation reads

$$L[\sigma_y] + \det(\sigma_y)W(\sigma_y,\phi) - (1/2)\left(\frac{\delta V}{\delta\phi}\right)^2 = 0 \tag{2.1}$$

where

$$L[\sigma_y] \equiv -N_{abcd}[\sigma_y]\frac{\delta V}{\delta\sigma_{yab}}\frac{\delta V}{\delta\sigma_{ycd}},\ \text{and}\ \frac{\delta V}{\delta\sigma_{yab}} \equiv \left(\frac{\delta V}{\delta\sigma_{ab}}\right)\Bigg|_{\sigma_{ij}=\hat{\sigma}_{ij}(z_y)}$$

In regard to operate with (2.1), the term $L[\sigma_y]$ is the principal difficulty. However, in order to study the behaviour of solutions to (2.1) as $y\to\infty$, the term $L[\sigma_y]$ may be replaced by other expressions without modifying the type of behaviour.

Henceforth one suppose that the function $z(y,x)$ has been fixed. With $\mathbb{F}\left(D^2_{\zeta(S)},\prod_{i\le j}D^2_{\zeta(S)}\right)$ we denote the respective space of continuous functions endowed with the point open topology; i.e. $\{f_n\}$ converges to g in $\mathbb{F}$ if and only if $\{f_n\}$ converges pointwise to g. Let $\mathbb{M}$ a metric space and let $\hat{\sigma}:\ \lambda\in\mathbb{M}\longrightarrow\hat{\sigma}_{\lambda}\in\mathbb{F}$ be a continuous map (therefore if $\lambda\to\mu$, for each y, $\hat{\sigma}_{\lambda}(z_y)\to\hat{\sigma}_{\mu}(z_y)$ in the distance d'). We define $O\equiv\{\lambda\in\mathbb{M}|\hat{\sigma}_{\lambda y}\in{}_{\zeta}M,$ for all $y\}$; in consequence for every $\lambda\in O$ there is the respective $L[\sigma_{\lambda y}]$. We need to replace N_{abcd} by a suitable expression.

Let $j_{abcd}:{}_{\zeta}M\longrightarrow D^2_{\zeta(S)}$, $a,b,c,d\in\{1,2,3\}$ be a family of continuous functions, and let $\lambda_0 \in \bar{\mathcal{O}}$ satisfying the following conditions:

iii) For every $\varepsilon>0$ there exists a deleted neighbourhood of λ_0, such that $|N_{abcd}(\sigma_{\lambda y}(x))-j_{abcd}(\sigma_{\lambda y})(x)|<\varepsilon\cdot|j_{abcd}(\sigma_{\lambda y})(x)|$, for all x,y,a,b,c,d, and all λ in that neighbourhood.

iv) There exists $k>0$ such that if ϕ is a field with $|\phi(x)|<k$ for all x, then if V is solution to the EHJ equation

$$\lim_{\lambda\to\lambda_0} N_{abcd}(\sigma_{\lambda y})\left(\frac{\delta V}{\delta\sigma_{\lambda yab}}\right)\Big|_{\phi}\left(\frac{\delta V}{\delta\sigma_{\lambda ycd}}\right)\Big|_{\phi}\neq 0, \text{ for each } y.$$

Then, under the preceding hypotheses, for every λ in a deleted neighbourhood of λ_0 the asymptotic behaviour, as $y\to\infty$ on $\{(\sigma,\phi)\in I(\hat{\sigma}_\lambda,z)\,|\,|\phi(x)|<k \text{ for all } x\}$, of solutions to the EHJ equation will be of the same type as that of solutions to

$$M[\sigma_{\lambda y}]+\det(\hat{\sigma}_{\lambda y})W(\sigma_{\lambda y},\phi)-(1/2)\left(\frac{\delta V}{\delta\phi}\right)^2=0, \tag{2.2}$$

where $M[\sigma_{\lambda y}] \equiv -j_{abcd}[\sigma_{\lambda y}]\dfrac{\delta V}{\delta\sigma_{\lambda yab}}\dfrac{\delta V}{\delta\sigma_{\lambda ycd}}$.

(The remark about zero-signature limit, in Sect. 4, shows that the condition iv) is necessary to can substitute $L[\sigma_{\lambda y}]$ for $M[\sigma_{\lambda y}]$ without modifying the type of behaviour).

Example 1. Let $\lambda_{ij},\tau_{ij}:\zeta(S)\longrightarrow\mathbb{R}$ $(i,j=1,2,3)$ be two families of C^2 functions such that $\tau_{ij}=\tau_{ji}$, $\lambda_{ij}=\lambda_{ji}$, $\operatorname{rank}(\tau_{ij}(x))=1$ and $\tau_{ij}(x)\neq0$ for all $x\in\zeta(S)$. We define $\sigma_{yij}(x)=\tau_{ij}(x)+z_y(x)\big(\tau_{ij}(x)+\lambda_{ij}(x)\big)$. Let us suppose that $\{\sigma_{yij}\}\in{}_\zeta M$; then a curve on ${}_\zeta M$ is defined. In Appendix I we shall construct a set $\{j_{abcd}\}$, such that the condition iii) holds for $\{j_{abcd}\}$ on $\sigma_{\lambda y}$ (Proposition A.3). The property iv) holds also for this set of curves. (As an example of families which satisfy the preceding conditions we can take $\tau_{ij}(x)\equiv1$, $\lambda_{ij}(x)\equiv\delta_{ij}$).

Example 2. Let S the 3-sphere. On S the usual coordinates χ, ϑ, φ are employed. By setting $z_y(x) = y^2$ and

$$\begin{pmatrix} y^2 & 0 & 0 \\ 0 & y^2\sin^2\chi & 0 \\ 0 & 0 & y^2\sin^2\chi\,\sin^2\vartheta \end{pmatrix} \equiv \sigma_y,$$

we have a curve in ${}_\zeta M$.

3 ASYMPTOTIC BEHAVIOURS

The following step is to seek the asymptotic relation that correspond to equation (2.2).

We employ the usual symbols ~ and « for the well-known asymptotic relations: "is asymptotic to" and "is much smaller than" [10].

Let us suppose that for the family $\{\sigma_{\lambda y}\}_\lambda$ of curves in ${}_\zeta M$, and the set $\{j_{abcd}\}$ of functions the following properties hold

v) $\quad {}^3R_{\lambda y}(x) \sim \dfrac{\text{function}(x)}{z_y(x)}$, as $y\to\infty$.

vi) $\quad \det(\sigma_{\lambda y}(x)) \sim z_y^3(x)\cdot h_\lambda(x)$, as $y\to\infty$.

vii) Given an arbitrary $\lambda\in O$ $M_{\lambda y}(x) \sim -(1/2)z_y^2(x)\left(\dfrac{\delta V}{\delta z_y(x)}\right)^2_\lambda$, as $y\to\infty$, for all V regular.

For the families of curves defined in the Example 1 the conditions v)-vii) hold (see Appendix I).

From (2.2) the following relation is obtained, as $y\to\infty$

$$(1/2)z_y^2(x)\left(\frac{\delta V}{\delta z_y(x)}\right)^2 + h(x)z_y^3(x)\left(2\Lambda+(1/2)m^2\phi^2(x)\right) \sim$$

$$\sim -(1/2)\left(\frac{\delta V}{\delta\phi(x)}\right)^2 \qquad (3.1)$$

According to the remarks of the Sect. 2 we have

Proposition 2. For all λ belonging to a certain neighbourhood B in $\mathbb{M}$, the asymptotic behaviour, on the curve $\{\sigma_{\lambda y}\}_{y\in(b,\infty)}$, of solutions to the EHJ equation is of the same type as that of solutions to (3.1). □

Remark. The meaning of the expression "the same type of behaviour" is explained in Remark to Theorem 2.

Hence for each pair consisted of a function z and a family $\{\hat{\sigma}_{\lambda ab}\}$ which satisfy i)–vii) we have a set $E(\hat{\sigma}_\lambda, z)\equiv\{\sigma_{\lambda y}|\lambda\in B\}$ of curves in ${}_\zeta M$. To simplify, henceforth we assume that $z_y(x)=p(y)\cdot\Omega(x)$, being $\Omega:\zeta(S) \longrightarrow \mathbb{R}^+$ a C^2 function, and $p:(b,\infty)\longrightarrow\mathbb{R}^+$ a continuous function with $\lim_{y\to\infty} p(y)=\infty$.

The union of the sets $E(\hat{\sigma}_\lambda, z)$ gives a class E of curves in ${}_\zeta M$. And given $\{\sigma_{yij}\}_{y\in(b,\infty)}\in E$ we can analyse the behaviour of solutions to (3.1), according to Proposition 2.

We denote $A(z_y,\phi,x) \equiv (1/2)z_y^2(x)\left(\dfrac{\delta V}{\delta z_y(x)}\right)^2$

$$B(z_y,\phi,x) \equiv h(x)z_y^3(x)\left(2\Lambda+(1/2)m^2\phi^2(x)\right)$$

$$C(z_y,\phi,x) \equiv -(1/2)\left(\frac{\delta V}{\delta\phi(x)}\right)^2$$

Taking into account that the terms of (3.1) are regular distributions; given a field ϕ, there are the following dominant balances to consider, as $y\to\infty$:

(a) $A(z_y,\phi,x)\sim -B(z_y,\phi,x)$, $|C(z_y,\phi,x)|\ll|A(z_y,\phi,x)|$, almost everywhere on $\zeta(S)$.

(b) $B(z_y,\phi,x) \sim C(z_y,\phi,x)$, $|A(z_y,\phi,x)|\ll|B(z_y,\phi,x)|$, almost everywhere on $\zeta(S)$.

(c) $A\sim C$, $|B|\ll|A|$.

(d) $A\sim B\sim C$.

(e) There are subsets of $\zeta(S)$: S_a, S_b, S_c, S_d (and at least two of them with nonzero measure), such that in S_r the dominant balance (r), $r\in\{a,b,c,d\}$, holds.

3a. In this case we have

$$(1/2)z_y^2(x)\left(\frac{\delta V}{\delta z_y(x)}\right)^2\sim -h(x)z_y^3(x)\left(2\Lambda+(1/2)m^2\phi^2(x)\right),$$

almost everywhere on $\zeta(S)$; then

$$V[z_y,\phi] \sim \pm(2/3)\int_{K_1}\left(-(4\Lambda+m^2\phi^2(x))h(x)z_y^3(x)\right)^{1/2}dx \pm$$

$$\pm(2/3)i\int_{K_2}\left((4\Lambda+m^2\phi^2(x))h(x)z_y^3(x)\right)^{1/2}dx + Y[\phi].$$

$Y[\phi]$ is a functional of ϕ. $K_1\equiv\{x\in\zeta(S)\,|\,4\Lambda+m^2\phi^2(x)\le 0\}$, and $K_2\equiv\zeta(S)-K_1$.

If, for example, $x\in K_2$

$\frac{\delta V}{\delta\phi(x)}\sim\pm(2/3)im^2\phi^2(x)\left(h(x)z_y^3(x)\right)^{1/2}\left(4\Lambda+m^2\phi(x)\right)^{-1/2}+\frac{\delta Y}{\delta\phi(x)}$, then the relation $|C(z_y,\phi,x)|\ll|B(z_y,\phi,x)|$ does not hold as $y\to\infty$. In consequence the dominant balance (a) is inconsistent.

3b. Now we have $\left(4\Lambda+m^2\phi^2(x)\right)h(x)z_y^3(x) \sim -\left(\frac{\delta V}{\delta\phi(x)}\right)^2$. If $m^2\phi^2(x)$ is much smaller than $|\Lambda|$ almost everywhere on $\zeta(S)$,

$$V[z_y,\phi] \sim \pm i\int_{\zeta(S)}\left(4\Lambda h(x)z_y^3(x)\right)^{1/2}\phi(x)dx + N[z_y],\ \text{for } \Lambda>0$$

$$V[z_y,\phi] \sim \pm\int_{\zeta(S)}\left(-4\Lambda h(x)z_y^3(x)\right)^{1/2}\phi(x)dx + N[z_y],\ \text{for } \Lambda<0.$$

But $|A(z_y,\phi,x)|\ll|B(z_y,\phi,x)|$ does not hold as $y\to\infty$.

If $m^2\phi^2(x)$ is much bigger than $|\Lambda|$ almost everywhere on $\zeta(S)$, then $-m^2\phi^2(x)h(x)z_y^3(x) \sim \left(\frac{\delta V}{\delta\phi(x)}\right)^2$; and hence

$$V[z_y,\phi] \sim \pm(1/2)i\int_{\zeta(S)}(m^2h(x)z_y^3(x))^{1/2}\phi^2(x)dx + N[z_y].$$

But $|A(z_y,\phi,x)|\ll|B(z_y,\phi,x)|$ does not hold as $y\to\infty$. In consequence, for a field ϕ such that ϕ^2 is either big enough or small enough with relation to $|\Lambda|m^{-2}$, the dominant balance (b) is inconsistent.

3c. Now

$(1/2)z_y^2(x)\left(\frac{\delta V}{\delta z_y(x)}\right)^2 \sim -(1/2)\left(\frac{\delta V}{\delta\phi(x)}\right)^2$. Then $z_y(x)\frac{\delta V}{\delta z_y(x)} \sim \pm i\,\frac{\delta V}{\delta\phi(x)}$.

The general solution is

$$V[z_y,\phi]\sim F\left(\int_{\zeta(S)}z_y(x)\cdot\exp(\pm i\phi(x))dx\right),$$

being F a differentiable function such that, as $y\to\infty$

$$\left|F'\left(\int_{\zeta(S)}z_y(x)\cdot\exp(\pm i\phi(x))dx\right)\right|^2 \gg \left|h(x)z_y(x)(2\Lambda+(1/2)m^2\phi^2(x))\right| \qquad (3.2)$$

almost everywhere on $\zeta(S)$ (in this way $|A|\gg|B|$).

Since $z_y(x)\equiv p(y)\cdot\Omega(x)$, the conditon (3.2) is equivalent to $|F'(a\cdot u)|\gg u^{1/2}$, as $u\to\infty$, with $a\equiv\int_{\zeta(S)}\Omega(x)\cdot\exp(\pm i\phi(x))dx$.

3d. In this case

$$z_y^2\left(\frac{\delta V}{\delta z_y}\right)^2 + \left(\frac{\delta V}{\delta\phi}\right)^2 \sim -(4\Lambda + m^2\phi^2)hz_y^3 \qquad (3.3)$$

Since A ~ B and C ~ B,

$$\frac{\delta V}{\delta z_y(x)} \sim \text{function}(x)\cdot(z_y(x))^{1/2}, \text{ and}$$

$$\frac{\delta V}{\delta\phi(x)} \sim \text{function}(x)\cdot(z_y(x))^{3/2}.$$

Therefore the solutions of (3.3) have the form

$V[z_y,\phi] \sim \int_{\zeta(S)}(z_y(x))^{3/2}H(\phi(x))dx$, with H a differentiable function of a real variable such that

$$(9/4)(H(\phi(x)))^2 + \left(\frac{dH}{d\phi(x)}\right)^2 = -(4\Lambda + m^2\phi^2(x))h(x) \qquad (3.4)$$

The asymptotic relation (3.3) implies the equation (3.4) between the coefficients of z_y^3. We analyse (3.4) in two cases:

(I) For fields ϕ such that $\phi^2(x)$ is, almost everywhere, much bigger than $|\Lambda|m^{-2}$ and 1. There are four dominant balances to consider in (3.4):

$$\phi^2 \ll |H|^2; \quad |H|^2 \ll \left|\frac{dH}{d\phi}\right|^2; \quad H^2 \sim \left(\frac{dH}{d\phi}\right)^2 \sim m^2\phi^2 h; \quad \left|\frac{dH}{d\phi}\right|^2 \ll |H(\phi)|^2.$$

A straightforward calculation shows that only the last possibility is consistent. Hence $H(\phi(x)) \sim \pm(2/3)i(m^2h(x))^{1/2}\phi(x)$, and

$$V[z_y,\phi] \sim \pm(2/3)i\int_{\zeta(S)}(m^2h(x)z_y^3(x))^{1/2}\phi(x)dx$$

$$\sim \pm(2/3)i\int_{\zeta(S)}(m^2\det(\sigma_{yi}\{x))))^{1/2}\phi(x)dx.$$

This asymptotic behaviour is possible under the hypotheses of (d), A~B~C, and if $|\phi|$ is big enough.

(II) In order to solve (3.4) for fields ϕ with $|\phi|$ small, a perturbation method can be used (see Appendix II). By applying Theorem A.1 we obtain

$$V[z_y,\phi] \sim \int_{\zeta(S)}(z_y(x))^{3/2}\cdot\left(\sum_0^\infty m^{2n}(h(x))^{1/2}y_n(\phi(x))\right)dx \quad . \qquad (3.5)$$

Conditions on ϕ which guarantee the convergence of the series are given in Theorem A.1

3e. For a field ϕ such that $\phi^2(x)$ is small enough with relation to $|\Lambda|m^{-2}$ for all $x\in\zeta(S)$, S_a and S_b are sets of measure zero, in accordance with the results of 3a and 3b.

$$S_c \equiv \{x\in\zeta(S)\,|\,A(z_y,\phi,x)\sim C(z_y,\phi,x);\ |B(z_y,\phi,x)|\ll|A(z_y,\phi,x)|\}$$

$$S_d \equiv \{x\in\zeta(S)\,|\,A(z_y,\phi,x)\sim B(z_y,\phi,x)\sim C(z_y,\phi,x)\}\ .$$

If moreover $|\phi(x)|<T$ for all $x\in\zeta(S)$, and $K_T<4(27m^2)^{-1}$ (Theorem A.1), we have according to 3c and 3d

$$V[z_y,\phi] \sim F_\phi\left(\int_{S_c} z_y(x)\cdot\exp(\pm i\phi(x))dx\right) +$$

$$+\int_{S_d}(z_y(x))^{3/2}\cdot\left(\sum_0^\infty m^{2n}(h(x))^{1/2}y_n(\phi(x))\right)dx \quad .$$

We can summarize the results in

Theorem 1. Let V be a regular solution of (3.1), with $\Lambda\neq0$; and σ_y a curve belonging to *E*. Then there is a number $\delta>0$ such that, given a field ϕ with $|\phi(x)|<\delta$ for all x, as $y\to\infty$

$$V[\sigma_{yab},\phi] \sim F_\phi\left(\int_{S_\phi}(\det(\sigma_{yij}(x)))^{1/3}\exp(\pm i\phi(x))dx\right) + \\ + \int_{\zeta(S)-S_\phi}\left(\det(\sigma_{yij}(x))\right)^{1/2}Q(\phi(x))dx \tag{3.6}$$

Being S_ϕ a subset of $\zeta(S)$; F_ϕ a differentiable function which satisfies $|F_\phi'(u)|\gg u^{1/2}$, as $u\to\infty$; and Q solution of the differential equation

$$(9/4)Q^2 + \left(\frac{dQ}{dv}\right)^2 = -(4\Lambda + m^2v^2). \quad \square$$

Remark 1. Bearing in mind the Theorem A.1

$Q(\phi(x)) = \sum_0^\infty m^{2n}y_n(\phi(x))$, and $y_0(v)=c$, or $y_0(v)=c\cdot\sin\left((3/2)v+d\right)$ (d constant). Where $c \equiv \pm(4/3)(-\Lambda)^{1/2}$, if $\Lambda<0$, and
$c \equiv \pm(4/3)i\Lambda^{1/2}$, if $\Lambda>0$.

Remark 2. If S can not be covered with only one chart, there is however a finite atlas $\{(A_\alpha,\zeta_\alpha)\}$ for S, and coverings $\{U_\alpha\}$ of S, with $\bar{U}_\alpha\subset A_\alpha$ and $\bar{U}_\alpha$ compact. So on $\zeta_\alpha(\bar{U}_\alpha)$ all the preceding results are valid. We can obtain a finite covering $\{W_\alpha\}$ of S with $W_\alpha\subset U_\alpha$, W_α compact, and such that $W_\alpha\cap W_\beta$ are sets of measure zero (if $\alpha\neq\beta$). Denoting with $\sigma_{\alpha ij}$ the coordinates of σ in $\left(W_\alpha,\zeta_{\alpha|W\alpha}\right)$, V is a functional of $\sigma_{\alpha ij}$, and V can be written $V=\sum_\beta V_\beta[\sigma_{\beta ij}]$, where each V_β is a functional with the properties stated in Theorem 1.

Remark 3. If $\int_{\zeta(S)}(h(x)\Omega^3(x))^{1/3}\exp(\pm i\phi(x))dx = 0$ $(\neq 0)$, the first (resp. second) term on the right of (3.6) can be deleted.

Taking into account the definition of the class *E* we have

Theorem 2. Let V be a regular solution of the EHJ equation, with $\Lambda\neq 0$, and σ_y a curve belonging to *E*, with $\det(\sigma_{yij}(x))\sim(p(y))^3\cdot b(x)$, as $y\to\infty$. Then there is a number $\delta>0$ such that, given a field ϕ with $|\phi(x)|<\delta$ for all x, either $V[\sigma_{yab},\phi]\sim\alpha_\phi\cdot(p(y))^{3/2}$, or $V[\sigma_{yab},\phi]\sim F_\phi(p(y))$, as $y\to\infty$. Where α_ϕ is a constant, and F_ϕ is a differentiable function, with $|F_\phi(u)|\gg u^{3/2}$ as $u\to\infty$. $\square$

Remark. Owing to (3.5), there is a real number ρ_ϕ such that

$$\alpha_\phi=\rho_\phi\cdot\int_{\zeta(S)}(b(x))^{1/2}\left(\sum m^{2n}y_n(\phi(x))\right)dx. \tag{3.7}$$

(That is really the explanation of that we mean by "the same type of behaviour" in Proposition 2).

4 PURE QUANTUM GRAVITY

In this case the relation (3.1), if $\Lambda\neq0$, is reduced to

$$\left(\frac{\delta V}{\delta z_y(x)}\right)^2\sim -4\Lambda h(x)z_y(x).$$

And then, if V is solution to the EHJ equation

$$V[\sigma_{yab}]\sim\rho c\int_{\zeta(S)}\left(\det(\sigma_{yij}(x))\right)^{1/2}dx \tag{4.1}$$

The right term of (4.1) is purely imaginary (real) for $\Lambda>0$ (resp. $\Lambda<0$). If ψ is a vawe function of the universe, its semiclassical approximation $D[\sigma_{ab}]\cdot\exp(iV[\sigma_{ab}])$ has on $\{\sigma_{yij}\}$ a dissipative (dispersive) [10] behaviour as $y\to\infty$, for $\Lambda>0$ (resp. $\Lambda<0$).

In the limit of zero signature [18] [19] the EHJ equation reads

$$N_{abcd}(\sigma)\,\frac{\delta V}{\delta\sigma_{ab}}\,\frac{\delta V}{\delta\sigma_{cd}}+2\Lambda\det(\sigma)=0 \tag{4.2}$$

We seek a solution in the form [20] $V_0=\beta\int\left(\det\sigma_{ij}(x)\right)^{1/2}dx.$

Taking into account that $\displaystyle\sum_a\sigma_{ca}(x)\left(\frac{\delta V_0}{\delta\sigma_{ab}(x)}\right)=(\beta/2)\left(\det(\sigma_{ij}(x))\right)^{1/2}\delta_{cb}$

$$N_{abcd}(\sigma)\,\frac{\delta V_0}{\delta\sigma_{ab}}\,\frac{\delta V_0}{\delta\sigma_{cd}}=-(3/8)\beta^2\det\sigma,$$

we obtain the following condition for β: $-(3/8)\beta^2+2\Lambda=0$.

In consequence, a difference between the case of $\Lambda>0$ and $\Lambda<0$ also appears in this particular solution; although here if $\Lambda>0$ V_0 is real, unlike the behaviour in (4.1). In spite of this similarity, our method can not be applied to solutions of (4.2). If $\{\sigma_\lambda\}$ is a family of metrics with $\lim_{\lambda\to\lambda_0}\det(\sigma_\lambda(x))=0$ for all x, then from (4.2) we deduce $L[\sigma_\lambda]\equiv N_{abcd}(\sigma_\lambda)\frac{\delta V}{\delta\sigma_{\lambda ab}}\frac{\delta V}{\delta\sigma_{\lambda cd}}\to0$, as $\lambda\to\lambda_0$. Thus the corresponding condition iv) (Sect. 2) is not satisfied for solutions to (4.2). Since $L[\sigma_\lambda]\to0$ as $\lambda\to\lambda_0$, $L[\sigma_\lambda]$ and the respective $M[\sigma_\lambda]$ (Sect. 2) can have opposite signs, for λ close to λ_0; and hence for the equation (4.2) the corresponding Proposition 2 is not valid.

With relation to pure gravity new degrees of freedom are introduced in the theory of gravity coupled to a scalar field. Although the latter is independent of pure gravity, one may hope that the pure gravity's results, about the asymptotic behaviour of wave functions, are recovered by making $\phi=\vartheta$ ($\vartheta(x)=0$ for all $x\in\zeta(S)$) in the formulas of gravity coupled to a scalar field. If we admit this "pure gravity recovery postulat" (PGRP), aditional consequences can be obtained from Theorem 2.

Proposition 3. If the PGRP is admited and $\Lambda\neq 0$, then there is a neighbourhood U of ϑ (in the distance d_1) such that, if ϕ belongs to U $V[\sigma_y,\phi]\sim\alpha_\phi\cdot(p(y))^{3/2}$; being α_ϕ purely imaginary (real) if $\Lambda>0$ (resp. $\Lambda<0$).

Proof. Let us suppose that in every neighbourhood of ϑ there is a field ϕ such that $V[\sigma_y,\phi]\sim F_\phi(p(y))$, with $|F_\phi(u)|\gg u^{3/2}$. Then we can construct a sequence of fields $\phi_1,..,\phi_i,..$ with $\lim_{i\to\infty}\phi_i=\vartheta$ (in the distance d_1).

$$\lim_{y\to\infty}\frac{|V[\sigma_y,\phi_i]|}{(p(y))^{3/2}}=\infty. \tag{4.3}$$

Owing to the PGRP
$$\lim_{y\to\infty}\frac{|V[\sigma_y,\vartheta]|}{(p(y))^{3/2}}\neq\infty. \tag{4.4}$$

Then, since V is continuous, (4.3) and (4.4) are contradictory. Therefore there exists a neighbourhood U of ϑ such that, for $\phi\in U$ $V[\sigma_y,\phi]\sim\alpha_\phi\cdot(p(y))^{3/2}$; and α_ϕ is given by (3.7). In accordance with Appendix II $y_0(v)=c$ or $y_0(v)=c\cdot\sin((3/2)v+d)$. If $\Lambda>0$, in any case, $y_1(v)$ and $y_1'(v)$ are purely imaginary, according to (II.2) and (II.3); and recalling (II.4) we conclude that $y_n(v)$ is purely imaginary for all n, and for all v. Hence α_ϕ is purely imaginary. Similarly for $\Lambda<0$. □

We have proved

Theorem 3. Given V a regular solution to the EHJ equation. If the PGRP is admited, $\Lambda\neq 0$ and σ_y is a curve belonging to E, with $\det(\sigma_{yij}(x))\sim b(x)(p(y))^3$, as $y\to\infty$. Then there is a neighbourhood U of ϑ such that, for $\phi\in U$

$$V[\sigma_{yij},\phi] \sim \alpha_{\phi}\cdot(p(y))^{3/2} \qquad \text{as } y\to\infty.$$

Being this expression purely imaginary (real) if $\Lambda>0$ (resp. $\Lambda<0$).□

5 APPLICATION TO WAVE FUNCTIONS OF THE UNIVERSE

Since $y\in(b,\infty)\longrightarrow\{\sigma_{yij}\}\in_{\zeta}M$ is continuous, if we employ another coordinates $y\in(b,\infty)\longrightarrow\{\bar{\sigma}_{yij}\}\in_{\bar{\zeta}}M$ is also continuous, and $\lim_{y\to\infty}\det(\bar{\sigma}_{yij}(x))=\infty$. Therefore it is reasonable to refer to the <u>curve</u> $y\in(b,\infty)\longrightarrow\sigma_y\in M$, <u>in</u> <u>$M$</u>

Given ψ wave function of the universe; ψ satisfies the Supermomentum equations (1.2), and hence ψ does not depend on the coordinates in terms of which $\sigma\in M$ and $q\in F$ are expressed. The action V also satisfies (1.2).

In accordance with the semiclassical approximation $\psi[\sigma,q]\cong D[\sigma,q]\cdot\exp(iV[\sigma,q])$, being D a slowly varying functional. Therefore, under the hypotheses of Theorem 3, we can hope that for $\sigma_y\in E$ there is a neighbourhood of field zero, such that if q belongs to it, on (σ_y,q) ψ exhibits a dissipative (dispersive) behaviour, as $y\to\infty$, if $\Lambda>0$ (resp. $\Lambda<0$).

Will the preceding behaviours be valid on a general curve β_y in M, with $\lim_{y\to\infty}\det(\beta_y)=\infty$?. The answer is negative (see Sect. 6). However the following rough calculation seems to suggests the vality of the preceding results for the curves of a family E' more extensive than E.

If $\beta_{yij}(x)\simeq\lambda_y(x)$ as $y\to\infty$, $\det(\beta_{yij})\simeq\lambda_y^3$, and $R_y\simeq\lambda_y^{-1}$. From the EHJ equation for pure gravity we obtain $\left(\frac{\delta V}{\delta\lambda_y(x)}\right)^2 \simeq -\Lambda\,\lambda_y(x)$.

$$\text{So if } \Lambda<0, \quad V \simeq \pm\int\left(-\Lambda\cdot\det(\beta_{yij}(x))\right)^{1/2}dx;$$

$$\text{if } \Lambda>0, \quad V \simeq \pm i\int\left(\Lambda\cdot\det(\beta_{yij}(x))\right)^{1/2}dx$$

And then ψ presents an oscillatory behaviour if $\Lambda<0$.

One can adopt the known interpretation that $|\psi(\sigma,q)|^2$ is proportional to the probability of finding the metric σ and the matter field q on S [21]. This probability interpretation runs into well-known difficulties, owing to the impossibility of defining a reasonable measure $d\sigma\cdot dq$ on an infinitely dimensional manifold [14],[11].

If $\Lambda<0$ a new and additional difficulty appears in the "integration" $\int|\psi|^2 d\sigma\cdot dq$, for every ψ regular, because on the asymptotic region defined by E' $\psi\simeq$(slowly varying functional)$\times$ $\times$(oscillatory functional). In consequence, the preceding considerations suggest that the possibility $\Lambda<0$ in pure quantum gravity be rejected. A similar remark (based on the study of minisuperspace models with $\Lambda<0$) is pointed in [3] and [7].

6 APPLICATION TO MINISUPERSPACES

We study first the behaviour of V along a curve, whose "points" are metrics of 3-spheres. The curve σ_y defined in Example 2 (Sect. 2) does not belong to E, because the condition vii) (Sect. 3) is not satisfied for this curve. Therefore the preceding results can not be applied to this case.

If (α_{ab}) is diagonal matrix, the $N_{abcd}(\alpha)$ non zero are

$$N_{aaaa}(\alpha)=(1/2)\alpha_{aa}\cdot\alpha_{aa}\text{; } N_{abab}=N_{abba}=-N_{aabb}=(1/2)\alpha_{aa}\cdot\alpha_{bb}\quad (a\neq b).$$

Hence, if $\sigma_{yab}(x)= z_y(x)\cdot\alpha_{ab}(x)$

$$N_{abcd}(\sigma_y)\left(\frac{\delta V}{\delta\sigma_{yab}}\right)\left(\frac{\delta V}{\delta\sigma_{ycd}}\right) = (1/2)z_y^2\cdot\sum_a\left(\alpha_{aa}\frac{\delta V}{\delta\sigma_{yaa}}\right)^2 +$$

$$+ (1/2)z_y^2\cdot\sum_{a\neq b}\alpha_{aa}\alpha_{bb}\left(2\frac{\delta V}{\delta\sigma_{yab}}\cdot\frac{\delta V}{\delta\sigma_{yab}} - \frac{\delta V}{\delta\sigma_{yaa}}\cdot\frac{\delta V}{\delta\sigma_{ybb}}\right).$$

Setting $\frac{\delta V}{\delta\sigma_{yab}(x)} \equiv r_{yab}(x)\cdot\frac{\delta V}{\delta\sigma_{yee}(x)}$, where e is an arbitrary element of $\{1,2,3\}$.

$$N_{abcd}(\sigma_y)\left(\frac{\delta V}{\delta\sigma_{yab}}\right)\left(\frac{\delta V}{\delta\sigma_{ycd}}\right) = (1/2)z_y^2\left(\frac{\delta V}{\delta\sigma_{yee}}\right)^2\cdot r_y,\text{ with}$$

$$r_y(x) \equiv \sum_a(\alpha_{aa}(x)r_{yaa}(x))^2 + \sum_{a\neq b}\alpha_{aa}(x)\alpha_{bb}(x)\left(2(r_{yab}(x))^2-r_{yaa}(x)r_{ybb}(x)\right).$$

On the other hand $\frac{\delta V}{\delta z_y} = \sum\left(\frac{\delta V}{\delta\sigma_{yab}}\right)\alpha_{ab} = \frac{\delta V}{\delta\sigma_{yee}}\cdot\sum_a\alpha_{aa}\cdot r_{yaa}$

$$N_{abcd}(\sigma_y)\left(\frac{\delta V}{\delta\sigma_{yab}}\right)\left(\frac{\delta V}{\delta\sigma_{ycd}}\right) = (1/2)z_y^2\cdot s_y\left(\frac{\delta V}{\delta z_y}\right)^2,\text{ where}$$

$$s_y(x) \equiv \frac{r_y(x)}{\left(\sum\alpha_{aa}(x)\cdot r_{yaa}(x)\right)^2}.$$

Hence, on $\{\sigma_{yab}\}$, the EHJ equation for V is

$$-(1/2)z_y^2(x)s_y(x)\left(\frac{\delta V}{\delta z_y(x)}\right)^2 + z_y^3(x)\det(\alpha_{ab}(x))\Big(R_y(x)-2\Lambda-$$

$$-(1/2)\sigma_y^{ab}(x)\phi_{,a}(x)\phi_{,b}(x)-(1/2)m^2\phi^2(x)\Big) - (1/2)\left(\frac{\delta V}{\delta\phi(x)}\right)^2 = 0.$$

If $(\alpha_{ab})=\begin{pmatrix}1 & 0 & 0\\ 0 & \sin^2\chi & 0\\ 0 & 0 & \sin^2\chi\,\sin^2\vartheta\end{pmatrix}$, and $z_y(x)=y^2$ (see Example 2),

$R_y(x) = z_y^{-1}(x)$. And as $y\to\infty$ we obtain the relation

$$(1/2)z_y^2(x)s_y(x)\left(\frac{\delta V}{\delta z_y(x)}\right)^2+h(x)z_y^3(x)(2\Lambda+m^2\phi^2(x)) \sim$$

$$\sim -(1/2)\left(\frac{\delta V}{\delta\phi(x)}\right)^2 \qquad (6.1)$$

Being $h(x)\equiv\det(\alpha_{ab}(x))$. The relation (6.1) is similar to (3.1).

For pure gravity

$$\left(\frac{\delta V}{\delta z_y(x)}\right)^2 \sim -4\Lambda\cdot\frac{h(x)}{s_y(x)}\cdot z_y(x).$$

Let us assume that, as $y\to\infty$, $s_y(x)\to s(x)>0$. (6.2)

Then $V[z_y] \sim c\displaystyle\int_{\zeta(S)}\left(\frac{\det\sigma_y(x)}{s_y(x)}\right)^{1/2}dx \sim c\cdot y^3\int_{\zeta(S)}\left(\frac{h(x)}{s(x)}\right)^{1/2}dx.$

That is, $V[y] \sim \pm iy^3$(constant), if $\Lambda>0$

$V[y] \sim \pm y^3$(constant), if $\Lambda<0$. (The constant is real)

If we admit the PGRP, $V[y,\phi] \sim \pm iy^3G[\phi]$, if $\Lambda>0$

$V[y,\phi] \sim \pm y^3G[\phi]$, if $\Lambda<0$;

for ϕ in a neighbourhood of the field zero. (And being G a real functional).

Hence, for $\Lambda<0$, $\psi[y,\phi] \cong D\cdot\exp(\pm iy^3G[\phi])$, as $y\to\infty$. (6.3)

If we suppose that ψ is real, in order to represent its asymptotic behaviour, we must form a real linear combination of the two behaviours (6.3).

$$\psi[y,\phi] \cong a_1D\exp(iy^3G[\phi]) + a_2D\exp(-iy^3G[\phi]) =$$

$$= \overline{D}\cos(y^3G[\phi] + J[\phi]), \text{ as } y\to\infty.$$

This is the type of asymptotic behaviour obtained by using minisuperspace models in [1,2,4,6-9].

For $\Lambda>0$ we have $\psi[y,\phi] \cong D\exp(-(y^3)G[\phi])$. On the other hand the volume of the 3-geometry is proportional to y^3; therefore we reproduce the result of [3].

The supposition (6.2) is a boundary condition on the functional derivatives of V. This condition holds if, for example, V is a symmetric functional of its variables σ_{ab} (in this case $r_{yab}=1$). If V depended on only one "metrical" variable the condition (6.2) would be trivial also. Therefore in the minisuperspace model this condition for the wave functions has no significance.

We consider now a curve whose points are spatially homogeneous metrics. In the parametrization due to Misner [22], one can write a spatially homogeneous 3-metric as $\sigma=e^{-2\Omega}(e^{2\beta})_{ab}\cdot dx^a dx^b$, with $(\beta_{ab})=\mathrm{diag}(\beta_+ + 3^{1/2}\beta_-,\ \beta_+ - 3^{1/2}\beta_-,\ -2\beta_+)$. If σ_y is a such curve $\sigma_{yab}=e^{-2\Omega(y)}\alpha_{yab}$, where $\alpha_{yab}=(e^{2\beta(y)})_{ab}$. This curve does not belong to E either.

Using the notations of preceding example

$$N_{abcd}(\sigma_y)\left(\frac{\delta V}{\delta\sigma_{yab}}\right)\left(\frac{\delta V}{\delta\sigma_{ycd}}\right)=(1/2)\exp(-4\Omega_y)\left(\frac{\delta V}{\delta\sigma_{yee}}\right)^2\cdot r_y .$$

Since $\det(\sigma_y)=\exp(-6\Omega)$ and ${}^3R_y=0$, the EHJ equation for pure gravity reads

$$(1/2)\exp(-4\Omega_y)\left(\frac{\delta V}{\delta\sigma_{yee}(x)}\right)^2\cdot r_y(x) + 2\Lambda\exp(-6\Omega_y)=0 \qquad (6.4)$$

$$\frac{dV}{dy}=\sum_{a,b}\frac{d\sigma_{yab}}{dy}\cdot\int_{\zeta(S)}\frac{\delta V}{\delta\sigma_{yee}(x)}r_{yab}(x)\,dx.$$

Let us assume that $\frac{\delta V}{\delta\sigma_{yab}(x)}$ does not depend on x, for all y,a,b. Then $r_{yab}(x)$ is constant as function of x, and

$$\frac{dV}{dy}=\mathrm{Vol}(\zeta(S))\frac{\delta V}{\delta\sigma_{yee}(x_0)}\sum_{a,b}\frac{d\sigma_{yab}}{dy}\cdot r_{yab}(x_0).$$

The equation (6.4) can be written

$$\hat{s}_y\left(\frac{dV}{dy}\right)^2+4\Lambda\exp(-6\Omega_y)=0, \text{ where}$$

$$s_y=\frac{r_y\cdot\exp(4\Omega_y)}{(\mathrm{Vol}(\zeta(S)))^2\left(\sum_{a,b}\left(-2\,\frac{d\Omega}{dy}\,\alpha_{yab}+\frac{d\alpha_{yab}}{dy}\right)r_{yab}\right)^2}=\hat{s}_y\cdot\exp(4\Omega_y).$$

If as $y\to\infty$, $\hat{s}_y\to s$ (being s a real number);

$$\text{for } s>0 \qquad V \sim \pm\int^{y}\left(-\frac{4\Lambda}{s}\det\sigma_t\right)^{1/2}dt\ , \quad \text{if } \Lambda<0$$

$$V \sim \pm i\int^{y}\left(\frac{4\Lambda}{s}\det\sigma_t\right)^{1/2}dt\ , \quad \text{if } \Lambda>0. \tag{6.5}$$

(Analogously for s<0).

Those behaviours are valid if V satisfies

a) $\frac{\delta V}{\delta\sigma_{yab}(x)}$ is independent of x, for all a,b,y. (6.6)

b) As $y\to\infty$,
$$\frac{\sum_a(\alpha_{yaa}r_{yaa})^2+\sum_{a\neq b}\alpha_{yaa}\alpha_{ybb}\left(2(r_{yab})^2-r_{yaa}r_{ybb}\right)}{\left(\sum\left(-2\frac{d\Omega}{dy}\alpha_{yab}+\frac{d\alpha_{yab}}{dy}\right)r_{yab}\right)^2}$$
is positive. (6.7)

Since σ_y is spatially homogeneous, it is admissible to demand that the realistic solutions to the EHJ equation satisfy (6.6). On the other hand (6.7) is a mathematical boundary condition on the functional derivatives without physical meaning.

Thus, if we choose Ω such that $\lim_{y\to\infty}\Omega(y)=-\infty$, we have found a curve, which does not belong to the family E' (Sect.5).

In a Bianchi I minisuperspace model, the function V depend on three "metrical" variables Ω, β_+, β_-; therefore the condition (6.7) will have significance in this minisuperspace, unlike the preceding example.

By replacing in the EHJ equation the functional derivatives by partial derivatives, and removing $\frac{\partial V}{\partial\sigma_{ab}}$ if $a\neq b$, we obtain the corresponding equation in the Bianchi I minisuperspace model.

$\sigma_{11}=\exp\left(-2\Omega+2\beta_+ +2(3)^{1/2}\beta_-\right)$, $\sigma_{22}=\exp\left(-2\Omega+2\beta_+ -2(3)^{1/2}\beta_-\right)$,

$\sigma_{33}=\exp\left(-2\Omega-4\beta_+\right)$, $\sigma_{11}\cdot\frac{\partial V}{\partial\sigma_{11}}=12^{-1}\left(-2\frac{\partial V}{\partial\Omega}+\frac{\partial V}{\partial\beta_+}+(3)^{1/2}\cdot\frac{\partial V}{\partial\beta_-}\right)$, etc.

$$N_{abcd}(\sigma)\frac{\partial V}{\partial\sigma_{ab}}\frac{\partial V}{\partial\sigma_{cd}}=(1/2)\left(\sum\left(\sigma_{aa}\frac{\partial V}{\partial\sigma_{aa}}\right)^2-\sum_{a\neq b}\sigma_{aa}\sigma_{bb}\frac{\partial V}{\partial\sigma_{aa}}\cdot\frac{\partial V}{\partial\sigma_{bb}}\right),$$

(the terms with $\frac{\partial V}{\partial\sigma_{ab}}$, $a\neq b$, have been removed).

By means of a straightforward calculation we get

$$N_{abcd}(\sigma)\left(\frac{\partial V}{\partial\sigma_{ab}}\right)\left(\frac{\partial V}{\partial\sigma_{cd}}\right)=(1/24)\left(-\left(\frac{\partial V}{\partial\Omega}\right)^2+\left(\frac{\partial V}{\partial\beta_+}\right)^2+\left(\frac{\partial V}{\partial\beta_-}\right)^2\right).$$

Thus the EHJ equation in this minisuperspace model reads

$$-\left(\frac{\partial V}{\partial \Omega}\right)^2+\left(\frac{\partial V}{\partial \beta_+}\right)^2+\left(\frac{\partial V}{\partial \beta_-}\right)^2 = -48\Lambda e^{-6\Omega} \qquad (6.8)$$

(This is the equation that is obtained also by using variational principles applied to the action integral, p. 808, and 1186 [13]).

Given a curve in the minisuperspace $(\Omega(y),\beta_+(y),\beta_-(y))$, such that $\lim_{y\to\infty} \Omega_y = -\infty$.

$$\frac{dV}{dy} = \frac{\partial V}{\partial \Omega}\left(\frac{d\Omega}{dy} + \frac{d\beta_+}{dy}\, m_+(y) + \frac{d\beta_-}{dy}\, m_-(y)\right), \text{ where } \left(\frac{\partial V}{\partial \beta_\pm}\right)\Big|_y \equiv \left(\frac{\partial V}{\partial \Omega}\right)\Big|_y \cdot m_\pm(y).$$

From (6.8) we get $\left(\frac{dV}{dy}\right)^2 s_y = -48\Lambda \exp(-6\Omega_y)$, with

$$s_y \equiv \frac{-1 + m_+^2(y) + m_-^2(y)}{\left(\frac{d\Omega}{dy} + \frac{d\beta_+}{dy}\, m_+(y) + \frac{d\beta_-}{dy}\, m_-(y)\right)^2}.$$

Let us assume that as $y\to\infty$, $s_y\to k>0$. (This supposition is the preceding condition (6.7) expressed in the minisuperspace). Then

$$V \sim \pm\int^y \left(-48\Lambda k^{-1} \det\sigma_t\right)^{1/2} dt, \text{ for } \Lambda<0$$

$$V \sim \pm i\int^y \left(48\Lambda k^{-1} \det\sigma_t\right)^{1/2} dt, \text{ for } \Lambda>0.$$

Analogously for $k<0$.

In the case of $m_+(y)$ and $m_-(y)$ are real as $y\to\infty$, then for $\Lambda<0$

$V(y)$ is real as $y\to\infty$, if the boundary condition $m_+^2 + m_-^2>1$

$V(y)$ is purely imaginary as $y\to\infty$, if $m_+^2 + m_-^2<1$.

Similarly for $\Lambda>0$.

In short, the curves defined in these examples do not belong to *E'*; and we must impose boundary conditions on the solutions to deduce, on these curves, the behaviours that have been obtained along the curves of *E*, without imposing any boundary conditions. (On $\sigma_y \in E$ the term $N_{abcd}\cdot\frac{\delta V}{\delta\sigma_{ab}}\,\frac{\delta V}{\delta\sigma_{cd}}$ turns into $(1/2)z_y^2\left(\frac{\delta V}{\delta z_y}\right)^2$ and R_σ turns into $1/z_y$; then the behaviour of V on σ_y is only determined by the structure of the EHJ equation).

APPENDIX I

We shall study in this Appendix the Example 1 of Section 2. Now

$\sigma_{yij}(x)=\tau_{ij}(x)+z_y(x)(\tau_{ij}(x)+\lambda_{ij}(x))$, with $\mathrm{rank}(\tau_{ij}(x))=1$ and $\tau_{ij}(x)\neq 0$ for all $x\in\zeta(S)$.

Taking into account that $\{\sigma_{yij}\}\in_\zeta M$ and that $\mathrm{rank}(\tau_{ij})=1$, a straightforward calculation proves

Proposition A.1 $\det(\sigma_{yij}(x))=z_y^2(x)f(x)+z_y^3(x)h(x)$, where

$h(x)=\det\left(\tau_{ij}(x)+\lambda_{ij}(x)\right)>0$, for all $x\in\zeta(S)$. □

Lemma $N_{abcd}(\sigma)=N_{abdc}(\sigma)=N_{bacd}(\sigma)=N_{cdab}(\sigma)$ for all a,b,c,d and for all σ. □

By using the Lemma it is easy to prove

Proposition A.2 $N_{abcd}(\tau) = (1/2)\tau_{ab}\tau_{cd}$. □

We introduce in this example the notation employed in Sect. 2. Each λ_{ab} is an element of $D^2_{\zeta(S)}$ (see Sect.1), and the distance d' makes $\prod_{i\leq j} D^2_{\zeta(S)}\equiv\mathbb{M}$ a metric space. We suppose that $\{\tau_{ij}\}$ and z have been fixed; and by means of λ we denote the family $\{\lambda_{ij}\}\in\prod_{i\leq j}D^2_{\zeta(S)}$. So $\sigma_{\lambda yij}\equiv\tau_{ij}+z_y\cdot(\tau_{ij}+\lambda_{ij})$, and $O\equiv\{\lambda\,|\,\{\sigma_{\lambda yi}\}\in_\zeta M\}$. We have the family zero $\lambda_0\equiv\{\lambda_{ij}=0\}$, and λ_0 belongs to $\overline{O}$.

On the other hand

$$N_{abcd}(\sigma_y) = N_{abcd}\left((1+z_y)\tau + z_y\lambda\right) = (1+z_y)^2 N_{abcd}(\tau) + z_y^2\, N_{abcd}(\lambda) +$$
$$+(1/2)z_y(1+z_y)(\tau_{ac}\lambda_{bd}+\lambda_{ac}\tau_{db}+\tau_{ad}\lambda_{bc}+\lambda_{ad}\tau_{bc}-\tau_{ab}\lambda_{cd}-\lambda_{ab}\tau_{cd}) .$$

$$N_{abcd}(\sigma_y) =(1/2)(1+z_y)^2\cdot(t_{abcd}(\tau,\lambda,z_y)) + j_{abcd}(\tau,\lambda,z_y)$$

Being $j_{abcd}(\tau,\lambda,z_y) = (1/2)(1+z_y)^2\left(\tau_{ab}+\lambda_{ab}\right)\left(\tau_{cd}+\lambda_{cd}\right)$ and

$$t_{abcd}(\tau,\lambda,z_y) = 2\left(1+z_y\right)^{-2}(N_{abcd}(\sigma_y)-j_{abcd}) .$$

By using the Proposition A.2 we obtain

$$t_{abcd}(\tau,\lambda,z_y) = z_y^2\left(1+z_y\right)^{-2}\cdot\left(\lambda_{ac}\lambda_{bd} + \lambda_{ad}\lambda_{bc} - \lambda_{ab}\lambda_{cd}\right) +$$
$$+ z_y\left(1+z_y\right)^{-1}\cdot\left(\tau_{ac}\lambda_{bd} + \lambda_{ac}\tau_{bd} + \tau_{ad}\lambda_{bc} + \lambda_{ad}\tau_{bc} - \tau_{ab}\lambda_{cd} - \lambda_{ac}\tau_{bd}\right)-$$
$$-\left(\tau_{ab}\lambda_{cd} + \lambda_{ab}\tau_{cd} + \lambda_{ab}\lambda_{cd}\right). \tag{I.1}$$

As consequence of (I.1), given $\varepsilon>0$ there are $\{\lambda_{ij}\}$ with $|\lambda_{ij}(x)|$ small enough such that $|t_{abcd}(\tau,\lambda,z_y)|<\varepsilon|(\tau_{ab}+\lambda_{ab})(\tau_{cd}+\lambda_{cd})|$ for all x and y. That is, there exists a deleted neighbourhood B of λ_0, such that, if λ belongs to B

$|N_{abcd}(\sigma_{\lambda y}(x))-j_{abcd}(\sigma_{\lambda y})(x)|=(1/2)(1+z_y)^2|t_{abcd}(\tau,\lambda,z_y)|<\varepsilon\cdot|j_{abcd}|$.

So we have

Proposition A.3. The family $\{j_{abcd}\}$ satisfies on $\{\sigma_{\lambda y}\}$ the condition iii) of the Section 2. □

$$^3R_y = \sigma_y^{ac}\sigma_y^{bd}\left[(1/2)\left(-\frac{\partial^2\sigma_{yac}}{\partial x^b\partial x^d}+\cdots\right)+\sigma_{yef}(\Gamma^e_{yda}\cdot\Gamma^f_{ybc}-\cdots)\right].$$

Taking into account that $z_y(x) = p(y)\cdot\Omega(x)$.

$$\frac{\partial^2\sigma_{yac}}{\partial x^b\partial x^d} \sim p(y)\,\frac{\partial^2\big(\Omega\cdot(\tau_{ac}+\lambda_{ac})\big)}{\partial x^b\,\partial x^d}\,,\quad \text{as } y\to\infty.$$

Since $\mathrm{rank}(\tau_{ij}(x)) = 1$, then $\sigma_y^{ab} \sim \dfrac{\text{function}(x)}{z_y(x)}$, as $y\to\infty$.

$$\sigma_{yef}(\Gamma^e_{yda}\cdot\Gamma^f_{ybc}) = (1/4)\sigma_y^{rs}\left(\frac{\partial\sigma_{yrd}}{\partial x^a}+..\right)\left(\frac{\partial\sigma_{ysb}}{\partial x^c}+..\right) \sim (\text{funct.})\cdot z_y.$$

Hence $R_y(x) \sim \dfrac{\text{function}(x)}{z_y(x)}$. Thus the family $\{\sigma_{\lambda y}\}$ satisfies the condition v) of Sect. 3. The property vi) holds, as consequence of the Proposition A.1. On the other hand, given $\{\lambda_{ij}\}$

$$\left(\frac{\delta V}{\delta z_y}\right)_\lambda = \left(\frac{\delta V}{\delta\sigma_{\lambda yab}}\right)(\tau_{ab}+\lambda_{ab}).\ \text{Then}$$

$$M_{\lambda y}= -j_{abcd}\left(\frac{\delta V}{\delta\sigma_{yab}}\right)\left(\frac{\delta V}{\delta\sigma_{ycd}}\right)= -(1/2)(1+z_y)^2\left(\frac{\delta V}{\delta z_y}\right)^2_\lambda$$

Therefore the condition vii) holds for $\{\sigma_{\lambda y}\}$ and $\{j_{abcd}\}$.

Finally, if $\sup_{x,i,j}|\lambda_{ij}(x)|=O(\varepsilon)$, then $\det(\sigma_{\lambda y}(x))=O(\varepsilon^2)$ and $R_{\lambda y}(x)\cong\varepsilon^{-3}$ and for each y. For the case of pure gravity, if V is solution to the EHJ equation, we have $L[\sigma_{\lambda y}]\cong\varepsilon$; consequently this family of curves $\{\sigma_{\lambda y}\}$ satisfies the condition iv) in the case of pure gravity. If we admit the PGRP, iv) holds also for gravity coupled to a scalar field.

APPENDIX II

In order to solve the equation

$$(9/4)H^2+\left(\frac{dH}{d\phi}\right)^2 = -(4\Lambda+m^2\phi^2)h$$

for fields ϕ with $|\phi(x)|$ small, we use a perturbation method.

First we suppose $\Lambda\neq0$.

We define $y \equiv h^{-1/2}H$, and $\varepsilon\equiv m^2$. The equation (3.4) may be written

$$(9/4)\cdot y^2+\left(\frac{dy}{dv}\right)^2 = -(4\Lambda+\varepsilon v^2).$$

We express
$$y = \sum_0^\infty \varepsilon^n\cdot y_n\,, \tag{II.1}$$

and obtain $(y_0')^2+(9/4)(y_0)^2=-4\Lambda$, $\quad 2y_0'y_1'+(9/4)2y_0y_1=-v^2, \ldots,$

$$\sum_{j=0}^{n} y'_{n-j}y'_j+(9/4)\sum_{j=0}^{n} y_{n-j}y_j=0, \quad \text{for } n>1. \quad \text{Where } y_j' = \frac{dy_j}{dv}.$$

The equation $(y_0')^2+(9/4)(y_0)^2=-4\Lambda$ has the solutions $y_0(v)=c$, and $y_0(v) = c\cdot\sin\big((3/2)v+d\big)$, being d a constant;

$$c\equiv\pm(4/3)(-\Lambda)^{1/2} \text{ if } \Lambda<0, \text{ and } c\equiv\pm(4/3)i\Lambda^{1/2} \text{ for } \Lambda>0.$$

<u>We analyse now the case</u> $y_0(v)=c\cdot\sin\big((3/2)v+d\big)$.

We impose $y(0)=y_0(0)$, $y_n(0)=0$ for $n>0$.

The equation $y_1'=-(9/4)y_0(y_0')^{-1}y_1-(1/2)v^2(y_0')^{-1}$ has the "resolvent"

$$R(v,v_0) = \cos\big((3/2)v+d\big)\cdot\big(\cos\big((3/2)v_0+d\big)\big)^{-1}$$

Hence $$y_1(v)=-\cos\big((3/2)v + d\big)\cdot\int_0^v \tau^2\big(3c\cdot\cos^2\big((3/2)\tau + d\big)\big)^{-1}d\tau \tag{II.2}$$

$$y_1'(v)=(3c)^{-1}(3/2)\cdot\sin\big((3/2)v + d\big)\cdot\int_0^v \tau^2\big(\cos\big((3/2)\tau + d\big)\big)^{-2}d\tau \;-$$
$$-(3c)^{-1}\cdot v^2\big(\cos\big((3/2)v +d\big)\big)^{-1} \tag{II.3}$$

For $n>1$, we have $\quad y_n'=-(9/4)y_0(y_0')^{-1}y_n+(2y_0')^{-1}f_n$;

$$f_n\equiv-\big(y'_{n-1}y'_n+\ldots+y'_1y'_{n-1}+(9/4)(y_{n-1}y_1+\ldots+y_1y_{n-1})\big).$$

Hence $$y_n(v)=(3c)^{-1}\cos\big((3/2)v+d\big)\cdot\int_0^v f_n(\tau)\,\big(\cos\big((3/2)\tau+d\big)\big)^{-2}d\tau. \tag{II.4}$$

To study the convergence of (II.1) we need the following

Lemma Let $\{a_n\}$ be the following sequence of complex numbers: $a_1=b, \ldots$ $a_n=a_{n-1}a_1 + \ldots + a_1a_{n-1}$, for $n>1$. Then $|a_n|\le|b|^n\cdot\binom{3n-4}{n}$, for $n>1$; and $\lim\limits_{n\to\infty}\sup\dfrac{|a_n|}{|a_{n-1}|}\le(27/4)|b|$.

<u>Proof</u> $a_2=a_1a_1=bb$, $\quad a_3=a_2a_1+a_1a_2=(bb)b+b(bb)=2b^3$

$a_4=a_3a_1+a_2a_2+a_1a_3=\big((bb)b+b(bb)\big)b+(bb)(bb)+b\big((bb)b+b(bb)\big)=5b^4$.

It can be displayed in a diagram

a_4	3+1	(2+1)+1	((1+1)+1)+1
		(1+2)+1	(1+(1+1))+1
	2+2		(1+1)+(1+1)
	1+3	1+(2+1)	1+((1+1)+1)
		1+(1+2)	1+(1+(1+1))

The number of branches in this diagram is equal to the coefficient of b^4. In general, each term b^n which appears in the development of a_n can be thought as a sequence of n "ones" separated by means of n-2 brackets. We can construct $\binom{n+2(n-2)}{n}$ "sequences" which have n "ones" and 2(n-2) strokes of brackets. Hence $\frac{a_n}{b^n} \le \binom{3n-4}{n}$, for n>1. And

$$\lim_{n\to\infty} \sup \frac{|a_n|}{|a_{n-1}|} \le |b| \lim_{n\to\infty} \frac{(3n-4)!\ (n-1)!\ (2n-6)!}{(3n-7)!\ n!\ (2n-4)!} = (27/4)|b|. \ \square$$

Let T be a real number>0, and $K_T \in [0,\infty]$ satisfying

$$\max\left(|y_1'(v)|, |y_1(v)|\right) \le K_T, \text{ for all } v\in[-T,T].$$

Proposition A.4 The series (II.1) is absolutely convergent for each $v\in[-T,T]$, if $\varepsilon<4\cdot(27K_T)^{-1}$.

Proof. Owing to the Lemma

$$|f_n(\tau)| \le |y_{n-1}'(\tau)y_1'(\tau)+\ldots+y_1'(\tau)y_{n-1}'(\tau)| +$$

$$+ \frac{9}{4}|y_{n-1}(\tau)y_1(\tau)+\ldots+y_1(\tau)y_{n-1}(\tau)| \le \binom{3n-4}{n}\left(|y_1'(\tau)|^n + \frac{9}{4}|y_1(\tau)|^n\right).$$

$$\lim_{n\to\infty} \sup \frac{|f_n(\tau)|}{|f_{n-1}(\tau)|} \le (27/4) \lim_{n\to\infty} \frac{|y_1'(\tau)|^n + (9/4)|y_1(\tau)|^n}{|y_1'(\tau)|^{n-1} + (9/4)|y_1(\tau)|^{n-1}} =$$

$$= (27/4)\max\left(|y_1'(\tau)|, |y_1(\tau)|\right) \le (27/4)K_T. \tag{II.5}$$

From (II.4), taking into account (II.5), we deduce

$$\lim_{n\to\infty} \sup \frac{|y_n(v)|}{|y_{n-1}(v)|} \le (27/4)K_T, \text{ for each } v\in[-T,T]. \quad \square$$

Now we consider the other possibility, $y_0(v)=c$.

Then $y_1=-(2/9)v^2(y_0)^{-1}$, and $y_n=(2/9)(y_0)^{-1}f_n$. We have analogously

$$\lim_{n\to\infty} \sup \frac{|f_n(v)|}{|f_{n-1}(v)|} \le (27/4)\max\left(|y_1(v)|, |y_1'(v)|\right),$$ and the series (II.1) is absolutely convergent for all $v\in[-T,T]$, if $\varepsilon<4(27K_T)^{-1}$, with $K_T = \sup_{[-T,T]}\left(\max\left(|y_1(v)|, |y_1'(v)|\right)\right)$.

We return to equation (3.4), and we have

Theorem A.1 The solutions to

$$(9/4)\left(H(\phi(x))\right)^2 + \left(\frac{dH}{d\phi(x)}\right)^2 = -\left(4\Lambda + m^2\phi^2(x)\right)h(x)$$

can be written $$H\left(\phi(x)\right) = \sum_0^\infty m^{2n}\left(h(x)\right)^{1/2} y_n\left(\phi(x)\right),$$

provided that $|\phi(x)|<T$ for all $x\in\zeta(S)$, and $27m^2K_T<4$, with $K_T=\sup_{[-T,T]}\left(\max\left(|y_1(v)|,|y_1'(v)|\right)\right)$ (besides K_T is$<\infty$ if $T>0$ is small enough). Moreover

$$y_0(v)=c \text{ or } y_0(v)=c\cdot\sin\left((3/2)v+d\right),\ (d \text{ constant}),\ \text{if } \Lambda\neq 0.$$
$$y_0(v)=(\text{constant})\cdot\exp\left(\pm(3/2)iv\right),\ \text{if } \Lambda=0.\quad \square$$

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