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Localizability of zero-mass particles

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Abstract. It is shown that the elementary relativistic system of zero mass associated with the photon is localizable, in the sense of Wightman, on the homogeneous manifold $SE(3)/SE(2)$. A possible physical interpretation of this configuration space is discussed.

§1. Introduction

It is reasonable to suppose that for the elementary relativistic systems there exists a notion of localizability. (For instance, such a notion is needed when the cross section is constructed starting from the scattering matrix).

It is generally accepted that the most reasonable point of view is that of Wightman [1], (see also [2]) which is a reformulation of physical ideas contained in [3].

The main ideas are the following. Suppose the configuration space of a certain physical system is the differentiable manifold Q, and let $\beta(Q)$ be the natural Borel structure on Q. One can argue (see [1], p. 847) that the position observable is a projection valued measure based on Q:

$$\beta(Q) \ni E \mapsto P_E \in \mathcal{P}(\mathcal{H})$$

Here $\mathcal{H}$ is the underlying Hilbert space of the system and $\mathcal{P}(\mathcal{H})$ is the set of orthogonal projectors in $\mathcal{H}$. The states in the range of P_E are interpreted as localized in $E \subset Q$.

Suppose now that the group G is a group of symmetries for the system i.e. one has in $\mathcal{H}$ a projective unitary representation U of G. If Q is a G-space, i.e. G acts naturally on Q, then a natural compatibility condition is:

$$U_g P_E U_g^{-1} = P_{g \cdot E} \tag{1}$$

Here $g \cdot E$ is the image of E under the action of G (see [2], p. 145–146). The couple (U, P) is called a system of imprimitivity. The couples (U, P) and (U', P') must be considered equivalent from the physical point of view if one has an unitary operator $V: \mathcal{H} \to \mathcal{H}$ such that:

$$U'_g = VU_gV^{-1}, \quad \forall g \in G; \qquad P'_E = VP_EV^{-1}, \quad \forall E \in \beta(Q) \tag{2}$$

We recall that for $Q = \mathbb{R}^3$, one can construct from P, via the spectral theorem, three commuting self-adjoint operators (the position operators) and in this way one can connect the point of view of Wightman with that of Newton and Wigner [3].

To apply these ideas to relativistic systems, one recalls that the elementary relativistic systems are, according to Wigner, irreducible projective unitary representations of the Poincaré group. Given such a representation, one obtains by restriction, a projective unitary representation U of the euclidian group. On the other hand, the euclidian group $SE(3)$ acts naturally on $\mathbb{R}^3$, which can be considered as the natural configuration space for a pointlike system. So, the problem of localizability appears naturally: does there exists a projection valued measure P based on $\mathbb{R}^3$ such that (U, P) is a system of imprimitivity? This question has a positive answer for representations associated with nonzero mass systems. Unfortunately, the answer is negative for the representation associated with the photon.

There are a number of attempts in the literature to solve this problem. We list in some detail those which are close in spirit to the axioms of Wightman, and do not claim to exhaust all the existing points of view in the literature. (A rather complete list of references up to 1970 can be found in [4]).

(1) One can abandon the requirement that the map

$$\beta(\mathbb{R}^3) \ni \mathrm{E} \mapsto P_E \in \mathcal{P}(\mathcal{H})$$

is a projection valued measure as in [5], [6], where one admits that, in general, the projection P_E, $P_{E'}$ do not commute for $E, E' \in \beta(\mathbb{R}^3)$.

(2) The definition of Wightman localizability is apparently noncovariant, i.e. is relative to an observer characterized by the space-like hyperplane $t = 0$. One can construct a covariant notion of localizability as follows [7]: the configuration space $\mathbb{R}^3$ is replaced by an arbitrary space-like hyperplane Π and the group G is taken to be a subgroup of the Poincaré group which lets Π invariant and is isomorphic with the euclidian group. The plane Π represents the physical space of a moving observer. If the observer moves with the speed of light, i.e. Π is tangent to the light cone, then one has Wightman localizability for the photon.

(3) One can replace the euclidian group by another subgroup of the Poincaré group which also acts transitively on $\mathbb{R}^3$ [8].

(4) One can suppose that for the photon, the relevant notion is that of degree of localizability.

This amounts to suppose that the map $E \mapsto P_E$ is a positive operator valued measure [9].

Then there exists an appropriate generalization of the imprimitivity theorem and one can generalize the analysis of Wightman (see [10] and the literature cited there for further mathematical developments).

In all these approaches relation (1) remains true and one has also the principal advantages of Wightman's approach: (a) clarity of interpretation; (b) one works only with bounded operators.

We mention that there exist in the literature a number of attempts to solve the problem of localizability for relativistic systems in the spirit of the Newton and Wigner approach, namely to construct a three (or four) dimensional position operator which verifies an analogue of (1). This point of view is affected by the usual domain problems associated with unbounded operators, and we mention only [11] and [12] which can be connected with (4) above via a generalization of the spectral theorem.

We do not insist on this point of view, because in this paper we propose a new solution which is in the spirit of Wightman.

The starting point is the observation that one usually identifies the configuration space Q with the 'physical space' $\mathbb{R}^3$ because of the implicit assumption that the elementary systems are pointlike. But if we adopt the point of view of Wigner, namely that an elementary system is an irreducible unitary projective representation of the Poincaré group, then the identification $Q = \mathbb{R}^3$ is harder to justify, and in principle one could accept that Q does not coincide with $\mathbb{R}^3$ (as homogeneous $SE(3)$-manifolds).

It is clear that the configuration space differs from the 'physical space' for many systems: for instance, for a scalar field theory the configuration space is $\mathscr{S}'(\mathbb{R}^3)$. Also, from the point of view of special relativity, the assumption that $\mathbb{R}^3$ is the 'physical space' is not very natural.

The next step is to decide what configuration spaces Q are admissible for an elementary relativistic system.

It would be interesting to analyse the problem in full generality and to have an axiomatic point of view. Here we make only a tentative proposal, namely that the configuration space is an homogeneous connected $SE(3)$ manifold. We did not yet succeed to analyse all the possibilities, but, following a suggestion from [13] (p. 584) we establish in §2 that the elementary relativistic system associated with the photon is localizable, in the sense of Wightman, on a certain $SE(3)$-homogeneous manifold Q_0 (which of course is not $SE(3)$-diffeomorphic with $\mathbb{R}^3$ as an $SE(3)$-space).

The problem of finding a physical interpretation for this new configuration space is not completely clear. A possible interpretation is based on the observation that Q_0 can be realized as the manifold of bidimensional planes in $\mathbb{R}^3$ with the natural action of the euclidian group; then a photon could be interpreted as a bidimensional plane – a plane wave – travelling with the speed of light. This interpretation cannot be considered completely satisfactory as it stands, and may be a better one can be found. Anyway we can consider our result, at least as an interesting mathematical fact connected with Wightman localizability.

Let us comment on the connection between our proposal and other approaches listed above. The idea that one could change the configuration space appears in [7], but as an $SE(3)$-space the proposal from this reference is diffeomorphic with $\mathbb{R}^3$. Also in [11] one can find the idea of localizing photons on planes or on curves, but in [11] as well as in [1], the photon is considered as a pointlike particle.

We have found points of support for the interpretation of the photon as a bidimensional object in the analysis of classical dynamical systems of Souriau [14] (especially the footnote of p. 191).

Finally, we mention that an analysis of localizability in the framework of hamiltonian formalism (cf. [15]) seems to corroborate the quantum picture presented in this paper. We will comment more on this connection in §3, where we present also some open problems. In §4 we make some final comments.

§2. Localizability for the photon

1. The elementary relativistic systems of zero mass for the proper orthochronous Poincaré group $\mathscr{P}_+^\uparrow$ are described, in the notation of [2], by the projective unitary irreducible representations $U^{\pm,n}(n \in \mathbb{Z})$. If one considers now the orthochronous Poincaré group $\mathscr{P}^\uparrow$, then the representations $U^{\pm,0}$ and $U^{\pm,n} \oplus U^{\pm,n}(n \in \mathbb{N}^*)$ of $\mathscr{P}_+^\uparrow$ have unique extensions (denoted by the same symbols) to irreducible representations of $\mathscr{P}^\uparrow$. The photon is associated with $U^{+,2} \oplus U^{+,-2}$.

2. The euclidian group is by definition the set $SE(3) = S0(3) \times \mathbb{R}^3$ with the composition law:

$$(R, \vec{a}) \cdot (R', \vec{a}') \equiv (RR', \vec{a} + R\vec{a}')$$

Here the vectors $\vec{a}, \vec{a}'$ from $\mathbb{R}^3$, are considered as column matrices and we use matrix multiplication. Let us consider the following subgroup of $SE(3)$:

$$K \equiv \{(R, \vec{a}) \in SE(3) \mid Re_3 \equiv \pm e_3, a_3 = 0\}$$

(K is the euclidian group of the plane. Explicitely:

$$K = \{(R(e_3, \varphi), \vec{a}) \mid \varphi \in [0, 2\pi), a_3 = 0\} \cdot \{(\mathbb{1}, \vec{0}), (R(e_1, \pi), \vec{0})\}$$

where $R(\vec{v}, \varphi)$ is the rotation of angle φ around axis $\vec{v}$).

In the following we will study the question of localizability of the photon on the manifold

$$Q_0 \equiv SE(3)/K$$

For practical computation it is convenient to note that

$$Q_0 \simeq (S^2 \times \mathbb{R})/\sim$$

where $(\vec{v}, q) \sim (\vec{v}', q')$ *iff* $\vec{v}' = \varepsilon\vec{v}$ and $q' = \varepsilon q$ ($\varepsilon = \pm 1$) and the action of $SE(3)$ is:

$$(R, \vec{a}) \cdot [\vec{v}, q] \equiv [R\vec{v}, q + (\vec{a}, R\vec{v})_{\mathbb{R}^3}] \tag{1}$$

Here $(,)_{\mathbb{R}^3}$ is the euclidian inner product in $\mathbb{R}^3$, and $[\vec{v}, q]$ is the equivalence class of $(\vec{v}, q) \in S^2 \times \mathbb{R}$.

It will be shown at the end of this section that Q_0 has a (natural) geometric realization as the manifold of bidimensional planes in $\mathbb{R}^3$.

3. We describe now the representation $U^{+,2} \oplus U^{+,-2}$ of $\mathscr{P}^\uparrow$ using the Hilbert space bundle formulation (see [2], p. 208–209, [16], p. 214–215). Consider the vector space:

$$V \equiv \left\{\varphi : X_0^+ \to \mathbb{C}^4 \mid \varphi \text{ is Borel}, \{p, \varphi(p)\} = 0, \forall p \in X_0^+, \right.$$
$$\left. \int_{X_0^+} d\alpha_0^+(p) B(\varphi(p), \varphi(p)) < \infty \right\}$$

Here $X_0^+ \equiv \{p \in \mathbb{R}^4 \mid p_0 = \|\vec{p}\|_{\mathbb{R}^3}\}$ is the future light cone, $\{,\}$ is the Minkowski bilinear form in $\mathbb{R}^4$:

$$\{x, y\} \equiv x_0 y_0 - (\vec{x}, \vec{y})_{\mathbb{R}^3},$$

α_0^+ is a Lorentz invariant measure on X_0^+, and B is the Lorentz–Hermite form in $\mathbb{C}^4$:

$$B(v, v') \equiv -\bar{v}_0 v_0' + \bar{v}_1 v_1' + \bar{v}_2' v_2' + \bar{v}_3 v_3'$$

We identify in V functions which coincide α_0^+-almost everywhere. Then V is complete, but

$$\|\varphi\| \equiv \left(\int_{X_0^+} d\alpha_0^+(p) B(\varphi(p), \varphi(p))\right)^{1/2}$$

is only a seminorm. Define on V an equivalence relation: $\varphi \sim \varphi'$ *iff* $\|\varphi - \varphi'\| = 0$. Then $\mathscr{H} \equiv V/\sim$ is a Hilbert space.

4. In V we have the following representation of $\mathscr{P}^\uparrow$:

$$(\mathcal{U}_{L,a}\varphi)(p) \equiv e^{i\{a,p\}} L\varphi(L^{-1}p) \; L \in \mathscr{L}^\uparrow, \qquad a \in \mathbb{R}^4 \tag{2}$$

Here L is an element of the orthochronous Lorentz group $\mathscr{L}^\uparrow$ i.e. a 4×4 real matrix verifying:

$$L^t G L = G$$

(diag $G = (1, -1, -1, -1)$) and $L_{00} > 0$. As before we use consistent matrix notations. Because $\|\mathcal{U}_{L,a}\varphi\| = \|\varphi\|$, $\mathcal{U}$ factorizes to a unitary representation of $\mathscr{P}^\uparrow$ in $\mathscr{H}$, denoted also by $\mathcal{U}$. It can be shown that $\mathcal{U} \simeq U^{+,2} \oplus U^{+,-2}$ (see [2]).

5. By restriction to $SE(3)$, we have from (2):

$$(\mathcal{U}_{R,\vec{a}}\varphi)(p) = e^{-i(\vec{a},\vec{p})_{\mathbb{R}^3}} R\varphi(R^{-1}p) \tag{3}$$

Here we identify elements R of $S0(3)$ with elements of $\mathscr{L}^\uparrow$ of the form $\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right)$. We will perform a number of unitary transformations that will bring U in a form in which localizability on Q_0 is easy to verify.

6. First we note that $\|\varphi\| = 0$ *iff* φ is of the form:

$$\varphi(p) = \Lambda\,(p)p$$

with $\Lambda : X_0^+ \to \mathbb{C}$ a Borel function. We can eliminate this gauge degree of freedom imposing the Coulomb gauge (see [5], sect. V):

$$\varphi_0(p) = 0$$

Then $\mathscr{H}$ becomes:

$$\mathscr{H} \equiv \left\{\vec{\varphi} : X_0^+ \to \mathbb{C}^3 \mid \vec{\varphi} \text{ is Borel, } (\vec{p}, \vec{\varphi}(p))_{\mathbb{C}^3} = 0,\ \forall p \in X_0^+, \right.$$

$$\left. \int_{X_0^+} d\alpha_0^+(p)\, \|\vec{\varphi}(p)\|^2_{\mathbb{C}^3} < \infty \right\}.$$

and (3) goes into:

$$(U_{R,\vec{a}}\vec{\varphi})(p) = e^{-i(\vec{a},\vec{p})_{\mathbb{R}^3}} R\vec{\varphi}(R^{-1}p) \tag{4}$$

Here $(,)_{\mathbb{C}^3}$ is the usual inner product in $\mathbb{C}^3$.

7. It is convenient now to identify functions defined on X_0^+ with functions defined on $\mathbb{R}^3$. We make also the rescaling $\vec{\varphi}(p) \to (2\,\|\vec{p}\|_{\mathbb{R}^3})^{1/2}\vec{\varphi}(\vec{p})$. Then $\mathcal{H}$ becomes:

$$\mathcal{H} \equiv \left\{ \vec{\varphi} : \mathbb{R}^3 \to \mathbb{C}^3 \mid \vec{\varphi} \text{ is Borel, } (\vec{p}, \vec{\varphi}(\vec{p}))_{\mathbb{C}^3} = 0, \forall \vec{p} \in \mathbb{R}^3, \right.$$
$$\left. \int_{\mathbb{R}^3} \frac{d\vec{p}}{\|\vec{p}\|_{\mathbb{R}^3}} \|\vec{\varphi}(\vec{p})\|^2_{\mathbb{C}^3} < \infty \right\}$$

and (4) takes the form:

$$(U_{R,\vec{a}}\vec{\varphi})(\vec{p}) = e^{-i(\vec{a},\vec{p})_{\mathbb{R}^3}} R\vec{\varphi}(R^{-1}\vec{p}) \tag{5}$$

8. We pass now to radius-angle variables i.e. a function of $\vec{p}$ is considered as a function of $H = \|\vec{P}\|_{\mathbb{R}^3}$ and $\vec{v} \equiv \vec{p}/H$. The Hilbert space becomes:

$$\mathcal{H} \equiv \left\{ \vec{\varphi} : S^2 \times \mathbb{R}_+ \to \mathbb{C}^3 \mid \vec{\varphi} \text{ is Borel, } (\vec{v}, \vec{\varphi}(\vec{v}, H))_{\mathbb{C}^3} = 0, \forall(\vec{v}, H) \right.$$
$$\left. \int_{S^2 \times \mathbb{R}_+} d\sigma \oplus dH \, \|\vec{\varphi}(\vec{v}, H)\|^2_{\mathbb{C}^3} < \infty \right\}$$

Here $d\sigma$ is the rotational invariant measure on S^2. The representation (5) becomes:

$$(U_{R,\vec{a}}\vec{\varphi})(\vec{v}, H) = e^{-iH(\vec{v},\vec{a})_{\mathbb{R}^3}} R\vec{\varphi}(R^{-1}\vec{v}, H) \tag{6}$$

9. One observes now that a Borel function defined on $S^2 \times \mathbb{R}_+$ can be extended to a Borel function on $S^2 \times \mathbb{R}$ by imposing the condition of parity:

$$\vec{\varphi}(\vec{v}, H) = \varphi(-\vec{v}, -H) \tag{7}$$

(The zero measure set $H = 0$ does not count).

The Hilbert space becomes:

$$\mathcal{H} \equiv \left\{ \vec{\varphi} : S^2 \times \mathbb{R} \to \mathbb{C}^3 \mid \vec{\varphi} \text{ is Borel, } (\vec{v}, \vec{\varphi}(\vec{v}, H))_{\mathbb{C}^3} = 0, \right.$$
$$\vec{\varphi}(\vec{v}, H) = \vec{\varphi}(-\vec{v}, -H), \forall(\vec{v}, H) \in S^2 \times \mathbb{R},$$
$$\left. \tfrac{1}{2} \int_{S^2 \times \mathbb{R}} d\sigma \otimes dH \, \|\vec{\varphi}(\vec{v}, H)\|^2_{\mathbb{C}^3} < \infty \right\}$$

The representation U is given again by (6).

10. We perform the "partial" Fourier transform $\mathcal{F} : \mathcal{H} \to \mathcal{H}$

$$(\mathcal{F}\vec{\varphi})(\vec{v}, q) \equiv \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} dH e^{-iqH} \vec{\varphi}(\vec{v}, H)$$

It is easy to see that $\mathcal{F}$ is well defined and that the representation (6) becomes by transport:

$$(U_{R,\vec{a}}\vec{\varphi})(\vec{v}, q) = R\vec{\varphi}(R^{-1}\vec{v}, q - (\vec{a}, \vec{v})_{\mathbb{R}^3}) \tag{8}$$

11. Finally, it is clear that we can consider:

$$\mathcal{H} \equiv \left\{ \vec{\varphi} : Q_0 \to \mathbb{C}^3 \mid \vec{\varphi} \text{ is Borel}, (\vec{v}, \vec{\varphi}([\vec{v}, q]))_{\mathbb{C}^3} = 0, \forall [\vec{v}, q] \in Q_0, \right.$$
$$\left. \int_{Q_0} d\mu \, \|\vec{\varphi}([\vec{v}, q])\|^2 < \infty \right\}$$

Here μ is the measure $d\sigma \otimes dq$ factorized to Q_0. The representation (8) becomes:

$$(U_{R,\vec{a}}\vec{\varphi})([\vec{v}, q]) = R\vec{\varphi}([R^{-1}\vec{v}, q - (\vec{a}, \vec{v})_{\mathbb{R}^3}]) \tag{9}$$

12. Define now the projection valued measure P in $\mathcal{H}$ based on Q_0 by:

$$(P_E\vec{\varphi})([\vec{v}, q]) = \chi_E([\vec{v}, q])\vec{\varphi}([\vec{v}, q]) \tag{10}$$

Then a simple computation establishes that:

$$U_{R,\vec{a}} P_E U_{R,\vec{a}}^{-1} = P_{(R,\vec{a})\cdot E} \tag{11}$$

where the action of $SE(3)$ on Q_0 is given by (1). In other words we have (1) from §1, and so (U, P) given by (9) + (10) is a system of imprimitivity in $\mathcal{H}$ based on Q_0. So we have the following:

Theorem. *The elementary system associated with the representation $U^{+,2} \oplus U^{+,-2}$ of $\mathcal{P}^{\uparrow}$ (i.e. the photon) is localizable (in the sense of Wightman) on the configuration manifold Q_0.*

Remark. (1) It is easy to see that Q_0 can be interpreted as the manifold of bidimensional planes in $\mathbb{R}^3$. Indeed, a plane in $\mathbb{R}^3$ is described by an equation of the following form:

$$(\vec{v}, \vec{x})_{\mathbb{R}^3} = q$$

Here $\vec{v} \in S^2$, $q \in \mathbb{R}$ and the couples $(\vec{v}, q)$ and $(-\vec{v}\text{-}, -q)$ are identified. This could lead us to the interpretation of the photon as a plane, as proposed in the introduction. But we must note that the identification $Q_0 \simeq (S^2 \times \mathbb{R})/\sim$, is not canonical, so in principle, one could realize Q_0 as another object from projective geometry with a different (and may be more appropriate) interpretation. So it is better to leave the question of interpretation of the configuration space Q_0 open.

(2) To compute the most general solution for P, one could proceed on the same lines as in [2]. Unfortunately, one needs an analogue of corollary 12.15 (with Q_0 instead of $\mathbb{R}^3$). We did not succeed to solve this problem.

13. We will show now that the system of imprimitivity (9) + (10) is irreducible. For this we apply th. 9.20 from [2]. We define:

$$B \equiv \{([\vec{v}, q], \vec{v}) \in Q_0 \times \mathbb{C}^3 \mid (\vec{v}, \vec{v})_{\mathbb{C}^3} = 0\}$$

and denote by π the canonical projection on Q_0. Let $SE(3)$ act on B according to:

$$(R, \vec{a}) \cdot ([\vec{v}, q], \vec{v}) \equiv ((R, \vec{a}) \cdot [\vec{v}, q], R\vec{v}) \tag{12}$$

Then B is a $SE(3)$. Hilbert space bundle (v. [2] p. 87). Also the representation (9) is in the standard form given by formula (111) from [2]. To apply th. 9.20 from [2], we take $x_0 = [e_3, 0]$. Then, $SE(3)_{x_0} = K$ from paragraph 1. If we identify $B_{x_0} \simeq \mathbb{C}^2$ by:

$$B_{x_0} \ni \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$$

then, the restriction of (12) to K is given by:

$$(R(e_3, \varphi), \vec{a})\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos \varphi v_1 + \sin \varphi v_2 \\ -\sin \varphi v_1 + \cos \varphi \sigma_2 \end{pmatrix} (a_3 = 0)$$

$$(R(e_1, \pi), \vec{0})\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}$$

which is irreducible. According to th. 9.20 and corrolary 9.13, (U, P) is irreducible.

Remarks. One can prove on the same lines that the systems of imprimitivity based on $\mathbb{R}^3$ and describing the localizability of positive mass systems (see [1], [2]) are also irreducible. Although the hypothesis of irreducibility is not made explicitly by Wightman, this is considered natural by some authors, e.g. [17], p. 78, [6] p. 155–156.

§3. Some open problems associated with the notion of localizability

1. The results obtained in §2 justify the following attempt: one can consider all homogeneous $SE(3)$, manifolds (not only Q_0) and decide which of them could be a configuration space for a given elementary relativistic system. At the mathematical level the problem seems rather difficult. Guided by a similar analysis performed in the framework of analytical mechanics [15] one could conjecture that the only possible configuration spaces are $\mathbb{R}^3$, Q_0 (and eventually the universal covering of Q_0); also in this way one could hope to show that elementary systems as the tahions, particles of zero mass and infinite spin etc., do not appear in nature because they are not localizable on some configuration space in the sense of Wightman. Below we show that unfortunately this is not true.

2. Consider S^2 as an *in* $SU(2)$ – homogeneous manifold relative to the following action:

$$(A, \vec{a})\vec{v} = \delta(A)\vec{v} \tag{1}$$

Here $\delta : SU(2) \to SO(3)$ is the covering homomorphism and *in* $SU(2)$ is the set $SU(2) \times R^3$ with the composition law:

$$(A_1, \vec{a}_1)(A_2, \vec{a}_2) = (A_1 A_2, \vec{a}_1 + \delta(A_1)\vec{a}_2).$$

Then, we have the following:

Proposition. *The elementary systems associated with the representations* $U^{m,\pm,j}$ $(m \in \mathbb{R}_+, j = 0, \frac{1}{2}, 1, \ldots)$ *of in* $SL(2, \mathbb{C})$ *(see* [2] *for notations) are localizable, in the sense of Wightman, on* S^2.

Proof. As in [2] we realize $U^{m,\pm,j}$ in $L^2(X_m^\pm, \mathbb{C}^{2j+1}, d\alpha_m^\pm)$ where $X_m^\pm \equiv \{p \in \mathbb{R}^4 \mid \{p, p\} = m^2, \text{ sign } p_0 = \pm\}$ and $\alpha_m^\pm$ is the Lorentz invariant measure on $X_m^\pm$. More conveniently we identify $L^2(X_m^\pm, \mathbb{C}^{2j+1}, d\alpha_m^\pm) \simeq L^2(\mathbb{R}^3, \mathbb{C}^{2j+1}, d\vec{p}/2\|\vec{p}\|_{\mathbb{R}^3})$; then $U \equiv U^{m,\pm,j}|$ *in* $SU(2)$ is given by:

$$(U_{A,\vec{a}}\varphi)(\vec{p}) = e^{-i(\vec{a},\vec{p})_{\mathbb{R}^3}} D^{(j)}(A)\varphi(\delta(A)^{-1}\vec{p}) \tag{2}$$

Here $D^{(j)}$ is the irreducible representation of weight j of $SU(2)$. We have a natural isomorphism $L^2(\mathbb{R}^3, \mathbb{C}^{2j+1}, d\vec{p}/2\,\|\vec{p}\|_{\mathbb{R}^3}) \simeq L^2(S^2, \mathcal{K}, d\sigma)$ where: $\mathcal{K} = L^2(\mathbb{R}_+, \mathbb{C}^{2j+1}, H\,dH/2)$. Transporting (2) in the new representation, we get:

$$((U_{A,\vec{a}}\varphi)(\vec{v}))(H) = e^{-iH(\vec{a},\vec{v})_{\mathbb{R}^3}} D^{(j)}(A)(\varphi(\delta(A)^{-1}\vec{v}))(H) \tag{3}$$

Define now the projection valued measure P based on S^2 by:

$$((P_E\varphi)(\vec{v}))(H) = \chi_E(\vec{v}) \cdot \varphi(\vec{v})(H) \tag{4}$$

Then we immediately have:

$$U_{A,\vec{a}} P_E U_{A,\vec{a}}^{-1} = P_{(A,a)\cdot E} \tag{5}$$

i.e. (U, P) is a system of imprimitivity. ∎

3. Let us comment this result. At a first glance one is tempted to find a physical interpretation for this new configuration space; e.g. a system localized on S^2 can be thought as a rotator with a fixed point. Unfortunately it is easy to see that the argument in the proposition above works as well for all elementary relativistic systems (including tahions and particles of zero mass and infinite spin).

So guided by the analogy with classical mechanics it is plausible to conjecture that by supplementing the requirements of Wightman localizability by a new condition one could eliminate these relativistic systems on grounds of nonlocalizability.

Using the result from §1, paragraph 13, one could impose the requirement that (U, P) is irreducible. Indeed, this new condition excludes the system of imprimitivity (3) + (4).

4. Another possible problem with the configuration space Q_0 found in §2 is connected with the understanding of the limit $m \to 0$. Indeed, for nonzero mass, the analysis in [1] shows that the corresponding elementary systems are localizable on $\mathbb{R}^3$, and for zero mass we have localizability on Q_0, so one can ask

what happens to a nonzero mass system when the mass is very small.

A possible solution is based on the observation that, proceeding on the same lines as in §2, one can prove that the elementary relativistic systems associated with $U^{m,\pm,j}(m \in \mathbb{R}_+,\ j=0,\ \frac{1}{2},\ 1, \ldots)$ are also localizable on the configuration space Q_0.

So, our proposal is the following. Define an elementary relativistic system to be a couple (U, Q) where U is a projective unitary irreducible representation of the Poincaré group, and Q is a configuration space, on which we have Wightman localizability. Then, the elementary systems $(U^{m,\pm,j}, \mathbb{R}^3)$ and $(U^{m,\pm,j}, Q_0)$ must be considered as physically distinct. The limit $m \to 0$ can be done only for the second ones, and it is plausible that for $(U^{m,+,1}, Q_0)$ one can obtain in such a way the photon as a limiting case.

§4. Conclusions

In this paper we have proposed a new solution for the problem of localizability for the photon, based on the observation that the configuration space of a system is not necessarily identical with the "physical space" $\mathbb{R}^3$. A number of problems remain open, namely: a) physical criteria for choosing a certain configuration space should be found; b) the physical interpretation of a certain configuration space in terms of operations performed in the 'physical space' $\mathbb{R}^3$ is desirable.

Nevertheless it would be interesting to solve the problem outlined in §3, paragraph 1, namely to take into consideration all $SE(3)$ – homogeneous manifolds as possible configuration spaces.

If we get 'too many solutions' as indicated by the analysis in §3, one must find supplementary conditions to the axioms of Wightman localizability. This can be done by imposing the irreducibility of the system of imprimitivity, or, perhaps, by using ideas from geometric quantization.

To make the connection with the point of view of Souriau, one needs a more comprehensive understanding of the connection between classical and quantal description of physical systems. An idea would be to find an analog of the concept of evolution space in the framework of quantum mechanics.

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