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REPRESENTATIONS OF KAC–MOODY ALGEBRAS USING TWISTED VERTEX OPERATORS

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CONTENTS

1. INTRODUCTION

1.1. Interplay of Physics and Mathematics

Many physicists hope that the superstring model [1,2] will yield a unified quantum theory of all fundamental interactions, including gravitation. Several difficult physical and mathematical problems must be solved before this hope will be substantiated. However it is already clear that the very rich mathematical structure of the model has significantly stimulated the collaboration between mathematicians and physicists. Here we are interested in the contributions to algebra.

In 1968 the Veneziano [3,4,1] or dual resonance model was invented. One year later two important tools for the understanding of the model were introduced : vertex operators and the Virasoro algebra. The latter is an extension of the infinite dimensional conformal algebra acting on a space of two dimensions. On the other hand, in 1967, Kac [5] and Moody [6] introduced the infinite dimensional affine Lie algebras, which turned out to be a discrete version of the current algebras considered by physicists in the early sixties. In 1980, Frenkel, Kac [7] and Segal [8] (FKS) constructed highest weight representations of the Kac-Moody algebras using the vertex operators of the dual model. The Virasoro operators provide labels for these representations. In 1985, the FKS [7,8] construction served to compactify the bosonic part of the heterotic string [9] from 26 to 10 dimensions and to display gauge groups of rank $26 - 10 = 16$. In 1969, the bosonic string provided a Lagrangian formulation of the dual model [10]. Together with the fermionic string it gave one of the first physical models of supersymmetry. Superalgebras were studied by Berezin and Kac in 1970 [11].

The dual model and its string and superstring versions failed to explain the phenomenology of strong interactions. Interest was revived after it was shown that gravitation could be included and that anomalies could be cancelled in a seemingly miraculous way for the gauge groups $E_8 \times E_8$ and $SO(32)$ [12]. One of the main problems is now to compactify the superstring from 10 to the physical 4 dimensions. One possible relatively easy way is to replace in the compactification procedure tori by orbifolds, which differ from the former by "twisted" boundary conditions [2]. Here again work was greatly stimulated by discussions between physicists and mathematicians, the latter being able to provide "twisted" constructions of Kac-Moody algebras [13,14]. The main tool are now "twisted" vertex operators. They play an important role in symmetry breaking. One can also speculate that they could be used to describe emission of "twisted" strings [15].

The aim of these lectures is to give a self-contained unified description of untwisted and twisted vertex operators and the corresponding constructions of

Kac-Moody algebras. We use materials from mathematicians [13,14,16] and recent preprints of mathematical physicists [17,18,19] but try to be comprehensible to the average physicist.

1.2. Summary

The main ideas are the following : 1) For a string moving on a torus T^d in d dimensions, the eigenvalues of the center of mass (cm) momentum operators p are discrete. They can be identified with the points of a rank d weight lattice of a finite, simply-laced Lie algebra g, provided $T^d = \mathbb{R}^d/Q$, where Q is the root lattice, and the set of components of p forms the Cartan subalgebra (CSA) of g. For a closed string, the vertex operator U is periodic in the string variable σ. The Laurent coefficients of U together with the set $\{p^I\}$ and the harmonic oscillators entering in the definition of U are the generators of the infinite-dimensional Kac-Moody algebra in the level 1 highest weight representations.

2) This construction has been generalized in several directions. a) Consider a string moving on an orbifold. Construct a new vertex operator U which is periodic up to an automorphism w of the root lattice. The Laurent coefficients of U, acting on a different Hilbert space, will be generators of a twisted Kac-Moody algebra. For $w = 1$ we recover the previous construction. This approach will be followed here.
b) Replace the simply-laced Lie algebra g by a non simply-laced algebra [20].
c) Introduce fermionic instead of bosonic oscillators [20]. In this case one finds an interesting connection with octonions [21].

2. LIE ALGEBRAS AND KAC-MOODY ALGEBRAS

2.1. Simple Finite Dimensional Lie Algebras [22]

The hermitean generators T_a of a simple compact Lie algebra g satisfy the commutation relations

$$[T_a, T_b] = i f_{abc} T_c \ . \tag{2.1.1}$$

The adjoint representation is defined on the vector space $\underline{g}$ spanned by the generators. The Killing form defines the scalar product

$$(T_a, T_b) = \text{const } \mathrm{Tr}(Ad\ T_a\ Ad\ T_b) \tag{2.1.2}$$

$$Ad\ T_a(T_b) = [T_a, T_b] \tag{2.1.3}$$

By a clever choice of the basis g can be divided into an abelian Cartan subalgebra (CSA) $\underline{h}$ spanned by the generators H^i such that $(H^i, H^j) = \delta_{i,j}$ and the step operators E_α, where $\alpha \in \underline{h}'$ (the dual of $\underline{h}$) is a root; (2.1.1) gives then

$$\begin{aligned} &[H^i, H^j] = 0 \\ &[H^i, E_\alpha] = \alpha(H^i)E_\alpha \equiv \alpha^i E_\alpha \\ &[E_\alpha, E_\beta] = \begin{cases} \varepsilon(\alpha,\beta)E_{\alpha+\beta}, & \text{if } \alpha+\beta \in \Delta \\ H_\alpha, & \text{if } \alpha+\beta = 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.1.4}$$

where Δ the set of roots. Here, to any root $\alpha \in \underline{h}'$ we have associated a generator $H_\alpha \in \underline{h}$ such that

$$(H_\alpha, H) = \alpha(H) \quad \forall\, H \in \underline{h} \tag{2.1.5}$$

Identifying $\underline{h}$ with its dual $\underline{h}'$, we can define also the scalar product on $\underline{h}'$ by

$$(\alpha,\beta) = (H_\alpha, H_\beta) \ . \tag{2.1.6}$$

One calls $\ell = \dim \underline{h}$ the rank of $\underline{g}$. The set Δ of the roots can be divided into two equal sets of positive and negative roots. A simple root α_i is a positive root which cannot be written as a sum of positive roots. The ℓ simple roots α_i form a basis of $\underline{h}'$.

We shall limit ourselves in the future to simply-laced Lie algebras for which all the roots have same length. It can be normalized to 2 : $(\alpha,\alpha) = \alpha^2 = 2$. The Cartan matrix is then defined by

$$A_{ij} = (\alpha_i, \alpha_j) \ . \tag{2.1.7}$$

The factor $\varepsilon(\alpha,\beta)$ is called a 2-cocycle and obeys

$$\varepsilon(\alpha,\beta+\gamma)\varepsilon(\beta,\gamma) = \varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) \tag{2.1.8}$$

$$\varepsilon(\alpha,\beta) = (-1)^{(\alpha,\beta)}\varepsilon(\beta,\alpha) \tag{2.1.9}$$

One obtains a representation of g by acting on vectors $|\lambda\rangle$:

$$H_\alpha|\lambda\rangle = (\alpha,\lambda)|\lambda\rangle \tag{2.1.10}$$

$\lambda \in \underline{h}'$ is called a weight. For compact simple Lie algebras all representations possess a highest weight Λ such that $(\Lambda, \alpha_i) \geq 0$ $i = 1, ...l$ and

$$\begin{aligned} &E_\alpha|\Lambda\rangle = 0 \quad \alpha > 0 \\ &H_\alpha|\Lambda\rangle = (\alpha,\Lambda)|\Lambda\rangle \end{aligned} \tag{2.1.11}$$

The whole representation space is then obtained from $|\Lambda\rangle$ by repeatedly acting on it with various $E_{-\alpha}$ $(\alpha > 0)$.

2.2. Untwisted Affine Simply Laced Kac-Moody Algebra [23,24]

To each finite Lie algebra g one assigns an infinite Lie algebra $\hat{g}$ or Kac-Moody algebra. The generators are $T_a \otimes t^n$, $T_a \in g$, $t \in \mathbb{C}$, with the commutation relations

$$[T_a \otimes t^n, T_b \otimes t^{n'}] = if_{abc}T_c \otimes t^{n+n'} + n(T_a, T_b)\delta_{n,-n'}k \tag{2.2.1}$$

where $n, n' \in \mathbb{Z}$; $a, b, c = 1, ...\dim \underline{g}$ and k is the central term

$$[k, T_a \otimes t^n] = 0 \tag{2.2.2}$$

One identifies $T_a \otimes t^0$ with T_a, the generators of g, satisfying (2.1.1).

It is useful to introduce a dilation or derivation operator d, (which we will in the following identify with the Virasoro operator $-L_0$)

$$[d, T_a \otimes t^n] = n\, T_a \otimes t^n \tag{2.2.3}$$

$$[d, k] = 0 \tag{2.2.4}$$

Corresponding to the basis (2.1.4), one writes

$$[H^i \otimes t^n, H^j \otimes t^{n'}] = n\, \delta^{i,j}\delta_{n,-n'}k \tag{2.2.5}$$

For $n \neq 0$, one gets an infinite set of <u>harmonic oscillators</u> (also called a Heisenberg algebra).

The other commutations relations are

$$[H_\alpha \otimes t^n, E_{\pm\beta} \otimes t^{n'}] = \pm(\alpha,\beta)E_{\pm\beta} \otimes t^{n+n'} \tag{2.2.6}$$

$$[E_\alpha \otimes t^n, E_\beta \otimes t^{n'}] = \begin{cases} \varepsilon(\alpha,\beta)E_{\alpha+\beta} \otimes t^{n+n'}, & \text{if } \alpha+\beta \in \Delta \\ H_\alpha \otimes t^{n+n'} + n\delta_{n,-n'}k, & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.7}$$

If rank $g = \ell$, $\hat{g}$ has $\ell + 1$ simple roots, namely those of $\underline{g}$ $(\alpha_i, i = 1, \ldots, \ell)$ and

$$\alpha_0 = \delta - \theta \tag{2.2.8}$$

θ is the highest root of g and δ the "imaginary root" with zero length

$$(\delta,\delta) = 0 = (\delta, \alpha_i) \quad i = 0, \ldots, \ell\ . \tag{2.2.9}$$

This shows that the metric in root space $\hat{h}'$ is not euclidean. Correspondigly the Cartan matrix

$$\hat{A} = (\alpha_i, \alpha_j) \quad i,j = 0,1,...l \tag{2.2.10}$$

is degenerate.

Example : for $\underline{g} = A_1 = su(2)$, $\underline{\hat{g}} \equiv A_1^{(1)}$ has the Cartan matrix

$$\hat{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \tag{2.2.11}$$

with det $\hat{A} = 0$. Here $\theta = \alpha_1$ and one verifies that $\delta^2 = (\alpha_0 + \alpha_1)^2 = 0$.
The root system of $\hat{g}$ is given by $\hat{\Delta} = \{j\delta + \gamma \mid j \in \mathbb{Z},\, \gamma \in \Delta\} \cup \{j\delta \mid j \in \mathbb{Z}^*\}$.

One can again identify $\underline{\hat{h}}$ and $\underline{\hat{h}'}$ with the correspondence : $H_{\alpha_i} \leftrightarrow \alpha_i$ ($i = 0,\ldots,\ell$); $k \leftrightarrow \delta$; $d \leftrightarrow \Lambda_0$. The dimension of $\underline{\hat{h}}$ is $\ell + 2$ and Λ_0 satisfies : $(\Lambda_0, \Lambda_0) = (\alpha_i, \Lambda_0) = 0$ ($i = 1,\ldots,\ell$) and $(\Lambda_0, \delta) = (\Lambda_0, \alpha_0) = 1$.

There exist highest weight representations (HWR), but for instance the adjoint representation is not a HWR. We shall consider HWR with the highest weight vector $|\Lambda\rangle$ satisfying

$$(\Lambda, \alpha_i) \geq 0, \; i = 0,1,...,l \tag{2.2.12}$$

$$E_\alpha \otimes t^n |\Lambda\rangle = 0 \tag{2.2.13}$$

for either $n > 0$ and (or) $\alpha > 0$, and

$$H_\alpha \otimes t^n |\Lambda\rangle = 0 \;\; \forall\, n > 0 \;\; . \tag{2.2.14}$$

The weights of a HWR have the general form [23]

$$\lambda = \bar{\lambda} + (\lambda, \delta)\Lambda_0 + (\lambda, \Lambda_0)\delta \tag{2.2.15}$$

where $\bar{\lambda}$ is a weight of a HWR of $\underline{g}$. (Λ, Λ_0) is arbitrary and corresponds to a choice of zero-point for the gradation of the corresponding HWR (it is usually set to zero). $(\Lambda, \delta) = k$ is the level of a HWR

$$k|\Lambda\rangle = (\Lambda, \delta)|\Lambda\rangle \;\; . \tag{2.2.16}$$

Since $[k, \hat{g}] = 0$, the central term acts as a scalar and the level has the same value on the whole (irreducible) representation. One can show that it has to be a positive integer for unitarity to be satisfied. From (2.2.12) and (2.2.8) it follows then that $(\bar{\Lambda}, \theta) \leq k$, where $\bar{\Lambda}$ is the projection of Λ on the root space of g. We shall be interested in level 1 representation for which

$$\Lambda = \bar{\Lambda} + \Lambda_0 \tag{2.2.17}$$

One can show [23] that $|\Lambda|^2 \geq |\lambda|^2$. Using (2.2.14)-(2.2.16) this implies that all weights of a HWR Lie inside the paraboloid

$$|\bar{\lambda}|^2 + 2k(\lambda|\Lambda_0) \leq |\Lambda|^2 \quad . \tag{2.2.18}$$

According to (2.2.15), δ is the axis of the paraboloid. Orthogonal to δ are the weights $\bar{\lambda}$ of the finite Lie algebra g.

2.3. Twisted Affine Kac-Moody Algebras [24,23]

We start again with the compact Lie algebra g with commutation relations (2.1.1). An automorphism τ of g leaves (2.1.1) unchanged

$$[\tau(T_a), \tau(T_b)] = if_{abc}\tau(T_c) \quad . \tag{2.3.1}$$

Suppose τ is of order m

$$\tau^m = 1 \quad . \tag{2.3.2}$$

In a complex basis one can divide $\underline{g}$ into eigenspaces $\underline{g}_k$ of τ

$$\underline{g} = \bigoplus_{k=0}^{m-1} \underline{g}_k \tag{2.3.3}$$

such that

$$\begin{aligned} \tau(T) &= r^k T \quad \text{if } T \in \underline{g}_k \\ r &= \exp\left(-2\pi i/m\right) \end{aligned} \tag{2.3.4}$$

This introduces a grading in $\underline{g}$:

$$[T, T'] \in \underline{g}_{j+k} \quad \text{if } T \in \underline{g}_j \ , \ T' \in \underline{g}_k \tag{2.3.5}$$

For the (non hermitean) $T_a \in \underline{g}_k$, one defines a twisted Kac-Moody algebra $\hat{\underline{g}}_\tau$ with generators labelled by fractional indices of the form :

$$\begin{aligned} T_a \otimes t^{n/m} \ ; \ n = jm + (n) \ \text{ if } T_a \in \underline{g}_{(n)} \\ j \in \mathbb{Z}, \quad 0 \leq (n) \leq m - 1 \end{aligned} \tag{2.3.6}$$

Apart from the range of the indices, the commutation relations are the same as before :

$$\begin{aligned} [T_a \otimes t^{n/m}, T_b \otimes t^{n'/m}] &= if_{abc} T_c \otimes t^{(n+n')/m} + \frac{n}{m}(T_a, T_b)\delta_{n,-n'}k \\ [d, T_a \otimes t^{n/m}] &= \frac{n}{m} T_a \otimes t^{n/m} \\ [d, k] &= 0 \end{aligned} \tag{2.3.7}$$

For $\tau = 1$, $m = 1$ and one recovers the untwisted algebra, for which the grading is the same for all elements of g. For this reason this is also called the homogeneous construction.

When τ is an inner automorphism, one gets in this way a Kac-Moody algebra isomorphic to the untwisted algebra [24]. In this case $\tau(T_a) = \gamma T_a \gamma^{-1}$, $\gamma \in G$, the Lie group of g. Since automorphisms produced by conjugate elements $\gamma' = \gamma_0 \gamma \gamma_0^{-1}$ give isomorphic twisted algebras, one can choose $\gamma = \exp(i\rho^i H^i)$, such that, acting on the Cartan subalgebra and the step operators

$$\begin{aligned} \tau(H_\alpha) &= H_\alpha \\ \tau(E_\alpha) &= \exp i(\rho, \alpha) E_\alpha \end{aligned} \tag{2.3.8}$$

If τ has order m, then

$$m(\rho, \alpha) = 2\pi n, \quad \forall\, \alpha \in \Delta,\ n \in \mathbb{Z} \tag{2.3.9}$$

To show that we reobtain in such a way a Kac-Moody algebra isomorphic to the untwisted one, let us redefine the generators in the following way :

$$\begin{aligned} E'_\alpha \otimes t^p &= E_\alpha \otimes t^{p+(\rho,\alpha)/2\pi} \\ H'_\alpha \otimes t^p &= H_\alpha \otimes t^p + k(\rho, \alpha)\delta_{p,0}/2\pi \\ d' &= d - (\rho, H)/2\pi \end{aligned} \tag{2.3.10}$$

which satisfy the untwisted commutation relations.

However, in a highest weight representation, the spectrum will look differently, because the derivation operator d ($d = -L_0$) is different. For example, the highest weight state (lowest energy state) will transform under a representation of $\underline{g}_0$, the subspace of $\underline{g}$ invariant under τ. Hence one gets <u>symmetry breaking</u>.

If the automorphism τ is outer, the twisted Kac-Moody algebra $\hat{g}_\tau \neq \hat{g}$ will be a subalgebra of $\hat{g}$ [23]: even more symmetry breaking.

3. VERTEX OPERATORS

3.1. Untwisted, Frenkel-Kac-Segal, Or Homogeneous Construction

The treatment of Kac-Moody algebras in Chapter 2 was abstract in the sense that it was based only on the commutation relations and the action on weight states. The aim of the present paragraph is to give a concrete realization of the generators as functions of harmonic oscillator operators acting on a Fock space, which will be identified with the Heisenberg generators, and momentum operators corresponding to the CSA of the underlying Lie algebra. This construction uses as an intermediate step the vertex operator of string theory, the moments of which yield the remaining step generators of the untwisted (homogeneous) Kac-Moody algebra.

We already noticed that the Kac-Moody algebra is equivalent to a current algebra with discrete momenta. In the context of string theory, this means that one compactifies the string on a torus. This happens for example for the bosonic part of the heterotic string for which 16 out of 26 dimensions are compactified. The Kac-Moody algebra in the Frenkel-Kac-Segal construction (FKS) then corresponds to the gauge algebra.

A torus T^1 is a circle which has the same topology as a finite segment of the real axis $\mathbb{R}$ with ends identified. Equivalently, $T^1 = \mathbb{R}/Q$, where Q is the set of points $n\ell$, $n \in \mathbb{Z}$ and ℓ some real number. In general, T^d is the product of d copies of T^1. A lattice Q in d dimensions is the set of points $\sum_{i=1}^{d} n_i\ell_i$, $n_i \in \mathbb{Z}$, ℓ_i some basis vectors of $\mathbb{R}^d$. Then

$$T^d = \mathbb{R}^d/Q \tag{3.1.1}$$

Q could be the root lattice of the finite Lie algebra $\underline{g}$ of rank d.

For a closed string on a torus the vertex operator obeys periodic boundary conditions. In the next paragraph, we shall consider twisted vertex operators. In this case, the torus is replaced by an orbifold. The vertex operator will only be periodic up to an automorphism of the lattice Q. Our orbifold $\mathcal{O}$ is defined as

$$\mathcal{O} = T^d/\text{action of } w, \quad w \in \text{Aut } Q \tag{3.1.2}$$

We now recall the construction of the vertex operator of a string moving on a torus $T^d = \mathbb{R}^d/Q$, where Q will be the root (or weight) lattice of a simply-laced finite, simple Lie algebra $\underline{g}$ of rank d.

The components of the Fubini-Veneziano operator are

$$X^I(z) = q^I - ip^I \ln z + i \sum_{n\in\mathbb{Z}^*} h_n^I \frac{z^{-n}}{n} \tag{3.1.3}$$

where $z = \exp i(\tau \pm \sigma)$, τ and σ are the string variables. The hermitean operators q and p are the position and momentum operators of the cm of the string (also called "zero modes"), and h_n ($h_n^\dagger = h_{-n}$, $h_0 \equiv p$) are the harmonic oscillators, with the usual commutation relations (compare with (2.2.5))

$$[q^I, p^J] = i\delta_{I,J} \tag{3.1.4}$$
$$[h^I_n, h^J_{n'}] = n\ \delta_{I,J}\delta_{n,-n'} \quad I, J = 1, ...d \quad n, n' \in \mathbb{Z} \tag{3.1.5}$$

A closed string is unchanged when $\sigma \to \sigma + 2\pi$. Hence X is unchanged up to an arbitrary vector in Q

$$X(ze^{2\pi i}) \equiv X(z) \mod Q \tag{3.1.6}$$

In the FKS construction, the operators p^I satisfying $[p^I, p^J] = 0$, $I, J = 1, \ldots, d$, generate the CSA. This relation, together with (3.1.5) corresponds to Eq. (2.2.5), and the oscillators $h_n (n \neq 0)$ form the Heisenberg algebra. We shall clearly identify $(h_n, \alpha) = h^I_n \alpha^I \equiv H_\alpha \otimes t^n$.

The vertex operator is then

$$\begin{aligned} U(\alpha, z) &= z^{\alpha^2/2} : e^{i(\alpha, X(z))} c_\alpha : \\ &\equiv z^{\alpha^2/2} e^{i(q,\alpha)} c_\alpha z^{(\alpha,p)} e^{i(X_-(z),\alpha)} e^{i(X_+(z),\alpha)} \end{aligned} \tag{3.1.7}$$

where α is a root of g and

$$X_\pm(z) = i \sum_{\pm n > 0} h_n \frac{z^{-n}}{n} \tag{3.1.8}$$

The operator c_α is needed to get the right commutation relations, as explained in Ref.[24] and in Paragraph 3.4. .

The Hilbert space on which the vertex operator acts is

$$\mathcal{H} = \mathcal{F} \otimes |P\rangle \tag{3.1.9}$$

$\mathcal{F}$ is the Fock space of the oscillators h_n with a ground state satisfying

$$h^I_n|\ \rangle = 0 \quad n > 0 \tag{3.1.10}$$

$|P\rangle$ is an ∞ dimensional space of states with momenta on the lattice P, caracterized by following considerations : The action of p and q on $|\lambda\rangle \in |P\rangle$ is

$$p^I|\lambda\rangle = \lambda^I|\lambda\rangle \tag{3.1.11}$$

$$e^{i(q,\alpha)}|\lambda\rangle = |\alpha+\lambda\rangle \tag{3.1.12}$$

In accordance with (3.1.6) we require the boundary condition

$$U(\alpha, ze^{2\pi i}) = U(\alpha, z) \tag{3.1.13}$$

which is obviously satisfied by the oscillator part. To satisfy it for states $|\lambda\rangle$ in $|P\rangle$ it is necessary that

$$(\lambda,\alpha) + \alpha^2/2 \in \mathbb{Z} \tag{3.1.14}$$

Suppose the ground state of p satisfies

$$p^I|\bar{p}\rangle = \bar{p}^I|\bar{p}\rangle \tag{3.1.15}$$

If $\alpha \in Q$, (3.1.14) implies

$$\bar{p} \in Q^* \tag{3.1.16}$$

so that $P \simeq Q$ consists of all the points of the root lattice Q of g shifted by $\bar{p}$.

We now use one instance of the Quantum equivalence theorem (Q.E.T) of Goddard and Olive [24] : the Virasoro and Sugawara constructions (see below) are equivalent only if $\Lambda = \bar{p} + \Lambda_0$ is the highest weight of a level one representation (see Eq. (2.2.17)). This strongly suggests that the construction of KM currents out of free bosonic fields (3.1.3) and their exponentials is only possible for such weights. Higher level vertex constructions have been considered in [16]. For the heterotic string, $\bar{p} = 0$ and $\Lambda = \Lambda_0$ belongs to the trivial representation of g. The general expression for c_α is [24]

$$c_\alpha = \sum_{\beta\in Q} \varepsilon(\alpha,\beta)|\beta+\bar{p}><\beta+\bar{p}| \tag{3.1.17}$$

where $\varepsilon(\alpha,\beta)$ is the cocycle entering the commutation relations (2.1.4).

One also defines the Klein factor $\hat{c}_\alpha$

$$\hat{c}_\alpha = e^{i(q,\alpha)}c_\alpha \tag{3.1.18}$$

$$\hat{c}_\alpha\hat{c}_\beta = \varepsilon(\alpha,\beta)\hat{c}_{\alpha+\beta} \tag{3.1.19}$$

$$= (-)^{(\alpha,\beta)}\hat{c}_\beta\hat{c}_\alpha \tag{3.1.20}$$

Finally, the generators of the untwisted (or homogeneous), affine simply-laced Kac-Moody algebra in a level one representation are given by

$$U^i(\alpha) = \frac{1}{2\pi i}\oint \frac{dz}{z} z^i U(\alpha, z) \tag{3.1.21}$$

together with the momentum operators p^I and the generators h^I_n of the Heisenberg algebra. Indeed, it will be shown in Paragraph 3.5 that $U^i(\alpha)$ have the same commutation relations (2.2.7) as $E_\alpha \otimes t^i$, for $k = 1$.

The Q.E.T. implies that the generators of a Virasoro algebra are either given by

$$L_n = \frac{1}{2} \sum_{I=1}^{\mathrm{rank} g} \sum_{n' \in \mathbb{Z}} : h^I_{n-n'} h^I_{n'} : \tag{3.1.22}$$

or, for a level 1 representation of the Kac-Moody algebra $\hat{g}$ (g simple, simply-laced) via the Sugawara form

$$L_n = \frac{1}{2 + Q_\theta} \sum_{a=1}^{\dim g} \sum_{m \in \mathbb{Z}} : U^a_{n-m} U^a_m : \tag{3.1.23}$$

Here U^a_n include all the generators of the Kac-Moody algebra.

Q_θ is the eigenvalue of the Casimir operator in the adjoint representation of g (with highest weight θ) and a normal ordering is necessary according to which U^a_m with positive m are moved to the right of those with negative m.

3.2. Automorphisms of the Root Lattice

As a preparation for the twisted vertex operator we consider automorphisms w of the root lattice Q which satisfy

$$\begin{gathered} \alpha \longrightarrow w\alpha \\ (w\alpha, w\beta) = (\alpha, \beta) \end{gathered} \tag{3.2.1}$$

This induces an automorphism on the CSA, since $\underline{h}$ and $\underline{h}'$ are isomorphic (see Paragraph 2.1)

$$wH_\alpha = H_{w\alpha} \tag{3.2.2}$$

Let the order of w be m

$$w^m = 1 \tag{3.2.3}$$

We divide the complexified vector spaces $\underline{h} \sim \underline{h}'$ into eigenspaces of w

$$\underline{h} = \bigoplus_{n=0}^{m-1} \underline{h}_n \tag{3.2.4}$$

The spaces $\underline{h}_n$ and $\underline{h}_{n'}$, orthogonal if $n + n' \not\equiv 0 \mod m$, correspond to the eigenvalues r^n,

$$r = \exp(-2\pi i/m) \tag{3.2.5}$$

Call $p_n(\alpha)$ the projection of α on $\underline{h}'_n$

$$\alpha = \sum_{n=0}^{m-1} p_n(\alpha) \tag{3.2.6}$$

$$wp_n(\alpha) = r^n p_n(\alpha) \tag{3.2.7}$$

The inner automorphisms of Q form the Weyl group $W(g)$ generated by the Weyl reflections about simple roots α_i of $\underline{g}$:

$$w_i\beta = \beta - (\beta, \alpha_i)\alpha_i \tag{3.2.8}$$

The quotient Aut $Q/W(g)$ is then equal to the automorphism group D of the Dynkin diagram. $D = \mathbb{Z}_2 = \{1, -1\}$ for the simple Lie algebras $A_\ell(\ell > 1)$, $D_\ell(\ell > 4)$ and E_6. $D = S_3$, the permutation group of 3 elements, for the algebra D_4. D is trivial for the algebras A_1, E_7, E_8, B_ℓ, C_ℓ, G_2, F_4.

There is a close, although not a one to one relation between $w \in$ Aut Q and $\tau \in$ Aut g (see Paragraph 2.3). If we set

$$\tau(H_\alpha) = wH_\alpha \tag{3.2.9}$$

it follows from $[E_\alpha, E_{-\alpha}] = H_\alpha$ that

$$\tau E_\alpha = \psi_\alpha E_{w\alpha}$$

with ψ_α some phase.
If $\alpha + \beta$ is a root, the Eq. (2.1.4), $[E_\alpha, E_\beta] = \varepsilon(\alpha, \beta)E_{\alpha+\beta}$, requires

$$\frac{\psi_\alpha \psi_\beta}{\psi_{\alpha+\beta}} = \frac{\varepsilon(\alpha, \beta)}{\varepsilon(w\alpha, w\beta)} \tag{3.2.10}$$

One easily sees that $\psi'_\alpha = \psi_\alpha f_\alpha$ is also a solution provided $f_\alpha f_\beta = f_{\alpha+\beta}$. For w-invariant roots $\alpha, \beta \in \Delta \cap \underline{h}'_0$, $\psi_\alpha \psi_\beta = \psi_{\alpha+\beta}$. With $\phi|_{\Delta \cap h'_0} = \psi$ and $\phi_\alpha \phi_\beta = \phi_{\alpha+\beta}$ everywhere, $\psi'_\alpha = \psi_\alpha \phi_\alpha^{-1}$ defines a τ such that $\tau(E_\alpha) = E_\alpha$ if $w\alpha = \alpha$. Consider $\hat{\alpha} = \sum_{n=0}^{m-1} w^n(\alpha)$. Then $\tau^m(E_\alpha) = \psi'_\alpha \psi'_{w\alpha} \cdots \psi'_{w^{m-1}\alpha} E_\alpha$. Since $w\hat{\alpha} = \hat{\alpha}$, $\psi'_{\hat{\alpha}} = 1$ and $\tau^m(E_\alpha) = (-)^{(\hat{\alpha}, \alpha)} E_\alpha$. Hence, if w has order m, τ has order m or $2m$ depending on whether $(\hat{\alpha}, \alpha) \in 2\mathbb{Z}$, $\forall\, \alpha \in \Delta$ or not [25].

3.3. Automorphisms : Example of su(3)

The impatient reader can go to paragraph 3.4. The finite Lie algebra $su(3) = A_2$ has the Dynkin diagram

$$\alpha_1 \circ - - - \circ \alpha_2 \tag{3.3.1}$$

and the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{3.3.2}$$

The automorphism w leaves A invariant. The Weyl group $W(g)$ of inner automorphisms is generated by the reflections w_i about the simple roots α_i $(i = 1, 2)$. Conjugate elements of W correspond to equivalent constructions. W has six elements which fall into 3 conjugacy classes which we denote H (homogeneous, untwisted), M (mixed, twisted), P (principal, twisted)

$$\begin{aligned} H &= \{1\} \\ M &= \{w_1, w_2, w_1 w_2 w_1\} \\ P &= \{w_1 w_2, w_2 w_1\} \end{aligned} \tag{3.3.3}$$

$w_1 w_2$ is also called the Coxeter element. In each class pick a representative.

Outer automorphisms are generated by the automorphism D of the Dynkin diagram

$$D \ : \ \alpha_1 \leftrightarrow \alpha_2 \tag{3.3.4}$$

or by a reflection

$$\begin{aligned} R \ &: \ \alpha_i \to -\alpha_i \\ R \ &= w_1 w_2 w_1 D \end{aligned} \tag{3.3.5}$$

We give some details for the principal automorphism P. We choose for w_p the Coxeter element, defined by

$$w_p \alpha_1 = \alpha_2 \ \ ; w_p \alpha_2 = -\alpha_3 = -\alpha_1 - \alpha_2 \tag{3.3.6}$$

This is a rotation of 120^0 of the root diagram. In the space spanned by α_1 and α_2, w_p is the matrix

$$w_p = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \tag{3.3.7}$$

Notice

$$\det\,(1 - w_p) = \det A \tag{3.3.8}$$

This is a general property of the Coxeter element.

Obviously

$$\underline{h}'_0 = 0 \tag{3.3.9}$$
$$\underline{h}'_1 = \mathbb{C}(-r^{-1}\alpha_1 + \alpha_2) \tag{3.3.10}$$
$$\underline{h}'_2 = \mathbb{C}(-r^{-2}\alpha_1 + \alpha_2) \tag{3.3.11}$$

We used $r = e^{-2\pi i/3}$ and $1+r+r^2 = 0$. From (3.3.9) also follows that $1+w+w^2 = 0$, in agreement with (3.3.6).

The projections on h'_n are :

$$\begin{aligned} p_1(\alpha_1) &= (r - r^2)^{-1}(-r^{-1}\alpha_1 + \alpha_2) \\ p_2(\alpha_1) &= -(r - r^2)^{-1}(-r^{-2}\alpha_1 + \alpha_2) \\ p_1(\alpha_2) &= r^{-2}p_1(\alpha_1) \\ p_2(\alpha_2) &= r^{-1}p_2(\alpha_1) \end{aligned} \tag{3.3.12}$$

For the <u>outer automorphism R</u>

$$w_R(\alpha_i) = -\alpha_i \quad ,i = 1,2 \tag{3.3.13}$$

$$w_R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad ,\det(1 - w_R) = 4 \tag{3.3.14}$$

$$\underline{h}'_0 = 0, \quad \underline{h}'_1 = \underline{h}' \tag{3.3.15}$$

The <u>mixed case</u> w_M, $w_M^2 = 1$ is defined by

$$w_M(\alpha_1) = -\alpha_1 \quad ; w_M\alpha_2 = \alpha_3 \tag{3.3.16}$$

The matrix representation is

$$w_M = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \; ; \quad \det\,(1 - w_M) = 0 \tag{3.3.17}$$

The eigenspaces are

$$\underline{h}'_0 = \mathbb{C}(\alpha_1 + 2\alpha_2) \;\; ; \;\; \underline{h}'_1 = \mathbb{C}\alpha_1 \tag{3.3.18}$$

The projection are

$$\begin{aligned} p_0(\alpha_1) &= 0 \qquad\qquad\quad ; \;\; p_1(\alpha_1) = \alpha_1 \\ p_0(\alpha_2) &= \frac{1}{2}(\alpha_1 + 2\alpha_2) \;\; ; \;\; p_1(\alpha_2) = -\frac{1}{2}\alpha_1 \end{aligned} \tag{3.3.19}$$

3.4. Twisted Vertex Operator

When a closed string moves on an orbifold (see eq.(3.1.2)), the boundary condition for the (modified) Fubini-Veneziano operator will be

$$X(ze^{2\pi i}) = wX(z) + \alpha \tag{3.4.1}$$

where w is an automorphism of Q of order m

$$w^m = 1 \tag{3.4.2}$$

The corresponding condition on the (twisted) vertex operator requires
a) oscillators with fractional indices
b) generalization of the Klein factor $\hat{c}(\alpha)$
c) generalization of the Hilbert space.

We immediately give the resulting vertex operator, which generalizes (3.1.7)

$$U(\alpha, z) = z^{p_0(\alpha)^2/2}\sigma(\alpha)z^{(p_0(\alpha),p)}e^{iX_-(\alpha,z)}e^{iX_+(\alpha,z)} \tag{3.4.3}$$

$p_0(\alpha)$ is the projection of the root α on the w-invariant subspace $\underline{h}'_0$. p is the momentum operator, and

$$\begin{aligned} X_\pm(\alpha, z) &= i \sum_{\pm n>0} h_\alpha(n) \frac{z^{-n/m} m}{n} \\ h_\alpha(n) &\equiv p_{(n)}(H_\alpha) \otimes t^{n/m}, \quad (n) \equiv n \bmod m \end{aligned} \tag{3.4.4}$$

so that the $h_\alpha(n)$ obey the commutation relations

$$[h_\alpha(n), h_\beta(n')] = \frac{n}{m}(p_n(\alpha), p_{n'}(\beta))\delta_{n,-n'} \tag{3.4.5}$$

We could also have started from a CSA basis $\{ h^I_n | \, h^I_n \in \underline{h}_n\}$, such that $(h^I_n, h^J_{n'}) = \delta_{n+n',m}\delta^{I,J}$; one would then consider the twisted Heisenberg algebra generators

$$h^I_{n \bmod m} \otimes t^{n/m} \equiv h^I_n \tag{3.4.6}$$

and

$$h_\alpha(n) = p_{(n)}(\alpha)^I h^I_n \tag{3.4.7}$$

Thus defined, $X_\pm(\alpha, z)$ satisfies the boundary condition (3.4.1), since

$$\begin{aligned} wX_\pm(\alpha,z) &= i\sum_{\pm n>0} wh_\alpha(n)\frac{z^{-n/m}m}{n} \\ &= i\sum_{\pm n>0} h_\alpha(n)\frac{(ze^{-2\pi i})^{-n/m}m}{n} \end{aligned} \tag{3.4.8}$$

It is also clear that for $w = 1$, (3.4.3) reduces to the homogeneous, untwisted, F.K.S. construction (3.1.7), with $p_0(\alpha) = \alpha$ and

$$\begin{aligned} \sigma(\alpha) = \hat{c}_\alpha &= e^{i(q,\alpha)}c_\alpha \\ h_\alpha(n) &= (h_n, \alpha) \end{aligned} \tag{3.4.9}$$

Another interesting case is the principal, twisted construction, when w is the Coxeter element

$$w = \prod_{i=1}^{\ell} w_i \tag{3.4.10}$$

where w_i is a Weyl reflection about the simple root α_i of the Lie algebra $\underline{g}$ with rank ℓ. Then $p_0(\alpha) = 0$ and $\sigma(\alpha)$ is no longer necessary (this will be justified in his general setting later, paragraph 3.6). In this case

$$U(\alpha,z) = V(\alpha,z) \equiv e^{iX_-(\alpha,z)}e^{iX_+(\alpha,z)} \tag{3.4.11}$$

Going back to the general case, our plan will be to show first what are the algebraic properties of the operators $\sigma(\alpha)$ necessary to get the commutation relations of the twisted Kac-Moody algebra. The next point will be to describe the Hilbert space on which $\sigma(\alpha)$ acts. It is clear that the oscillators will act on a Fock space.

It is convenient to change the variables :

$$x^m = z \ . \tag{3.4.12}$$

The Laurent coefficients of the twisted vertex operator

$$U^{i/m}(\alpha) = \frac{1}{2\pi i}\oint \frac{dx}{x}x^i U(\alpha, x^m) \tag{3.4.13}$$

will realize the generators $p_{(i)}(E_\alpha) \otimes t^{i/m}$ of the twisted Kac-Moody algebra.

A problem arises when we try to calculate the commutator $[U^i(\alpha), U^j(\beta)]$ because, as is well known the product $U(\alpha, x^m)U(\beta, y^m)$ is singular for $x = y$. This is specially true for the oscillator part $V(\alpha, x^m)$. The cure is to introduce the non singular normal ordered product, using the Baker-Hausdorff formula

$$V(\alpha, x^m)V(\beta, y^m) =: V(\alpha, x^m)V(\beta, y^m) : \times \exp\,[iX_+(\alpha, x^m), iX_-(\beta, y^m)] \tag{3.4.14}$$

Lemma [16]

$$[X_+(\alpha, x^m), X_-(\beta, y^m)] = \sum_{s=0}^{m-1} \ln\left(1 - \frac{r^s y}{x}\right)^{(\alpha, w^s\beta)} \qquad |y| < |x| \tag{3.4.15}$$

One knows [24] that in the F.K.S. construction ($w = 1$) the products $U(\alpha, x^m)U(\beta, y^m)$ and $U(\beta, y^m)U(\alpha, x^m)$ differ only by the range of x and y. Hence we shall choose $\sigma(\alpha)$ in such a way that the same is true in our general case.

Using (3.4.3), (3.4.11), (3.4.14) and (3.4.15) we get for $|x| > |y|$:

$$U(\alpha, x^m)U(\beta, y^m) = Z(\alpha, x, \beta, y) \prod_{s=0}^{m-1} \left(1 - \frac{r^s y}{x}\right)^{(\alpha, w^s\beta)} \Sigma(\alpha, x, \beta, y) \tag{3.4.16}$$

with

$$Z(\alpha, x, \beta, y) = x^{\frac{m}{2}|p_0(\alpha)|^2} y^{\frac{m}{2}|p_0(\beta)|^2} : V(\alpha, x^m)V(\beta, y^m) : = Z(\beta, y, \alpha, x) \tag{3.4.17}$$

$$\Sigma(\alpha, x, \beta, y) = \sigma(\alpha)x^{m(p_0(\alpha),p)}\sigma(\beta)y^{m(p_0(\beta),p)} \tag{3.4.18}$$

For $|x| < |y|$ we get similar expressions after the change $\alpha \leftrightarrow \beta$, $x \leftrightarrow y$. We want to symmetrize the last two factors on the R.H.S. of (3.4.16). Notice that

$$\prod_{s=0}^{m-1} \left(1 - \frac{r^s y}{x}\right)^{(\alpha, w^s\beta)} = \prod_{s=0}^{m-1} (x - r^s y)^{(\alpha, w^s\beta)} x^{-m(p_0(\alpha), p_0(\beta))} \tag{3.4.19}$$

This follows from

$$\sum_{s=0}^{m-1} w^s\alpha = mp_0(\alpha) \tag{3.4.20}$$

Indeed, the L.H.S. is invariant under w and hence is proportional to $p_0\alpha$. The factor m results from counting. Compare the example $su(3)$ in Paragraph 3.3. .

Next one finds, using $(\beta, w\alpha) = (w^{-1}\beta, \alpha)$,

$$\begin{aligned}\prod_{s=0}^{m-1}(y - r^s x)^{(\beta, w^s\alpha)} &= \prod_{s=0}^{m-1}(-r^s)^{(\beta, w^s\alpha)} \prod_{s=0}^{m-1}(x - r^{-s}y)^{(\beta, w^s\alpha)} \\ &= S(\alpha,\beta)\prod_{s=0}^{m-1}(x - r^s y)^{(\alpha, w^s\beta)}\end{aligned} \tag{3.4.21}$$

where the so-called symmetry factor $S(\alpha, \beta)$ is given by

$$S(\alpha,\beta) = \prod_{s=0}^{m-1}(-r^s)^{(\beta, w^s\alpha)} \tag{3.4.22}$$

$$= \prod_{s=0}^{m-1}(-r^{-s})^{(\alpha, w^s\beta)} \tag{3.4.23}$$

$$= (-1)^{m(p_0(\alpha), p_0(\beta)} r^{-\sum_{s=0}^{m-1} s(\alpha, w^s\beta)} \tag{3.4.24}$$

(3.4.23) shows that

$$S(\alpha,\beta) = S^{-1}(\beta,\alpha) \tag{3.4.25}$$

In analogy with the relation, true in F.K.S. construction :

$$x^{(\alpha,p)}e^{i(q,\beta)} = e^{i(q,\beta)}x^{(\alpha,p)}x^{(\alpha,\beta)} \tag{3.4.26}$$

we require

$$\begin{aligned}x^{m(p_0(\alpha),p)}\sigma(\beta) &= \sigma(\beta)x^{m(p_0(\alpha),p)}x^{m(p_0(\alpha),p_0(\beta))} \\ y^{m(p_0(\beta),p)}\sigma(\alpha) &= \sigma(\alpha)y^{m(p_0(\beta),p)}y^{m(p_0(\beta),p_0(\alpha))}\end{aligned} \tag{3.4.27}$$

Hence, with (3.4.18)

$$\begin{aligned}\Sigma(\alpha, x, \beta, y) &= \sigma(\alpha)\sigma(\beta)x^{m(p_0(\alpha),p)}y^{m(p_0(\beta),p)}x^{m(p_0(\alpha),p_0(\beta))} \\ \Sigma(\beta, y, \alpha, x) &= \sigma(\beta)\sigma(\alpha)y^{m(p_0(\beta),p)}x^{m(p_0(\alpha),p)}y^{m(p_0(\beta),p_0(\alpha))}\end{aligned} \tag{3.4.28}$$

Putting (3.4.19) and (3.4.28) into (3.4.16), we see that the factor $x^{m(p_0(\alpha),p_0(\beta))}$ cancels.

Taking (3.4.21) into account, it follows that (3.4.16) becomes completely symmetric provided

$$\sigma(\alpha)\sigma(\beta) = S(\alpha,\beta)\sigma(\beta)\sigma(\alpha) \tag{3.4.29}$$

This is the fundamental relation which together with the condition (3.4.27) and the expression of S given by (3.4.22) specifies the algebraic properties of $\sigma(\alpha)$, which will be used in the discussion of the Hilbert space on which $\sigma(\alpha)$ acts.

We now can write the commutator for the Kac-Moody generators in the following way

$$[U^{\frac{i}{m}}(\alpha), U^{\frac{j}{m}}(\beta)] = \frac{1}{(2\pi i)^2} \oint_{|y|<|x|} \oint - \oint_{|x|<|y|} \oint \frac{dx}{x} \frac{dy}{y} \times x^i y^j \, C(\alpha, x^m, \beta, y^m) \tag{3.4.30}$$

where

$$C(\alpha, x, \beta, y) = Z(\alpha, x, \beta, y) \prod_{s=0}^{m-1} (x - r^{-s} y)^{(\beta, w^s \alpha)} \times \sigma(\alpha)\sigma(\beta) x^{m(p_0(\alpha), p)} y^{m(p_0(\beta), p)} \tag{3.4.31}$$

It is also interesting to apply formulas (3.4.22) to (3.4.24) for $S(\alpha, \beta)$ in several cases. For $w = 1$, $m = r = 1$ so

$$S(\alpha, \beta) = (-1)^{(\alpha, \beta)} \tag{3.4.32}$$

Using (3.4.9) this agrees with (3.1.20) and (3.4.29).

For $\underline{h}_0 = 0$ (no w-invariant subspace of $\underline{h}$) (3.4.24) gives :

$$S(\alpha, \beta) = r^{-\sum_{s=0}^{m-1} s(\alpha, w^s \beta)} \tag{3.4.33}$$

If w is the Coxeter element Eq. (3.4.10), $\underline{h}_0 = 0$. In addition $\det(1 - w) = \det A$ (see Eq. (3.3.8)) where A is the Cartan matrix of $\underline{g}$. Then one can show [18] that

$$\sum_{s=0}^{m-1} s(\alpha, w^s \beta) \equiv 0 \mod m \tag{3.4.34}$$

Hence

$$S(\alpha, \beta) = 1 \quad \forall \, \alpha, \beta \, \in Q \tag{3.4.35}$$

In this case, $\sigma(\alpha)$ is not necessary and the vertex operator is $U(\alpha, x^m) = V(\alpha, x^m)$ (cf Eq. (3.4.11)).

Example : $g = su(m) = A_{m-1}$. There are $m-1$ simple roots α_i. The Coxeter element is realized by $w\alpha_i = \alpha_{i+1}$, $i = 1, \ldots, m-2$ and $w\alpha_{m-1} = -\sum_{i=1}^{m-1} \alpha_i \equiv -\theta$. One verifies that $w(-\theta) = \alpha$, so that $w^m = 1$. Putting $\alpha_i^2 = 2$, it follows that $(\alpha_i, \alpha_{i\pm1}) = -1$ and $(\alpha_i, \alpha_j) = 0$ for $i \neq j$, $j \pm 1$. Then one easily verifies (3.4.34).

3.5. The States and the Commutation Relations

Recall Eqs (3.4.3), (3.4.4), (3.4.13), to write the generators of the twisted Kac-Moody algebra as

$$U^{i/m}(\alpha) = \frac{1}{2\pi i}\oint \frac{dx}{x} x^i U(\alpha, x^m) \tag{3.5.1}$$

$$U(\alpha, z) = x^{mp_0(\alpha)^2/2}\sigma(\alpha)x^{m(p_0(\alpha),p)}e^{iX_-(\alpha,x^m)}e^{iX_+(\alpha,x^m)} \tag{3.5.2}$$

Acting on the vacuum (with $\bar{p} = 0$) one sees that the residum in (3.5.1) will be non zero for a positive n' such that

$$i + \frac{m}{2}|p_0(\alpha)|^2 + n' = 0 \tag{3.5.3}$$

Since $m(p_0(\alpha), p_0(\alpha)) = \sum_{j=0}^{m-1}(w^j\alpha, \alpha) \in \mathbb{Z}$, it follows that [17]

$$i \in \mathbb{Z} \ \text{ or } i \in \mathbb{Z} + \frac{1}{2} \tag{3.5.4}$$

To study the spectrum created by $U^{i/m}(\alpha)$ acting on the vacuum, consider the derivation operator d equal to minus the Virasoro operator L_0 :

$$L_0 = \frac{1}{2}|p_0(\alpha)| + \sum_{I=1}^{\mathrm{rank}\, g}\sum_{n'\in\mathbb{Z}} :h^I_{n-n'}h^I_{n'}: \tag{3.5.5}$$

then one finds

$$[d\ , U^{n/m}(\alpha)] = \frac{n}{m}U^{n/m}(\alpha) \tag{3.5.6}$$

Starting with $d = 0$, $U^{i/m}(\alpha)$ creates states with $d = i/m$. From (3.5.3) it follows that $i + (m/2)|p_0(\alpha)|^2 \leq 0$ so that the states are limited by the paraboloid

$$d = -\frac{1}{2}|p_0(\alpha)|^2 \tag{3.5.7}$$

To compute the commutation relation (3.4.30) one has to look for poles. The behaviour of the integrand is dictated by the factor

$$\prod_{s=0}^{m-1}(x - r^s y)^{(\alpha, w^s\beta)} \tag{3.5.8}$$

If α and β are roots of a simply-laced finite Lie algebra, $(\alpha, w^n\beta)$ can take the only values ± 2, ± 1, 0. Hence (3.5.8) can have simple and double poles for $x = r^n y$. Then, by deforming integration contours

$$\begin{gathered}[U^{i/m}(\alpha), U^{j/m}(\beta)] = \frac{1}{(2\pi i)^2} \sum_{n \in I} \oint \frac{dy}{y} \oint_{r^n y} \frac{dx}{x}\, x^i y^j C(\alpha, x^m, \beta, y^m) \\ I(-1,-2) = \{n |\ (\alpha, w^n\beta) \in \{-1,-2\},\ \ 0 \le n \le m-1\}\end{gathered} \tag{3.5.9}$$

Example : For $su(3)$ and the principal construction, $w\alpha_1 = \alpha_2$, $w\alpha_2 = -\alpha_1 - \alpha_2$. Hence, for $\alpha = \beta$ one gets simple poles for $n = 1, 2$ and for $\alpha = -\beta$ a double pole for $n = 0$.

After a lengthy calculation one finds [17],

$$\begin{aligned}[U^{i/m}(\alpha), U^{j/m}(\beta)] = & \sum_{s \in I(-1)} \eta(\alpha, -s)\varepsilon_w(w^{-s}\alpha, \beta) r^{is} \prod_{k=1}^{m-1} (1 - r^k)^{(\alpha, w^{k+s}\beta)} \\ & \times U^{(i+j)/m}(w^{-s}\alpha + \beta) \\ & + \sum_{s \in I(-2)} \eta(\alpha, -s)\varepsilon_w(\alpha, -\alpha) r^{is} \prod_{k=1}^{m-1} (1 - r^k)^{-(\alpha, w^k\alpha)} \\ & \times \left[i\delta_{i+j,0} - m h_\beta(i+j)\right]\end{aligned} \tag{3.5.10}$$

where we have used $(\alpha, w^s\beta) = -2$ implies $w^s\beta = -\alpha$; $\varepsilon_w(\alpha, \beta)$ will be defined in Eq. (3.6.2), and $\eta(\alpha, s)$ in Eq. (3.6.8)

To this one should add the commutator

$$[h_k^I, U^{\frac{i}{m}}(\alpha)] = p_{-k}(\alpha)^I U^{\frac{i+k}{m}}(\alpha) \tag{3.5.11}$$

For the w-invariant subalgebra g_0 see Refs [17],[26],[25]. See also the example $su(3)$ (paragraphe 3.8).

3.6. Extension of the Lattice Q and the Hilbert Space

Taking into account the algebraic properties of $\sigma(\alpha)$, we can discuss the Hilbert space on which it acts. But first we need the two notions of central extension $\hat{Q}$ and group algebra $|Q\rangle$. Recall Eq. (3.4.29), with S given by (3.4.22)

$$\sigma(\alpha)\sigma(\beta) = S(\alpha, \beta)\sigma(\beta)\sigma(\alpha) \tag{3.6.1}$$

In analogy with (3.1.19) we write

$$\sigma(\alpha)\sigma(\beta) = \varepsilon_w(\alpha, \beta)\sigma(\alpha + \beta) \tag{3.6.2}$$

where $\varepsilon_w(\alpha,\beta)$ satisifes the cocycle condition (2.1.8), but (2.1.9) corresponds now to

$$\varepsilon_w(\alpha,\beta) = S(\alpha,\beta)\varepsilon_w(\beta,\alpha) \tag{3.6.3}$$

(3.6.2) can be considered as a projective representation of the abelian group Q. Call $\hat{Q}$ the extension of Q by the cyclic group T generated by $(-)^m r$, $r = e^{-2\pi i/m}$. An element of $\hat{Q}$ is denoted by the pair (α, a), $\alpha \in Q$, $a \in T$. Multiplication is defined as

$$(\alpha_1, a_1)(\alpha_2, a_2) = (\alpha_1 + \alpha_2, \varepsilon_w(\alpha_1,\alpha_2)\ a_1 a_2) \tag{3.6.4}$$

The extension $\hat{Q}$ is actually uniquely fixed by a choice of S, which in turn, in our context, depends on the automorphism w of Q. On the other hand, if ε_w satisifies (3.6.3), and f is some map $f : Q \longrightarrow T$, then $\varepsilon'_w = \varepsilon_w f_\alpha f_\beta f^{-1}_{\alpha+\beta} \equiv \varepsilon_w c_f(\alpha,\beta)$ also satisfies (3.6.3), so that, for a given extension, one gets a class of 2-cocycles, differing by the (exact) 2-cocycles c_f generated by f. This allows to normalize ε_w to

$$\varepsilon_w(0,\alpha) = \varepsilon_w(\alpha,-\alpha) = 1 \tag{3.6.5}$$

The group algebra $|Q\rangle$ was already introduced in Eq. (3.1.9) : it is an infinite-dimensional vector space spanned by elements $\alpha \in Q$ such that $|\alpha\rangle \in |Q\rangle$. Similarly one defines $|\alpha, a\rangle \in |\hat{Q}\rangle$. Let σ be a map which associates to each $\alpha \in Q$ a representative in $\hat{Q}$: $\sigma(\alpha) = (\alpha, f_\alpha)$, $f_\alpha \in T$; then

$$|\alpha, a\rangle = \sigma(\alpha)|0\rangle \tag{3.6.6}$$

Then, with (3.6.2) and (3.6.4)

$$\begin{aligned}\sigma(\alpha_1)|\alpha_2, f_{\alpha_2}\rangle &= \varepsilon_w(\alpha_1,\alpha_2) f_{\alpha_1} f_{\alpha_2} f^{-1}_{\alpha_1+\alpha_2} |\alpha_1+\alpha_2, f_{\alpha_1+\alpha_2}\rangle \\ &= \varepsilon_w(\alpha_1,\alpha_2) c_f(\alpha_1,\alpha_2)\sigma(\alpha_1+\alpha_2)|0\rangle\end{aligned} \tag{3.6.7}$$

in accordance with (3.6.2). One can also check that (3.6.1) is valid. This slightly more abstract but equivalent language avoids the explicit use of the position operator q which is ill defined when, for instance, $\underline{h}_0 = 0$ (no w-invariant subspace).

The automorphism $w : Q \rightarrow Q$ can be "lifted" to an automorphism of $\hat{Q}$ [17] such that $\hat{w}a = a$, $a \in T$ and

$$\hat{w}^n((\sigma(\alpha)) = \eta(\alpha,n)\sigma(w^n\alpha) \quad \eta \in T \tag{3.6.8}$$

The <u>boundary condition</u> $X(ze^{2\pi i}) = wX(z) \mod Q$ becomes, for the vertex operator :

$$U(\alpha, ze^{2\pi i}) = \hat{w}U(\alpha, z) \tag{3.6.9}$$

and, using (3.4.3) and (3.6.8)

$$\eta(\alpha, n)\sigma(w^n\alpha) = \sigma(\alpha)r^{-(n/2)m|p_0(\alpha)|^2 - nm(p_0(\alpha),p)} \tag{3.6.10}$$

This relation is usefull in expressing the commutation relations (3.5.10).

We also need a feature which was trivial in the F.K.S. construction : the subset $L \subset Q$ which is not invariant under the automorphism w

$$\begin{gathered} L = \underline{h}'^{\perp}_0 \cap Q \\ \underline{h}'^{\perp}_0 = \sum_{i=1}^{m-1} \underline{h}'_i \end{gathered} \tag{3.6.11}$$

Call $\hat{L}$ the image of L by σ. Now we are in a position to discuss the Hilbert space $\mathcal{H}$ on which the twisted vertex operator acts. This difficult problem was solved by Kac and Peterson [13] and independently by Lepowsky [14]. We give here without proof a simplified version [17,18]. The result is

$$\mathcal{H} = \mathcal{F} \otimes |\hat{Q}\rangle \otimes_{|\hat{L}\rangle} \mathcal{V} \tag{3.6.12}$$

$\mathcal{F}$ is the Fock space of the oscillators (3.4.6). $\hat{Q}$ has been defined previously and $\mathcal{V}$ is a new space yet to be defined. The meaning of the tensor product in (3.6.12) is the following [17]

$$\sigma(\alpha)|0\rangle \otimes_{|\hat{L}\rangle} \xi \equiv |0\rangle \otimes_{|\hat{L}\rangle} \varphi(\sigma(\alpha))\xi \quad \forall\, \alpha \in L,\ \xi \in \mathcal{V} \tag{3.6.13}$$

φ is a projective representation of L or a linear representation of $\hat{L}$ (so that $\mathcal{V}$ is its carrier space), and will be explicited in Paragraph 3.7. This amounts in fact to induce a representation of $\hat{Q}$ from that of $\hat{L}$, φ, on $\mathcal{V}$. One can show [18] that $\hat{Q}/\hat{L} \cong Q_0$, the projection of Q on $\underline{h}'_0$ so that, in view of (3.6.13),

$$|\hat{Q}\rangle \otimes_{|\hat{L}\rangle} \mathcal{V} \cong \bigoplus_{j \in \frac{\hat{Q}}{\hat{L}}} \mathcal{V}_j \cong |Q_0\rangle \otimes \mathcal{V} \tag{3.6.14}$$

One can interpret it considering that shifts in momentum along the invariant $\underline{h}_0$-direction change the state in $|Q_0\rangle$ (see (3.1.12)), whereas those corresponding to L act on $\mathcal{V}$ through φ. In order to further characterize φ, introduce [14]

$$R = \{\alpha \in L \mid S(\alpha, \beta) = 1\ \forall \beta \in L\} \tag{3.6.15}$$

R is a subgroup of L such that $S(\alpha, \beta)$ is non degenerate on the quotient $N = L/R$. Since $S = 1$ on R, $\hat{R}$ is an abelian subgroup of $\hat{L}$ and hence its irreducible representations are one-dimensional. The order of L/R can be shown (see paragraph

3.7) to be a square $|L/R| = c_w^2$.
The dimension of the representation φ has then to be equal to c_w and to solve the problem one has to find the projective representations of N. The reader will find a complete justification of this statement in [18] while we will give in paragraph 3.7 only the final result.

One can also show [18] that if $\underline{h}_0 = 0$, $\det(1-w) = |L/R| \det A$, A the Cartan matrix. If $\det(1-w) = \det A$, as is for example the case for the Coxeter element (see Eq. (3.3.8)), then $|L/R| = 1$ and $\mathcal{V}$ is trivial, in accordance with $S(\alpha,\beta) = 1$. Hence $\sigma(\alpha)$ is not necessary (see Eq. (3.4.35) and the example $su(m)$ which follows).

3.7. Projective Representations of $\mathbf{N = L/R}$

Altschüler et al.[18] have given an algorithm which allows to calculate the irreducible representation of the finite abelian group N. We consider N, a finite abelian group with a bilinear form $S : N \times N \longrightarrow \mathbb{C}^*$ such that S is alternating, i.e. $S(x,y) = S^{-1}(y,x)$ (see Eq. (3.4.25)) and non degenerate : if $S(x,y) = 1$, $\forall x \in N$ then $y = 0$ (see comment after Eq. (3.6.15)). S is bimultiplicative, that is $S(x+y,z) = S(x,z)S(y,z)$ and $S(x,y+z) = S(x,y)S(x,z)$ (this follows from formula (3.4.22)).

Theorem 1 : For such an N we have the structure of a direct product

$$N_1 \times N_2 \times \ldots \times N_r \tag{3.7.1}$$

Each factor N_j is the product of two cyclic groups $\mathbb{Z}_j$ and $\mathbb{Z}'_j$ of the same order n_j

$$N_j = \mathbb{Z}_j \times \mathbb{Z}'_j \tag{3.7.2}$$

such that

$$S(x_j, x'_j) = \varepsilon_j \tag{3.7.3}$$

is a primitive $n_j - th.$ root of unity where x_j resp. x'_j is a generator of $\mathbb{Z}_j$, resp. $\mathbb{Z}'_j$ and

$$S(N_i, N_j) = 1 \text{ if } i \neq j \ . \tag{3.7.4}$$

It follows that the order of N is a square since the order of N_j is n_j^2.

Theorem 2 : For each $j = 1, \ldots, r$, let the $n_j \times n_j$ matrices be defined by

$$P_j = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{3.7.5}$$

$$Q_j = \begin{pmatrix} 0 & \varepsilon_j & 0 & 0 & \dots & 0 \\ 0 & 0 & \varepsilon_j^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \varepsilon_j^{n_j-1} \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{3.7.6}$$

if n_j is odd

$$Q_j = \begin{pmatrix} 0 & \delta_j & 0 & \dots & 0 \\ 0 & 0 & \delta_j^3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \delta_j^{2n_j-3} \\ \delta_j^{2n_j-1} & 0 & 0 & \dots & 0 \end{pmatrix} \tag{3.7.7}$$

if n_j is even, where δ_j is a primitive $2n_j - th.$ root of unity such that

$$\delta_j^2 = \varepsilon_j \ . \tag{3.7.8}$$

Then the map $\varphi : N \to GL_n(\mathbb{C})$ defined by

$$\varphi(x_i) = p_i \ ; \ \varphi(x_i') = q_i \tag{3.7.9}$$

where

$$\begin{aligned} p_i &= I_1 \otimes I_2 \otimes \cdots I_{i-1} \otimes P_i \otimes I_{i+1} \cdots \otimes I_r \\ q_i &= I_1 \otimes I_2 \otimes \cdots I_{i-1} \otimes Q_i \otimes I_{i+1} \cdots \otimes I_r \end{aligned} \tag{3.7.10}$$

$$I_i = \text{unit} \ \ n_i \times n_i \ \text{matrix} \tag{3.7.11}$$

is the unique, up to equivalence, projective, irreducible representation of N (with respect to the extension of N defined by S).

Theorem 2 is a consequence of Theorem 1 and a theorem of Morris [27] about projective representations of direct products of cyclic groups of the same order.

Example :

a) $N = \mathbb{Z}_2 \times \mathbb{Z}_2$, $n = 2$, N has 4 elements

$$\varphi(x) = p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \varphi(x') = q = \begin{pmatrix} 0 & \delta \\ \delta^3 & 0 \end{pmatrix} \quad \delta = i \tag{3.7.12}$$

and

$$\varphi(x)\varphi(x') = i\varphi(x + x') \tag{3.7.13}$$

$\varphi(\alpha)$ are the Pauli matrices.

b) $N = \mathbb{Z}_3 \times \mathbb{Z}_3$, $n = 3$

$$\varphi(x) = p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \ ; \varphi(x') = q = \begin{pmatrix} 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \\ 1 & 0 & 0 \end{pmatrix} \quad \varepsilon = e^{(2\pi i/3)} \tag{3.7.14}$$

3.8. Example of su(3)

For various automorphisms w of the root lattice Q we list a few useful properties

1) $w = 1$ (untwisted, homogeneous) $S(\alpha,\beta) = (-)^{(\alpha,\beta)}$; $Q_0 = Q$; $L = 0$; $N = 1$; no V;

2) $w : \alpha_1 \to -\alpha_1$; $\alpha_2 \to \alpha_1 + \alpha_2$; $w^2 = 1$ (twisted, mixed) $S(\alpha,\beta) = (-)^{(\alpha,\beta)}$ which is a general result for automorphism of order 2 ; $L = \{n\alpha_1\}$; $n \in \mathbb{Z}$; $R = L$; $N = 1$; no V;

3) $w : \alpha_1 \to \alpha_2$; $\alpha_2 \to -\alpha_1 - \alpha_2$; $w^3 = 1$ (twisted, principal) $S = 1$; $Q = L = R$; $N = 1$; no $|Q_0\rangle$, no V, no $\sigma(\alpha)$;

4) $w : \alpha \to -\alpha$; $w^2 = 1$ (twisted, outer automorphism) $S(\alpha,\beta) = (-)^{(\alpha,\beta)}$; $Q = L$; $R = 2Q$; $N = Q/2Q$: 4 elements. V has dimension 2.

5) $w : \alpha_1 \leftrightarrow \alpha_2$; $w^2 = 1$ (twisted, outer automorphism) $S(\alpha,\beta) = (-)^{(\alpha,\beta)}$; $L = \{n(\alpha_1 - \alpha_2)\}$; $n \in \mathbb{Z}$; $R = L$; $N = 1$; no V.

We next discuss the commutation relations of the twisted constructions of the Kac-Moody algebra based on the principal and an outer automorphism.
w principal : For the vertex operator we have $U(\alpha, x^3) = V(\alpha, x^3)$ (Eq.(3.4.11)) and from Eqs. (3.4.14-15)

$$V(\alpha,x^3)V(\beta,y^3) =: V(\alpha,x^3)V(\beta,y^3): \times \prod_{s=0}^{2}\left(1-\frac{r^s y}{x}\right)^{(\alpha,w^s\beta)} \tag{3.8.1}$$

For $\beta = \alpha$, one gets two simple poles when $x = ry$ and $x = r^2 y$. For $\beta = -\alpha$, there is one double pole for $x = y$. The contribution of the simple poles to the commutator (Eq. (3.5.10)) is

$$\begin{aligned}[V^{i/3}(\alpha), V^{j/3}(\alpha)] &= r^i(1-r)^{(\alpha,w^2\alpha)}(1-r^2)^{(\alpha,\alpha)}V^{(i+j)/m}(w^{-1}\alpha+\alpha) \\ &\quad + r^{2i}(1-r)^{(\alpha,\alpha)}(1-r^2)^{(\alpha,w\alpha)}V^{(i+j)/m}(w^{-2}\alpha+\alpha)\end{aligned} \tag{3.8.2}$$

Using $\alpha + w\alpha + w^2\alpha = 0$, and $V^{i/3}(w^n\alpha) = r^{ni}V^{i/3}(\alpha)$, with $n = 1, 2$, one finds

$$\begin{aligned}[V^{i/3}(\alpha), V^{j/3}(\alpha)] &= -(r^{2i+j}(2r^2+1) + r^{2(2i+j)}(2r+1)) \\ &\quad \times V^{(i+j)/3}(-\alpha)\end{aligned} \tag{3.8.3}$$

The contribution of the double pole is

$$[V^{i/3}(\alpha), V^{j/3}(-\alpha)] = 3(i\delta_{i+j,0} + 3h_\alpha(i+j)) \tag{3.8.4}$$

It is instructive to compare this commutators with the abstract relations Eq. (2.3.7) Now, according to (2.3.10), the structure realized by the principal construction should be isomorphic to the untwisted affine algebra; in particular, we should be able to exhibit the finite Lie algebra $su(3)$ which is not explicit in the commutation relations above because of the change of $\mathbb{Z}$-gradation produced by the twisted realization (see Chapter 7 of [23]). Notice that this problem, the solution of which will be outlined in the following, can be entirely investigated at the abstract level. The lifting of the automorphism w is easy because $\underline{h}_0 = 0$: we can choose all phases $\psi_\alpha = 1$. Acting on the generators H_α and E_α one gets

$$\begin{aligned} \tau H_{\alpha_1} &= H_{\alpha_2}; \tau H_{\alpha_2} = H_{-\alpha_1-\alpha_2}; \tau H_{-\alpha_1-\alpha_2} = H_{\alpha_1} \\ \tau E_{\alpha_1} &= E_{\alpha_2}; \tau E_{\alpha_2} = E_{-\alpha_1-\alpha_2}; \tau E_{-\alpha_1-\alpha_2} = E_{\alpha_1} \end{aligned} \tag{3.8.5}$$

Hence $su(3)$ can be divided into 3 eigenspaces. We give the eigenvectors

$$E_{\pm\alpha_1} + E_{\pm\alpha_2} + E_{\mp(\alpha_1+\alpha_2)} = 3p_0(\pm E_{\alpha_i}) \in \underline{g}_0 \tag{3.8.6}$$

so that

$$g_0 \cong u(1) \oplus u(1) \tag{3.8.7}$$

$$\begin{aligned} -r^{-n} H_{\alpha_1} + H_{\alpha_2} &\in \underline{g}_n \\ & \qquad n = 1,2 \\ E_{\pm\alpha_1} + r^{-n} E_{\pm\alpha_2} + r^{-2n} E_{\mp(\alpha_1+\alpha_2)} &\in \underline{g}_n \end{aligned} \tag{3.8.8}$$

Recalling the correspondance $V(\alpha, z) \leftrightarrow \sum p_n(E_\alpha) \otimes t^{n/m} z^{-n}$, we shall identify $V^0(\pm\alpha)$ with the generators of g_0 and choose them as the new basis vectors for the C.S.A. of $su(3)$, say $\hat{H}_1$ and $\hat{H}_2$. This is of course consistent with (3.8.4) for $i, j = 0$:

$$[V^0(\alpha), V^0(-\alpha)] = 0 \tag{3.8.9}$$

The next task is to identify the new step operators $\hat{E}_{\pm\alpha_i}$. To achieve that, one should somehow "diagonalize" the commutators (3.8.3) to (3.8.4) with respect to the new C.S.A. : $[\hat{H}_i, \hat{E}_\alpha] = \alpha(\hat{H}_i)\hat{E}_\alpha$. The closure of the algebra constitutes here an important constraint guiding the actual identification: the powers of t have to be matched carefully. For instance, $\hat{E}_{-\alpha_1-\alpha_2}$ should be chosen as some combination of operators corresponding to $t^{-2/3}$, $\hat{E}_{-\alpha_i}$ as some combinations with $t^{-1/3}$, $\hat{E}_{\alpha_i}$ would be expressed in term of operators $V^{1/3}(\alpha)$ etc. The commutation relations of $su(3)$ ensure then that we do not "propagate" indefinitely with powers of t.

Outer automorphism $\alpha \to -\alpha$

We can again set all the phases $\psi_\alpha = 1$ so that $\tau E_\alpha = E_{w\alpha}$. The 3 eigenvectors in the subspace $\underline{g}_0$ are then,

$$E_{\alpha_1} + E_{-\alpha_1},\ E_{\alpha_2} + E_{-\alpha_2},\ E_{\alpha_1+\alpha_2} + E_{-\alpha_1-\alpha_2} \tag{3.8.10}$$

This is isomorphic to the algebra $su(2)$. The 5 eigenvectors in $\underline{g}_1$, are :

$$\begin{gathered} H_{\alpha_1},\ H_{\alpha_2},\ E_{\alpha_1} - E_{-\alpha_1},\ E_{\alpha_2} - E_{-\alpha_2}, \\ E_{\alpha_1+\alpha_2} - E_{-\alpha_1-\alpha_2} \end{gathered} \tag{3.8.11}$$

They form a representation of $su(2)$. The vertex operator is

$$U(\alpha, x^2) = V(\alpha, x^2)\varphi(\alpha) \tag{3.8.12}$$

$V(\alpha, x^2)$ acts on the Fock space and $\varphi(\alpha)$ on the 2 dimensional space V on which the Pauli matrices σ_i together with the unit matrix form a projective representation of the group $N = Q/2Q$ (see Paragraph 3.7.). A natural choice of representatives for the cosets of Q mod $2Q$ is $\{0, \alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ and we assign $\varphi(\alpha_i) = \sigma_i$.

The generators $U^{k/2}(\alpha)$ of the twisted Kac-Moody algebra obey

$$U^{k/2}(-\alpha) = (-)^k U^{k/2}(\alpha) \tag{3.8.13}$$

To get the commutation relations, one looks at the pole structure of

$$(x-y)^{(\alpha,\beta)}(x+y)^{-(\alpha,\beta)} \tag{3.8.14}$$

Since $(\alpha_1, \alpha_2) = (\alpha_2, -\alpha_1 - \alpha_2) = (-\alpha_1 - \alpha_2, \alpha_1) = -1$ all these pairs will yield simple poles. Hence one sees that, for example,

$$[U^0(\alpha_1), U^0(\alpha_2)] = 2iU^0(\alpha_1 + \alpha_2) \tag{3.8.15}$$

This and the other commutators build up a structure isomorphic to $g_0 = su(2)$. On the other hand, g_1 forms a representation 5 of $su(2)$ and one can see that there is no way to get $su(3)$ as a subalgebra. Hence, the operators $U^{k/2}(\alpha)$ belong to a twisted Kac-Moody algebra different from the affine $su(3) \equiv A_2^{(1)}$. This one is called $A_2^{(2)}$. The lower index 2 means that one started with A_2. The upper means that the Dynkin automorphism is of order 2.

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