On the statistical interpretation of quantum mechanics

Helvetica Physica Acta, Vol. 60 (1987) 363–383

0018–0238/87/030363–21$1.50 + 0.20/0

On the statistical interpretation of quantum mechanics

By Hans Rudolf Tschudi[1])

(27. III. 1986, rev. 18. VII. 1986)

Abstract. A fundamental feature of physical statements is their universality: A theoretical prediction concerning measured observable values of a concrete physical system is likewise valid for every equal and equally prepared system. The predictions relate immediately to the infinite ensemble of equally prepared, equal systems and to the single concrete system only as far as it is a part of the ensemble. This implies a statistical interpretation of physical theories.

The physical experiment and the relationship between experimental and theoretical statements are analysed in the frame work of this statistical interpretation. It is shown that no general measurement problem occurs there. This problem and other 'fundamental' problems of quantum mechanics originate from the erroneous dogma that the wave function is a property of the physical object concretely present.

Finally, the problem of hidden variables in quantum mechanics is discussed. It is shown, under very general and natural assumptions, that quantum mechanical predictions cannot be reproduced by classical theories.

The author is convinced that the understanding of the quantum formalism concerns every physicist and should not be just the reserve of a few specialists. He has made a special effort to explain clearly the essential ideas, avoiding unnecessary technical sophistication. This paper is addressed to every interested physicist.

1. Introduction

More than 50 years after the formulation of quantum mechanics, the interpretation of a pure state (the wave function) still remains contested. The many different opinions concerning this topic may be divided into two groups: '(I) The *Statistical Interpretation* (of the quantum mechanical formalism), according to which a *pure state* (and hence also a general state) provides a description of certain statistical properties of an *ensemble* of similarly prepared systems, but need not provide a complete description of an individual system' and '(II) Interpretations which assert that *a pure state provides a complete and exhaustive description of an individual system* (e.g., an electron)' [1, p. 360]. For the sake of brevity, we designate by *Copenhagen Interpretation* every interpretation containing assumption (II) although this definition may not always do justice to the historical facts.

It is a characteristic feature of the Copenhagen interpretation that so-called fundamental problems occur – most notably the measurement problem. L. E. Ballentine [1] shows that these difficulties are direct consequences of assumption

[1]) Present address: Im Heugarten 7c, 8617 Mönchaltorf, Switzerland

(II) and that no such problems occur in the statistical interpretation. He argues, furthermore, that assumption (II) is by no means necessary for the interpretation of the quantum mechanical formalism and concludes that it should be dropped. We share this opinion. But we think, moreover, that accepting or rejecting assumptions (I) or (II) should rely upon better foundations than on mere arguments of convenience.

The present work aims to indicate a more fundamental foundation of the statistical interpretation. This foundation is based on a discussion of the characteristics of physical statements and on an analysis of the manner a concrete material system is described by physical theories (Section 2.1). It is shown that every state – and hence also a pure state – corresponds to a preparation instruction and, therefore, to an infinite ensemble of similarly prepared, similar systems (Section 2.5). We indicate how 'measurement problems' emerge from the erroneous assumption (II) and that no measurement problems occur in the statistical interpretation (Section 2.6).

The epistemological structure of the physical experiment and its function in the physical description of nature are analyzed in Section 2.1. It is shown that the preparation of the state of a system and the subsequent registration of an observable value are two distinct parts of an experiment. These different parts must satisfy different demands and they should not be merged into one diffuse notion 'measurement'. Preparation and registration are explained in detail using as examples the preparation of pure spin states and the registration of spin by an inhomogeneous magnetic field (Section 2.4).

The considerations of Section 2.1 are formalized in Section 2.2. This formalization leads to a general formal scheme (GFS) of quantitative physical theories corresponding to the axioms I to IV of Mackey's axiomatics of quantum mechanics [2]. Quantum and classical theories are special cases of the GFS – equipped, of course, with additional structures (Section 2.3).

The GFS includes quantum mechanics in its statistical interpretation. Physical objects appear there real and body-like, in contrast to the dual wave-particles of the Copenhagen interpretation. But they partially evade human manipulation so far as only probability statements about their future behaviour are possible. Statistical predictions also occur in classical theories but are caused there merely by unprecisely defined initial conditions. The question arises whether the statistical character of quantum mechanical statements can be reduced – as in classical physics – to inaccuracies of the preparation of initial conditions. Can quantum mechanical predictions be obtained from classical theories where an average of certain hidden variables has been taken?

The contributions to this problem of hidden variables in quantum theories are very controversial. They extend from von Neumann's proof that no dispersion free quantum state exists [1, p. 374] to explicit constructions of hidden variable models [3, 4, 5, 6]. We will not add new mathematical or formal points of view to the problem. But our analysis of the general structure of quantitative physical theories allows us to indicate precisely under which circumstances hidden variable models are possible and why these conditions are inappropriate (Section 3).

Our proof that no reasonable hidden variable models are possible (except for the case of quantum theories the Hilbert space of which is two dimensional) closely follows an important paper of Kochen and Specker [7]. Some objections raised against this proof by Belinfante [8] and Bell [9] are criticized.

2. The conceptual foundation of the statistical interpretation of quantum mechanics

2.1. *Measurement, experiment, observable, state*

The most typical trait of physical thinking consists of interpreting an individual observation as a special instance of a general law. As an example, let us consider an observation like 'There was a thunderstorm in Zurich on August 15th 1985 at 5 p.m.' A physicist will not take place or time of the thunderstorm as essential for its occurrence but rather given meteorological facts always producing thunderstorms when they occur. He expects nature to follow causal laws which can be formulated without explicit reference to historical or geographical circumstances. The causes can be realized – in principle – by anybody anywhere at any time and always yield, if realized, the same consequences. As the validity of causal laws is asserted for all future situations where they apply, these laws have the logical form of universal predictions.

The universality of physical predictions has a simple but important consequence: If something is affirmed for an actually performed measurement it is simultaneously asserted for each other similar measurement. Physical statements refer, therefore, to classes of similar measurements and to a measurement at present in performance only as far as it is considered as an element of such a class. Let us call a class of similar measurements an *experiment.* The most general universal prediction for an experiment is the prediction of relative frequencies of the possible results of the single measurements – therefore a statistical prediction with respect to the single measurement. These statistical predictions may also contain deterministic laws as special cases: The relative frequency of the deterministic result equals one.

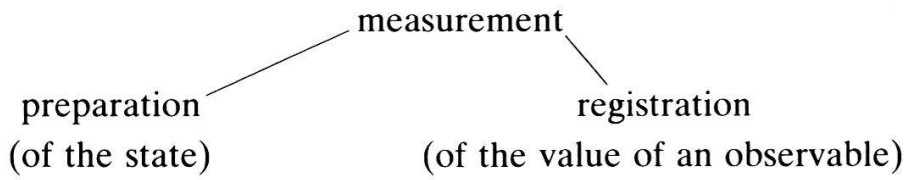
For the sake of clarity, one should precisely distinguish between experiments, notions and statements referring to experiments on the one hand, and measurements, their notions and statements on the other hand. We call notions and statements referring to the logical level of experiment *theoretical,* those referring to actually performed measurements *concrete* or *material.* Notions used for the description of other (theoretical or material) notions are clearly not material. In many cases, the same expression may be used theoretically or materially depending on the context ('Copper has good heat conductivity' but 'This is a piece of copper'). We will see in Section 2.5 that notions such as 'state', 'wave function', 'observable' are all theoretical – properties of experiments and not of concrete actual measuerments. Therefore, we do not agree with the Copenhagen interpretation of quantum mechanics asserting that a wave function is a complete description of a concretely present material system.

Physical predictions are typically quantitative. The predict numerical values of observables in comparison with given standard units. Materially, an observable corresponds to a prescribed series of measurement actions effected in a concrete measurement in order to ascertain the value of the observable. Let us call this ascertainment a *registration* of the observable value.

A useful physical theory should correctly predict the values of certain observables for a given experiment. These predictions are tested in actually performed measurements. Physical predictions can never be proved by the results of measurements because of their universality but at most refuted [10]. In general, such a refutation cannot be achieved by a *single* experiment because most theories contain empirical parameters such as masses, charges etc. But this complication is of no importance for the present considerations.

Physical predictions on a physical system S have the logical form of univeral implications: 'At all places, at any time, and for any observer the implication $\varphi(S) \rightarrow q$ is valid'. $\varphi(S)$ denotes the causes, initial and boundary conditions imposed on the system S, or the *state* of the system. The asserted consequences q consists in predictions of the observable values. We call the realization of a certain state of a concrete material system its *preparation*. To a state corresponds one or several *preparation instructions*, to an observable one or several *registration instructions*.

A measurement which can be confronted with a theoretical prediction divides, in a natural way, into two parts:



First of all, one must realize the state φ of the system S. Then one has to determine the value of the observable and confront it with the prediction q. In the case of statistical predictions, several measurements are necessary for a decisive confrontation.

The preparation of a state and the subsequent registration of the observable value are logically different parts of a measuerment. Both parts are necessary. Unfortunately, this distinction is not always made[2] and both notions are merged into one diffuse notion 'measurement'. This is the cause of much trouble in the discussion of the measurement problem in quantum mechanics. We will say more about the meaning of preparation and registration in Section 2.4. A survey of the terminology used is given in Table I.

---

[2]) A short and incomplete survey on this subject: Heisenberg implicitly refers to the mentioned distinction [11, p. 15, 19] but takes the registration to be insignificant. Pauli distinguishes between measurements of the first kind – where 'an immediate repetition of the measurement of the same quantity yields the same value as the first measurement' – and of the second kind where this is not true [12, p. 72, 73]. However, these different kinds of measurements are not seen in our context. We owe the clear distinction between the different functions of preparation and registration to K. Popper [13, p. 25].

Table I
A survey of the terminology used in the present work

			state	observable
theoretical notion	physical system	experiment	a given state $(\varphi, \psi, \ldots)$ corresponds to a preparation instruction	an observable $(E, \mathbf{p}, \mathbf{x}, \ldots)$ corresponds to a registration instruction
material notion	concrete, material system			values of an observable (10 m/sec)
realization on a material system		measurement	preparation	registration

2.2. *A general formal scheme of quantitative physical theories*

We may formalize the content of the last section in the following manner: Basic notions of quantitative physical theories are observables $A, B, \ldots$ corresponding to registration instructions and states $\varphi, \psi, \ldots$ corresponding to preparation instructions. A prediction attaches a real probability measure $W_{A,\varphi}(\Delta)$ to the observable A and to the state φ. The quantity W indicates the relative frequency with which the registered value of the observable A falls into the interval Δ if the system is prepared according to the instruction φ. In other words, the theoretical prediction $W_{A,\varphi}$ and the result of concrete measurements are connected as follows: The values $a_1, a_2, \ldots, a_n$ of the observable A registered on physical systems prepared according to instruction φ should form a random test of the statistical ensemble $W_{A,\varphi}$.

The following postulates of separation augment the economy of description:

SI: Observables separate states. Two states φ, ψ are identical if the corresponding probability measures $W_{A,\varphi}$ and $W_{A,\psi}$ are equal for every observable A.

SII: States separate observables. Two observables A, B can be identified if $W_{A,\varphi} = W_{B,\varphi}$ for all states φ.

This general formal scheme (GFS) of quantitative physical theories corresponds to the axioms I–IV of Mackey's axiomatics of quantum mechanics [2].

W is a real probability measure allowing for additional structures. It is possible in a natural way to define real functions of observables and convex linear combinations of states. Let $f: \mathbb{R} \to \mathbb{R}$ denote a real Borel function (i.e. the full inverse image $f^{-1}(\Delta)$ of every measurable set Δ is again a measurable set). The real function $f(A)$ of an observable A is then defined by

$$W_{f(A),\varphi}(\Delta) = W_{A,\varphi}(f^{-1}(\Delta)) \tag{2.1}$$

for each state φ. A registration of an observable A yielding the value a is, therefore, by definition also a registration of $f(A)$ with value $f(a)$.

Secondly, let us choose a countable sequence of states $\{\varphi_j\}$ of a physical

system and a countable sequence $\{p_j\}$ of real numbers such that

$$0 \leqslant p_j \leqslant 1 \quad \text{and} \quad \sum_{j=1}^{\infty} p_j = 1. \tag{2.2}$$

The convex linear combination

$$\sum_{j=1}^{\infty} p_j W_{A,\varphi_j} \equiv W_{A,\varphi} \tag{2.3}$$

is again a probability measure for each observable A and determines a state φ as a convex linear combination of states contained in the set $\{\varphi_j\}$. That φ is a convex linear combination of states φ_j signifies – in practice for measurements – that every sample of the statistical ensemble φ can be split up into samples of the states φ_j. The number p_j indicates the relative frequency with which the registered values fall into the sample belonging to the ensemble φ_j. By the way, it may not always be meaningful to interpret the convex linear combination φ in the sense that the system is in the state φ_j with probability p_j: The preparation instruction corresponding to φ must not have any relations to preparation instructions for the states φ_j (consider for instance Gibbs states!).

The given purely formal description of physical theories is by no means exhaustive. An understanding of physical theories as black boxes spitting out probability measures would be far too narrow. A theory interpreted this way could only and strictly be applied to the experiment for which it was designed. There would be no foundation for an extension to other even very similar situations. But it is a decisive strength of physical theories that – besides the prognoses for the group of experiments they should explain – they also give hints on how to attack different physical questions or even exhibit principles valid for all physical theories. General relativity is a deep and fundamental theory for these reasons and not because an extraordinarily large number of experiments exists whose outcomes are correctly predicted.

2.3. *Quantum and classical theories as examples of the GFS*

The GFS provides observables and states only on a symbolic level. In order to be able to compute probability measures one needs a developed theory. In such a theory, states and observables correspond to well defined mathematical objects and there exist explicit instructions how to calculate the probability measure $W_{A,\varphi}$. Quantum and classical theories are examples [2].

In the *quantum mechanical* description of a system with a finite number of degrees of freedom, the set of pure states forms a Hilbert space, general states φ correspond to density matrices ρ_φ, observables A to selfadjoint operators $\hat{A}$ on the Hilbert space. The real probability measure $W_{A,\varphi}$ is calculated as follows

$$W_{A,\varphi}(\Delta) = \text{Trace}\,(\rho_\varphi E_{\hat{A}}(\Delta)) \tag{2.4}$$

where $E_{\hat{A}}(\cdot)$ is the spectral decomposition of $\hat{A}$.

Two quantum mechanical observables whose spectral decompositions do not commute cannot be simultaneously predicted exactly for all pure states. A *classical theory* is, on the other hand, formally most simply characterized by the condition that all states are convex linear combinations of atomic states. In an atomic state, all observables attain sharp values. One presumes, in other words, that the system can be prepared in such a way that the values of all observables can be predicted with certainty – and that every preparation can be formally considered as convex linear combination of such dispersion free preparations. Observables appear as real functions on the atomic states, general states as probability measures on the set of atomic states.

A classical theory is, therefore, a triplet (G, F, Z) where
- the set of atomic states forms the phase space G. G must be a measure space.
- F represents a family of real functions on the phase space G. Each observable A corresponds to a real function $\tilde{A} \in F$.
- Z denotes the set of probability measures on G. A general state φ corresponds to a probability measure $\mu_\varphi \in Z$. Atomic states correspond to measures concentrated on one point.

The predictions $W_{A,\varphi}$ are calculated according to the formula

$$W_{A,\varphi}(\Delta) = \int_G \chi_\Delta \circ \tilde{A}\, d\mu_\varphi = \mu_\varphi(\tilde{A}^{-1}\Delta). \tag{2.5}$$

χ_Δ is the characteristic function of the interval $\Delta \in \mathbb{R}$, the set $\tilde{A}^{-1}\Delta$ is the full inverse image of the set Δ under the function $\tilde{A}$ (i.e. the set of all points in G whose image under $\tilde{A}$ lies in Δ). If the observable A is represented by the function $\tilde{A}$ then the function $f \circ \tilde{A}$ corresponds to the observable $f(A)$ because of the identity $\tilde{A}^{-1} \circ f^{-1} = (f \circ \tilde{A})^{-1}$.

It is characteristic that the notion state plays no important role in classical physics. Having the example of planetary motions in view, it seems natural to consider a classical system as a system in an atomic state and only to mention the support of the point measure (not the measure itself). This support already contains the whole information about the state of the system. The measure space structure does not appear explicitly. One only speaks of the observables whose sharply predicted values appear as real functions of the point support of the atomic measure (the initial conditions) and time.

2.4. *More about preparation and registration*

Let us explain preparation and registration in more detail with the aid of the Einstein–Podolsky–Rosen (EPR) thought experiment and of a quantum mechanical model of spin measurement. Let us first consider Bohm's version [14, p. 614ff] of the EPR experiment [15]: Two spin-1/2 particles are bound into an unstable state of total spin 0 and disintegrate under conservation of the total spin. No physical interaction between the two particles persists after disintegation. We

confine ourselves to a description of the spin space of the particles represented as a tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$ of two spin-1/2 spaces.

The spin variables of particles I and II have the form

$$s_j^{\mathrm{I}} = s_j \otimes \mathbb{1}, \qquad s_j^{\mathrm{II}} = \mathbb{1} \otimes s_j \quad j = 1, 2, 3 \tag{2.6}$$

and the state ψ of total spin 0 is given by

$$\psi = \sqrt{\tfrac{1}{2}}\,(\psi_+ \otimes \psi_- - \psi_- \otimes \psi_+) \tag{2.7}$$

$\{\psi_+, \psi_-\}$ being any orthonormal basis of $\mathbb{C}^2$. The expectation value of the observables s_j^{I} in the state ψ equals

$$(\psi, s_j^{\mathrm{I}}\psi) = \tfrac{1}{2}((\psi_+, s_j\psi_+) + (\psi_-, s_j\psi_-)) = \tfrac{1}{2}\,\mathrm{Trace}\, s_j. \tag{2.8}$$

The spin state of each particle after separation is, therefore, described by the density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.9}$$

It is worthwhile to inquire in greater detail into the transition from the total system to its separated parts. The fact that we consider the two particles as separated means that we want to describe physically the system of the two particles exclusively by observables of the form $A^{\mathrm{I}} \otimes \mathbb{1}^{\mathrm{II}}$ and $\mathbb{1}^{\mathrm{I}} \otimes A^{\mathrm{II}}$. Observables corresponding to correlations between the two partial systems are not contained in these sets. Correlations may nevertheless exist but they are neglected and they can be neglected exactly if there are no physical interactions between the particles. The set of observables $\{A^{\mathrm{I}} \otimes \mathbb{1}^{\mathrm{II}}, \mathbb{1}^{\mathrm{I}} \otimes A^{\mathrm{II}}\}$ no longer suffices to separate all states of the tensor product space $\mathscr{H}^{\mathrm{I}} \otimes \mathscr{H}^{\mathrm{II}}$. These states are divided according to separation postulat SI into classes of states, equivalent with respect to the reduced set of observables considered. Each of these classes contains an element of the form $\rho^{\mathrm{I}} \otimes \rho^{\mathrm{II}}$ solely determined if the sets of observables A^{I} and A^{II} separate all pure states of the Hilbert spaces $\mathscr{H}^{\mathrm{I}}$ and $\mathscr{H}^{\mathrm{II}}$. The density matrices ρ^{I}, ρ^{II} denote the states of the particles I and II if they are taken as isolated systems.

Several disintegrations are now observed and the experiment is arranged in such a way that one always gets a pair of particles I and II belonging to each other and moving in positive and negative x-direction. The z-spin component of each I-particle is registered. This is, at the same time, also a registration of the z-spin component of the corresponding II-particle because the spins of the two particles deriving from the same disintegration are strictly anticorrelated.

We now have the following situation: All II-particles together belong to a spin state described by the density matrix (2.9). As it was explained in Section 2.2, each sample belonging to the mixture ρ can be decomposed into samples to the pure states ψ_+ and ψ_-. Owing to the strict spin anticorrelation, this decomposition can be achieved for II-particles by registrations of the corresponding spin components on the I-particles – even without physically disturbing the II-particles.

The preparation of a pure spin state is done by *separation* and *filtering*. First, the II-particles are sorted out following the desired characteristic. Then one eliminates the undesired ones. The filtering may be achieved by an absorbing screen put into the particle path and a suitable timing. But it also may be done without machinery using the total II-particle jet for an experiment allowing, however, for only a part of the registrations made.

If we change the direction of the Stern–Gerlach magnet and measure the y-spin component of the I-particles we do not change anything on the II-particles. They still belong to the ensemble whose spin state is described by the density matrix (2.9). However, we are now able to decompose the sample belonging to the mixture ρ into two subsamples to eigenstates of the s_y^{II}-observable and to prepare a pure s_y-state by filtering. One has to decide, obviously, which characteristic (s_z or s_y) one wants to sort out into subsamples. Each separation yields a different pure state which cannot be decomposed further.

We see that the possibilities of *preparation* are here restricted. The (generalized) Heisenberg uncertainty relations are statements on what kind of preparations are possible. No such restrictions exist for registrations. We may consider the registration of the z-spin component on particle I also as a registration of the corresponding observable of particle II because of the strict spin anticorrelation and register, in addition, the y-spin component of particle II. Similarly, one can construct an apparatus registering sharply *position* and *momentum* of a mass point at the same time [1, §3].

The principles of separation and filtering in order to prepare an ensemble with certain properties was used implicitely at an other place in our example. The decay is isotropic. The particles I and II belonging to one another and moving in positive and negative x direction are sorted out by a system of impervious screens and diaphragms.

The simple – and for us natural – explanation of preparation is clearly only possible within the scheme of the statistical interpretation of quantum mechanics. Each interpretation understanding a pure state as a property 'pinned' on a concrete material system gets into serious difficulties by the EPR thought experiment. For it must admit the II-particle to be in a pure state after the registration of a spin component on particle I. *Which* state depends on which spin component was registered on particle I although no physical interaction between the two particles exists at that time.

These difficulties seem to originate from the fact that two systems, admittedly free at the moment of registration on particle I, were formerly in interaction and that certain correlations persist. The notion 'free system' becomes, therefore, problematic for this non statistical interpretation of pure states. There are, indeed, people pretending that this consideration shows the impossibility of separated, free systems in quantum mechanics:[3] Each piece of matter – even if it

---

[3]) See for instance B. d'Espagnat [16, p. 84ff]. However, his 'démonstration de la nonséparabilité quantique' rests essentially on the assumption that the representation of a state by a density matrix is unique. This is true if sufficiently many selfadjoint operators (for instance all projections) are considered to be observables. But this assumption is, as we have seen, *not* satisfied if two non interacting systems are taken to be separated.

is now separated for the sake of a measurement – was once in interaction with the rest of the universe and correlations persist from that former interaction.

We must object to this triumph of holism that the possibility of isolating a piece of matter independently of its position and its history and the possibility of preparing it into well defined initial and boundary conditions is a fundamental methodoligical principle of physics which constitute essentially the character of physical statements. These presuppositions cannot be cancelled in a later stage of theory formation. One cannot dispense with the possibility of realizing free systems in physics neither practically nor theoretically.

Let us now discuss preparation and registration using as an example the measurement of spin with an inhomogeneous magnetic field. We take the model of the Stern–Gerlach experiment from references [14, p. 593ff.] and [2, §2]. The model is discussed there in the Schrödinger picture. We solve it in the Heisenberg picture – the picture corresponding to the conceptual structure of the GFS. In the GFS, states denote preparation instructions (for instance of certain initial conditions of mechanical systems) which show, by their nature, no time evolution. In the Heisenberg picture, the meaning of preparation and registration becomes especially clear.

The model of the Stern–Gerlach experiment consists in exposing neutral spin-1/2 particles to an infinitely extended, inhomogeneous magnetic field of the form

$$\mathbf{B} = \theta(t) \cdot \theta(T-t) \cdot (0, 0, x_2 B') \quad \text{with} \quad B' = \text{const.} \tag{2.10}$$

i.e. the magnetic field acts only during the time interval $[0, T]$. The Hamilton operator of the particle in the external magnetic field is then given by

$$H = \frac{\mathbf{p}^2}{2m} - (\boldsymbol{\mu}, \mathbf{B}) = \frac{\mathbf{p}^2}{2m} - \theta(t) \cdot \theta(T-t) \cdot b x_2 s_3 \tag{2.11}$$

where

$$\boldsymbol{\mu} = \gamma \mathbf{s}, \qquad b = \gamma B' > 0. \tag{2.12}$$

The magnetic moment $\boldsymbol{\mu}$ of the particle is proportional to the particle spin $\mathbf{s}$, the magnetic field finite in time but infinite in space. Although a magnetic field constant in time but confined to a finite region in space would be more realistic, the problem could then not be solved in the same explicit manner as the present one. Our conclusions concerning registration and preparation are not altered by the simplifying assumptions.

We now must integrate the Heisenberg equations

$$\frac{dA}{dt} = \frac{i}{\hbar}[H, A] \tag{2.13}$$

for the observables $\mathbf{x}, \mathbf{p}, \mathbf{s}$ with regard to the commutation relations

$$\begin{aligned} &[p_j, p_k] = 0, \qquad [x_j, x_k] = 0 \\ &[s_j, p_k] = 0, \qquad [s_j, x_k] = 0 \quad j, k = 1, 2, 3 \\ &[p_j, x_k] = \frac{\hbar}{i} \delta_{jk} \end{aligned} \tag{2.14}$$

and the equations

$$s_1 s_2 = -s_2 s_1 = \frac{i\hbar}{2} s_3 \ (+\text{cycl.})$$
$$s_j^2 = \hbar^2/4 \quad j = 1, 2, 3 \tag{2.15}$$

valid for spin-1/2 operators. We first obtain the equations

$$\dot{\mathbf{x}} = \mathbf{p}/m \tag{2.16}$$
$$p_1 = \text{const.}, p_3 = \text{const.}, s_3 = \text{const.} \tag{2.17}$$
$$\dot{p}_2 = \theta(t) \cdot \theta(T - t) \cdot b s_3 \tag{2.18}$$

and from these equations

$$x_2(t) = \frac{t}{m}(p_2(0) + Tbs_3) - \frac{T^2 b}{2m} s_3 + x_2(0) \tag{2.19}$$

for $t \geqslant T$.

This operator equation signifies that a registration of x_2 at a time $t \geqslant T$ is also a registration of the observable represented by the right-hand side of the equation with identical measuring results. The observable s_3 has only the values $\pm\hbar/2$. If we prepare the particle by a system of diaphragms and screens in such a way that the absolute values of the observables $p_2(0)$, $x_2(0)$ are smaller than $\delta p_2/2$, $\delta x_2/2$ (where, obviously, the Heisenberg uncertainty relations must be satisfied) and if we choose the parameters t, T, δp_2, δx_2 such that

$$\delta p_2 \ll \hbar b T \tag{2.20}$$

and

$$t > \frac{m\delta x_2 + \hbar b T^2/2}{\hbar b T - \delta p_2} \tag{2.21}$$

then the possible values of the observable $x_2(t)$ fall into two disjoint regions corresponding unambiguously to the measured values $\hbar/2$, $-\hbar/2$ of the observable s_3. A detection of the particles in one of the two disjoint x_2-regions signifies a registration of its s_3-component.

The Stern–Gerlach magnet will, in fact, split up a thin particle jet into two thin jets of well defined particle spin. The time evolution described assures this separation independently of the initial spin state of the particles. This is exactly what a registration should yield namely to determine the observable value at the moment of registration independently of the history of the system and therefore independently of its state, as well.

If the particles are not detected but the particle detectors are simply replaced by apertures then the Stern–Gerlach magnet acts as a *preparation* apparatus for spin polarized particles. The apparatus is even optimized in so far as the intensity of a jet, already s_3-polarized, will not be weakened, the observable s_3 being conserved. Again, preparation occurs in two steps: Separation with respect to the desired property (in this case by spatial separation) and filtering (elimination of the undesired particles).

In general we cannot predict in which of the two partial jets a particle will appear after having entered the Stern–Gerlach apparatus. But we know that the value of the spin component s_3 is $+\hbar/2$ *if* the particle appears in the corresponding partial jet. This kind of questioning is characteristic for preparation. Characteristic for registration is the statement *that* the particle has appeared in the jet of $+\hbar/2$ particles. Its further future does not matter.

It is easy to determine and integrate the equations of motion of the observables s_1, s_2. The equations of motion are

$$\dot{s}_1(t) = \theta(t)\theta(T-t)bx_2^0(t)s_2(t) \tag{2.22a}$$

$$\dot{s}_2(t) = -\theta(t)\theta(T-t)bx_2^0(t)s_1(t). \tag{2.22b}$$

$x_2^0(t)$ denotes the free motion of the particle in absence of the external magnetic field

$$x_2^0(t) = \frac{t}{m}p_2(0) + x_2(0). \tag{2.23}$$

Integration of the equations of motion yields

$$s_1(t) = s_1(0)\cos A(t) + s_2(0)\sin A(t) \tag{2.24a}$$

$$s_2(t) = -s_1(0)\sin A(t) + s_2(0)\cos A(t) \qquad t \in [0, T] \tag{2.24b}$$

with

$$A(t) = \frac{bt^2}{2m}p_2(0) + btx_2(0). \tag{2.25}$$

We do not further discuss these operator equations but only remark that the time evolution mixes the spin components $s_1(0)$ and $s_2(0)$ in a complicated manner.

2.5. *A pure state is not a property of a concrete material system*

The considerations of Section 2.1 contain a very general argument against the opinion that a pure state provides a complete and exhaustive description of an individual system. We are now going on to elaborate the argument more in detail.

The universality of physical statements (the requirement of independence from position, time, and observer) leads to a remarkable and total disappearance of individual traits in these statements. We already mentioned that each physical prediction valid for a concrete piece of matter is also valid for every other similar and similarly prepared piece of matter. It relates to the whole ensemble of similarly prepared systems. All elements of this ensemble are identical as far as the physical prediction is concerned. Referring to this, they have no individual properties.

Preparations and registrations can be carried out following universal and generally valid instructions. Physical systems can be characterized by universal instructions such as, for instance, for the purification of copper, for the construction of a pendulum, or for the generation of a jet of electrons. No proper

names are really needed for these instructions. If proper names occur, they can be removed: The unit of time is no longer attached to the rotation of earth but defined by the number of oscillations of the light of a certain spectral line. The instructions of how to build an apparatus producing the light and allowing for the expression of a unknown time interval as a multiple of the light's period can be given without reference to proper names.

Clearly, physical statements on individuals are possible. But then their individuality is no object. In physical cosmology, the universe is not considered to be an individual but an example of a certain class of systems described by a world model. It has no physical consequences that our universe is the only known example. In scientific explorations of living beings, their individual properties, their history, figure at most as disturbances impairing the reproducibility of the results – disturbances which must be reduced as much as possible by suitable preparations and standartisations.

These abstractions from all individuality make the physical description of nature indirect and mediate in a particular way. Physical statements and the occurring expressions refer *directly* to physical systems, to experiments, to preparation and registration instructions. These notions can be conceived as classes of concrete material sytstems, of measurements, of preparations, and registrations on each of them. They refer to concrete material systems, measurements and so on, only as far as they appear as members of the corresponding classes.

One should not consider, therefore, physical observables (energy, momentum, position of a point mass) and given states (the ground state of a hydrogen atom) as properties of a concrete material system, as labels fixed to each of them. Properties of concrete material systems can be only the possible outcomes of registrations, the measured values of observables.[4] 'Velocity of 10 m/sec' is a property of a real body i.e. the statement 'This body has a velocity of 10 m/sec' is a meaningful sentence. 'Velocity' is not the property of a concrete body but a theoretical notion. What could be the meaning of a sentence like 'This body has a velocity'? That it is not at rest? The sentence is then short for 'This body has a velocity different from zero' – again a statement about measured values of an observable. Or that 'velocity' represents an observable measurable on a body? But this is a statement on the *notion* 'body' not on 'this body here', a real object. Another example: 'Colour' is not a property of a concretely present body but 'red', 'green', 'blue' are.

These remarks would hardly be necessary if one could not read sentences like 'A point mass in a pure momentum state has no position'. The corresponding correct statement is that 'A point mass in a momentum eigenstate can be found with equal probability anywhere in space and, as a consequence, its position

---

[4]) I am conscious that I do not use a precise linguistic terminology. I do not feel that it is a suitable place here to describe an exact terminology of speaking about language (see for instance [17] for this purpose).

cannot be predicted'. This is clearly different from 'having no position': A man is not unwealthy if he is not able to predict the amount of its wealth.

A given state is a symbol of a preparation instruction. In a developed theory, a state can also be considered as a symbol for all possible predictions of the results of observable registrations in virtue of the function $\varphi \to W_{A,\varphi}$. The sentence 'This body is in the state φ', means that the body was subject to the corresponding preparation process, that it belongs to the ensemble of similarly prepared bodies represented by the symbol φ, and that the prediction $W_{A,\varphi}$ is asserted for the observable A. 'The state φ' is not a property of 'this concrete body here' like 'the velocity of 10 m/sec'. This interpretation might be defendable if the sentence 'This body is in the state φ', can be considered to be synonymous to 'The observable A_1 has the value a_1, A_2 has the value a_2, . . . , for this body', i.e. if all observables can be sharply predicted.

This is the case for a system in a classical atomic state. But even then we must refuse as unsuitable the opinion that a pure state provides a complete and exhaustive description of an individual system. It mistakes the epistemological status of the notion 'state': A state corresponds to a preparation instruction and, therefore, to an ensemble of similarly prepared systems. For the non-dispersion free states of quantum mechanics, the Copenhagen interpretation leads to the known problems and paradoxes whereas no such problems occur in the statistical interpretation. This is briefly illustrated in the following section using the measurement problem as an example.

2.6. *Some remarks on the problem of measurement in quantum mechanics*

Once the fact is accepted that physical properties allow for a symbolic representation in some mathematical language no 'fundamental' measurement problem occurs in the GFS and, therefore, neither in classical physics nor in the statistical interpretation of quantum mechanics. Physical observables are measurable quantities *by definition*. What does – in the framework of the GFS – the statement mean that $W_{A,\varphi}(\Delta)$ is the probability to find the registered value of the observable A in the interval Δ if the system is prepared in the state φ? Firstly, that there exists, in fact, at least one apparatus suitable for the registration of A and, secondly, that the values registered with it form a sample of the statistical ensemble $W_{A,\varphi}(\cdot)$. The first part is almost tautological. What sense could be attributed to an observable for which no registration procedure exists? The definition of an observable must include instructions how it can be registered – at least on principle. The second part does not concern the possibility of registration as such, but the theoretical predictions.

Analogous reflections are valid for preparation. There must exist, at least on principle, preparation instructions for all states theoretically pretended to be preparable. It makes little sense to introduce principally unpreparable states into a theory.

An entirely different question is why a given apparatus allows a suitable preparation or registration. Such a discussion can be held theoretically only on

the level of a developed theory. Our model of the Stern–Gerlach apparatus is an example of this. It might seem to be unsatisfactory at first sight if we do not include a detailed theoretical description of the impervious screens and diaphragms forming the particle jets or of the particle detector in our model of spin registration. But it is enough to assure *empirically* that a given apparatus is a suitable preparation or registration device and it is not necessary to have a detailed insight into how it works. It suffices to ascertain empirically that diaphragms made from empirically rigid metal sheets sort out a ray of light which can be split up into its colours by a glass prism and that a photographic plate (or the eye!) are suitable light detectors. Such empirical knowledge is, however, indispensable for methodological reasons: Only the above-mentioned means allow us to develop and to test theories like Maxwell's theory of light or theories concerning the structure of matter and the interaction between light and matter. Only then is a theoretical understanding of the mechanisms acting in photographic plates or glass prisms possible.

How do 'fundamental' measurement problems occur in the Copenhagen interpretation? Due to the idea that a pure state is attached to the single concrete system, preparation can no longer be understood as separation and filtering process on the ensemble. The preparation of a pure state must be treated on the level of the single particle. The difficulties arising appear especially clear and paradoxical in the EPR thought experiment which we have already mentioned. Registration problems occur for all pure states not being eigenstates of the corresponding observable because the pure state relates then to more than one value of the observable. One has, therefore, to pack the statistical dispersion of the values into the single concrete system which then becomes the carrier of contradictory properties. These undesirable consequences are valid not only for microscopic but also for macroscopic systems by reason of the superposition principle (Schrödinger's cat paradox).

We must forgo further discussions of the measurement problem as well as an appreciation of the literature dealing with it. The interested reader may consult Ballentine's well written article [1, §4].

3. The problem of hidden variables, Kochen and Specker's theorem

In classical theories, simultaneous sharp predictions for all observables are possible (the system is then in an atomic state) and statistical statements are mere consequences of an inaccurate preparation. In a quantum theory, there are no atomic states. Classical theories connect causes and effects in a more rigid manner than quantum theories – deterministic classic statements are stronger than the probability statements of quantum theories. It is, therefore, an interesting question whether the statistical character of quantum mechanical predictions is intrinsic, not reducible to inaccuracies of preparation.

In other words: Let us assume that a quantum mechanical description of a physical system is given. Is there a classical theory yielding the same probability

predictions for all observables and states of interest as the quantum theory considered? The classical theory may contain parameters which principally cannot be controlled and which one has to average over. Such a theory is called a classical model (with hidden variables) of the quantum theory.

A classical model of a quantum theory should respect the epistemological structure of the GFS. Therefore, we define such a model to be a classical theory (G, F, Z) – as described in Section 2.3 – with a mapping from the operators of the quantum theory into the measurable real functions on the phase space G

$$A \to F_A \in F \tag{3.1a}$$

and a mapping from the density matrices ρ of the quantum theory into probability measures on G

$$\rho \to \mu_\rho \in Z \tag{3.1b}$$

such that

$$W_{A,\rho}(\Delta) = \text{trace } \rho E_A(\Delta) = \mu_\rho(F_A^{-1}(\Delta)). \tag{3.1c}$$

(We no longer discern between states and density matrices, observables and operators). Moreover, the observable $f(A)$ shall always be represented by the function $f \circ F_A$ if A is represented by the function F_A:

$$F_{f(A)} = f \circ F_A. \tag{3.1d}$$

The following conception lies behind this definition: Each quantum state corresponds to a classical mixed state which is interpreted as a probability distribution over atomic states. A classical system always is in an atomic state, but one does not know in which one. Hidden variables specify the actually realized classical state.

Such classical models do not exist. This is shown by

Kochen and Specker's theorem. There is no classical model (in the sense of conditions (3.1)) of a quantum theory whose Hilbert space has a dimension higher than two, if a sufficient number of operators are observables.

Proof. As a preparation, let us determine the structure of classical discrete observables. An observable is discrete if it has only a finite or countably infinite number of different values $a_1, a_2, \ldots, a_n, \ldots$. Discrete observables lead in classical models to a partition of the phase space. To each value a_j of the observable A corresponds a subset M_j of the phase space G

$$G \supset M_j = \tilde{A}^{-1}([a_j]) \quad j = 1, 2, \ldots, n, \ldots \tag{3.2}$$

The sets M_j being the total inverse images of the disjoint sets $[a_j]$ are disjoint

$$M_j \cap M_k = \emptyset \quad \text{if} \quad j \neq k \tag{3.3}$$

and they cover the total phase space

$$\bigcup_j M_j = G. \tag{3.4}$$

The discrete observable A has, therefore, the following classical representation

$$A(p) = \sum_j a_j \chi_j(p) \tag{3.5}$$

for $p \in G$ where $\chi_j(p)$ denotes the characteristic function of M_j. A question (a discrete observable with the two observable values $a_1 = 0$ and $a_2 = 1$) corresponds to a characteristic function.

We show now that no classical model of the spin variables of a particle of spin one exists. The Hilbert space of these spin variables is isomorphic to $\mathbb{C}^3$. Observables are represented by Hermitian 3×3 matrices if a fixed orthonormal basis is chosen. The commutation relations of the spin components

$$\mathbf{J} = \hbar \mathbf{M}, \quad [(\mathbf{e}, \mathbf{M}), (\mathbf{e}', \mathbf{M})] = i(\mathbf{e} \vee \mathbf{e}', \mathbf{M}) \tag{3.6}$$

imply the following matrix representation of the components of $\mathbf{M}$ [18]: Let us denote by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the unit vectors indicating the directions of the axes of a right handed Cartesian coordinate frame in $\mathbb{R}^3$. There exists then an orthonormal basis ψ_1, ψ_2, ψ_3 of $\mathbb{C}^3$ such that

$$(\mathbf{e}_1, \mathbf{M}) = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (\mathbf{e}_2, \mathbf{M}) = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$(\mathbf{e}_3, \mathbf{M}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.7}$$

Using this matrix representation of spin components one easily verifies that the opeators

$$P(S\mathbf{e}_j) = 1 - (S\mathbf{e}_j, \mathbf{M})^2, \quad j = 1, 2, 3 \tag{3.8}$$

satisfy the relationships

$$P(S\mathbf{e}_j)P(S\mathbf{e}_k) = \delta_{jk} P(S\mathbf{e}_j) \tag{3.9}$$

$$P(S\mathbf{e}_1) + P(S\mathbf{e}_2) + P(S\mathbf{e}_3) = 1 \tag{3.10}$$

for any orthogonal matrix S with $\det S = +1$. The projections $P(S\mathbf{e}_j)$ can be considered as function of an observable A_S for each right trihedral $\{S\mathbf{e}_1, S\mathbf{e}_2, S\mathbf{e}_3\}$:

$$A_S = a_1 P(S\mathbf{e}_1) + a_2 P(S\mathbf{e}_2) + a_3 P(S\mathbf{e}_3) \quad \text{with} \quad a_1 < a_2 < a_3 \tag{3.11}$$

and

$$P(S\mathbf{e}_j) = f_j(A_S), \qquad j = 1, 2, 3 \tag{3.12}$$

for instance

$$P(S\mathbf{e}_1) = f_1(A_S) = \frac{(A_S - a_2)(A_S - a_3)}{(a_1 - a_2)(a_1 - a_3)}. \tag{3.13}$$

We therefore relate a projection $P(\mathbf{e})$ to each direction $\mathbf{e} \in \mathbb{R}^3$ in such a way that the properties (3.9, 10, 12) are valid for every right trihedral $\{S\mathbf{e}_1, S\mathbf{e}_2, S\mathbf{e}_3\}$. The projection $P(\mathbf{e})$ corresponds to the question: Does the spin component $(\mathbf{e}, \mathbf{M})$ have the value zero?

There is no classical model for this set $\{P(\mathbf{e})\}$ of quantum mechanical observables. Suppose that such a model exists. The projection $P(\mathbf{e})$ is then represented by a characteristic function

$$\chi_{\mathbf{e}}: G \to \{0, 1\}. \tag{3.14}$$

It follows from equations (3.5, 1d, 9, 12) that

$$\chi_{\mathbf{e}_1}(p) + \chi_{\mathbf{e}_2}(p) + \chi_{\mathbf{e}_3}(p) = 1 \tag{3.15}$$

for all points p of the phase space G and for each orthogonal trihedral $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. But the existence of functions satisfying equations (3.14, 15) is prohibited by Gleason's theorem [19] which can be formulated as follows for our purposes:

Gleasons theorem: There exists a positive symmetric matrix R with trace 1 for each non negative function $\beta(\mathbf{e})$ on the unit sphere satisfying the condition $\beta(\mathbf{e}_1) + \beta(\mathbf{e}_2) + \beta(\mathbf{e}_3) = 1$ for each orthogonal right trihedral $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ such that $\beta(\mathbf{e}) = (\mathbf{e}, R\mathbf{e})$.

Equation (3.15) can, therefore, not be satisfied by characteristic functions having only 0 and 1 as functional values. QED

Kochen and Specker show by a direct and elegant proof that the equations (3.14, 15) are contradictory without the use of Gleason's deep and difficult theorem [7]. One only needs a finite number of trihedrals (at most 39 with 109 different directions [20]) to get a contradiction. They also construct a classical model for the spin variables of a spin-1/2 particle (a different model is given in [6]). The above construction breaks down in the case of two dimensions: A characteristic function on the unit circle satisfying equations analogous to (3.15) can be given freely on a quadrant.

Finally, let us consider circumstances under which hidden variable models are possible.

First of all, classical models may become possible if the relationship (3.1a) is weakened to

$$A \to F_{A,\rho} \in F \tag{3.16}$$

i.e. if the classical representative of an observable also depends on the quantum state ρ. Bohm's hydrodynamic interpretation of quantum mechanics [3, 4, 5] has this feature. However nothing is gained with this weaker relationship. On the contrary, the mapping (3.16) means that the mathematical representation of an observable – and also of the time evolution of the observables – depends on the manner the system was prepared. The fact that the properties of concrete physical systems (namely the values of its observables) can be determined independently

of the system's history by given registration devices is no longer expressed by the formal structure of the classical model.

That the time evolution of the observables depends on the state is even worse. The conceptual separation of the physical description into a law of motion and initial conditions corresponds to the idea that physical systems are objective entities, provided with a fixed structure, which may carry different properties or properties changing with time. In the theoretical description of physical systems (for instance within the framework of non-relativistic quantum mechanics), the algebra of observables – above all the form of the Hamiltonian – characterizes the general nature of the system. The state determines the accidental set of properties actually realized. This distinction allows to speak of hydrogen atoms being in different states. Otherwise, one would be forced to consider hydrogen atoms in s – states and hydrogen atoms in p – states to be totally different entities.

A 'classical model' mixing up observables and states in the above manner misses the very heart of the physical description of nature. In comparison with this conceptual clumsiness, it is of minor importance that Bohm's hydrodynamic model of quantum mechanics is not a classical model in our sense because it violates condition (3.1d).

Belinfante [8] and Bell [9] point out a tacit assumption in the construction of the set $\{P(\mathbf{e})\}$ which they criticize: Each projection $P(\mathbf{e})$ can be considered as a function of an observable A_S in many different ways

$$P(\mathbf{e}) = f_j(A_S) = f_{j'}(A_{S'}) = \cdots \tag{3.17}$$

all corresponding to different registration devices for the same observable $P(\mathbf{e})$. Belinfante and Bell argue now that it is natural to represent different registration devices of the same quantum mechanical observable by different observables of the classical model – these classical observables having the peculiar feature to yield the same probability distribution for all states preparable in reality.[5])

Is it really natural to do so? In fact, classical models become possible if one allows for a splitting up of quantum mechanical observables into several classical observables. This is easily shown by the following construction, a slight adaptation of an argument given by Kochen and Specker: Choose a subset $0'$ of the set 0 of all observables you are interested in in such a way that every observable $A \in 0$ can be obtained as a real function $f(A')$ of an observable $A' \in 0'$. Choose the space $\mathbb{R}^{0'}$ as the phase space G of the system. The observable A' is represented by the real function $\tilde{A}' : \mathbb{R}^{0'} \to \mathbb{R}$ which depends linearly on the coordinate belonging to A' and is independent of all other coordinates, A is represented by the function $f \circ \tilde{A}'$. The state φ is represented by the product

[5]) Actually, Belinfante and Bell formulate their objections slightly differently. They argue that the result of the registration of an observable may depend on the registration device used. However this conception implies that the observable value ceases to be an objective property of the inquired system alone – an idea which particularly lacks attractivity in connection with classical hidden variable models. The essential assumption is, in any case, that the observables $f_j(A_S)$ and $f_{j'}(A_{S'})$, identical from the quantum mechanical point of view, should be represented by different functions of the classical model.

measure

$$\prod_{A' \in 0'} W_{A',\varphi}(\cdot). \tag{3.18}$$

In this classical model, no relationships exist between observables except that an observable may be a real function of another one. This model is not much more than a list of the occurring probability measures arranged into the formal structure of a classical theory. However, science is more than a mere list of observations. The difference exactly lies in the neglected relationships between observables.

The above-mentioned argument appears now less satisfactory. If it is used without restrictions one is inevitably led to the trivial classical model just described. Therefore, at least certain different registration devices of a quantum observable should be represented by the same observable of the classical model. We know from theoretical insight into the registration methods which differences between registration devices are irrelevant and when two essentially different apparatuses register the same observable. These arguments are questioned if quantum observables are split up – and one should have reasons why they might be mistaken.

Every splitting up of quantum observables weakens the structure of the theoretical description. It invalidates structural equations between observables of the type (3.17) and is a step towards the undesirable trivial classical model. Kochen and Specker's theorem shows that classical models are possible only at the price of such a weakening.

4. Classical models are incomplete

We have shown in the last section that there are, in general, no reasonable classical models of quantum theories respecting the epistemological structure of the GFS. Quantum mechanics is remarkably unimportant for the proof. The mapping (3.1), the notions 'real function of an observable', 'sum of real functions of the same observable', 'projection' can already be entirely formulated in the GFS. We only need quantum mechanics to assure the existence of the set $\{P(\mathbf{e})\}$ of projections with the properties (3.10, 12). What we have shown, in fact, is that there is a system of observables in the GFS which can be described in quantum mechanics but not in classical physics – and which corresponds to observable properties of real physical systems [7, § 4].

This is a remarkable answer to the question 'Can Quantum Mechanical Description of Physical Reality Be Considered Complete?', the title of the famous paper of Einstein, Podolsky, and Rosen [15] where the authors show that a pure quantum state can hardly be taken as the complete physical description of an individual material physical system. From our point of view, the question of completeness must be asked differently. We explained in Section 2.5 why we *never* consider a state as the property of an individual concrete system (even not

in classical theories) – not to speak of a 'complete description'. We propose to call a class of physical theories complete if it allows for the formulation of any system of prognoses relevant in the field covered by the mentioned class of theories. We think that non-relativistic quantum mechanics of physical systems with a finite number of degrees of freedom is complete in this sense – we showed that classical theories are incomplete.

We conclude that nature reveals principally incalculable elements even in the field of physics. Quantum processes are intrinsically indeterministic – only probability predictions are in general possible. This fact is opposed to expectations in classical physics most pertinently expressed by Laplace's demon: A demon once knowing the forces between all atoms of the universe and their positions and velocities at a given moment could calculate the whole history of the universe, its earliest past and its most distant future. Kochen and Specker's theorem shows that this conception of the physical world is too narrow, that already – or just – the most simple physical systems behave differently. There are sets of observables corresponding to real physical phenomena which cannot be predicted in the manner of classical physics. Relationships of the kind of Heisenberg's uncertainty principle determine absolute frontiers of human command of nature.

REFERENCES

[1] L. E. BALLENTINE, Rev. of mod. Phys. *42*, p. 358 (1970).
[2] G. W. MACKEY, *The Mathematical Foundation of Quantum Mechanics*, W. A. Benjamin, New York (1963).
[3] D. BOHM, Phys. Rev. *85*, p. 166 (1952).
[4] D. BOHM, Phys. Rev. *85*, p. 180 (1952).
[5] H. FREISTADT, Nuovo Cimento Supp. Ser. 10, 5, p. 1 (1957).
[6] J. S. BELL, Rev. of mod. Phys. *38*, p. 447 (1966).
[7] S. KOCHEN and E. SPECKER, J. of Math. and Mech. *17*, p. 59 (1967).
[8] F. J. BELINFANTE, *A Survey of Hidden-Variables Theories*, Pergamon Press (1973).
[9] J. S. BELL, *Foundations of Physics 12*, p. 989 (1982).
[10] K. POPPER, *Logik der Forschung*, J.C.B. Mohr, Tübingen (1971).
[11] W. HEISENBERG, *Physikalische Prinzipien der Quantentheorie*, B. I. Hochschultaschenbuch 1, Bibliographisches Institut, Mannheim.
[12] W. PAULI, *Die allgemeinen Prinzipien der Wellenmechanik*, Handbuch der Physik V. 1, S. Flügge ed., Springer-Verlag, Berlin (1958).
[13] K. POPPER in: *Quantum Theory and Reality*, M. Bunge ed., Springer-Verlag, Berlin (1967).
[14] D. BOHM, *Quantum Theory*, Prentice Hall, Englewood Cliffs (N.J.) (1951).
[15] A. EINSTEIN, B. PODOLSKY and N. ROSEN, Phys. Rev. *47*, p. 777 (1935).
[16] B. D'ESPAGNAT ed., Proceedings of the International School of Physics 'Enrico Fermi', Course IL, Foundations of Quantum Physics, Academic Press, New York and London (1971).
[17] W. KAMLAH and P. LORENZEN, Logische Propädeutik, B. I. Hochschultaschenbuch 227, Bibliographisches Institut, Mannheim (1973).
[18] L. D. LANDAU and E. M. LIFSCHITZ, *Lehrbuch der theoretischen Physik III*, Quantenmechanik, §27, Akademie-Verlag, Berlin (1971).
[19] A. M. GLEASON, J. of Math. and Mech. *17*, p. 59 (1967).
[20] R. JOST, *Measures on the finite dimensional subspaces of a Hilbert space, Studies in Mathematical Physics, Essais in Honor of Valentine Bargmann*, Princeton University Press, Princeton (1976).