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Relativistic kinematics and dynamics: a new group theoretical approach

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Abstract. We reanalyze the relationships between physical states and space-time symmetries with a view to describing relativistic extended and interacting systems. We first propose for this description to introduce, in space-time, an additional observable, related to a natural notion of simultaneity. We justify the introduction of this new observable on the basis of the operational meaning of the relations between state descriptions and symmetries in this case. The Poincaré transformations are correspondingly split into two parts: the first one, kinematical, related to the symmetries of the description of the states, the other one, dynamical, related to the possible forms for the evolution. We show that the kinematical symmetries lead in a straightforward way to the expected classical and quantal state spaces for single particles of arbitrary spin and we show how the remaining symmetries can be related to the derivation of the possible forms for the dynamics. We find as a particular case the usual dynamics of single particles in external fields (with some satisfactory improvements due to the corresponding new interpretation) and we extend the method to the dynamics of N interacting particles. We also exhibit why this new approach and interpretation of relativistic states is necessary and how it allows a covariant description in the problems raised by the (recently measured) quantum correlations at-a-distance concerning the Einstein–Podolsky–Rosen paradox, something which seems quite impossible in the usual frameworks.

1. Introduction

The interpretation of the relativistic space-time symmetries and their relations to the description of physical systems is well known to be much less obvious than what might be thought at first sight. In fact, the construction of a relativistic framework for N interacting particles, or for a single extended object, is for example still the subject of an abundant literature, in classical as well as is quantum physics. In particular, the so-called no-interaction theorem [1] has shown, by the simplicity of its content, how fundamental the difficulties are: roughly speaking the theorem indeed just says that if we want to describe classical point particles by a set of Poincaré invariant world lines (this means more precisely with a time evolution generator along the world lines compatible with a Poincaré representation), then we are necessarily left with only free particles (straight trajectories).

The difficulties are also illustrated in quantum relativistic physics where the well known and successful description of elementary particles in terms of representations of the Poincaré group [2] is bound to contain its own dynamics and in such a way that these dynamics correspond necessarily to only free particles.

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Neither on this basis it has been possible to describe N interacting particle systems and even the single free particle theory is not free of difficulties and ambiguities. These latter facts are in our opinion not independent of the difficulties we then face in the corresponding field theories.

Abandoning either the worldline description or the Poincaré invariance in the above classical description is of course not an easy way to take, and most attempts for avoiding the conclusion of the no-interaction theorem use therefore quite roundabout ways. For example many authors [3] tried to use redundant kinematical variables which are then reduced by Dirac constraints (introducing indirectly and only implicitly the interactions) as well as, from another side, an eleventh generator beside the Poincaré algebra for the evolution, this generator being determined, at least partially, dynamically [4]. Another group of approaches [5] is based on the generalization of the notion of free center of mass (or center of energy) coupled with internal interacting variables, but these attempts also face fundamental difficulties as soon as $N > 2$ [6].

In our opinion however, the simplicity of the hypotheses and of the conclusion of the no-go theorem is a good indication that the difficulty does not lay in some additional sophisticated mathematical consideration. We shall on the contrary argue at the level of the simplest, but fundamental, relationship between the notion of the state of a physical system, the symmetry principles in its description, and its dynamics, pursuing thereby another line of approach which is based on the following main considerations [7–9].

We adopt the quite usual and realistic point of view that the state of a physical system is completely characterized at each stage of its evolution by the set of all its properties which are then actual, i.e., by the set of all the possible test measurements for which we can predict a positive result with certainty. This is equivalent to what Einstein calls the "elements of reality" of the system [10]. The evolution is then characterized by the fact that certain actual properties become potential whereas other ones, potential, become actual. In other words the state represents the "shape", the form, of the system and its evolution the motion of this shape [11]. Our purpose is just to analyze the relationships between motion, form and symmetry, and in particular to give a relativistic meaning to this notion of "actuality".

It follows by definition that the state is the collection of all actual properties and the state space contains the collection of all potential properties. A kinematical symmetry is then a relation between equivalent descriptions of these properties.

In terms of a single classical point particle for example, a property can be a space-time (or momentum) subset in which the particle is localized. Then the state corresponds to a point in phase space. The principle of Poincaré symmetries lies of course in the Poincaré equivalent ways of locating this point. By definition, a Poincaré transformation, like any such kinematical symmetry acts thus passively. The dynamics (the motion) enters in the play only in a second step, as the generator of the trajectories in the state space. By definition, this generator acts thus, like any dynamical symmetry, actively. This means that the role of the time as an observable (the date) is now necessarily clearly distinguished from its role of an evolution parameter (the label of the changes) and the kinematics (the description of the form, hence of the state and the state spaces) is clearly distinguished from the dynamics (the motion) hence from the introduction of the evolution and of the interactions. This is important because it implies that the latter cannot change the representation nor the interpretation of the observables and their relation to the properties, hence to the state.

In the last few years this approach has been successfully developed for the description of classical and quantal, relativistic and non-relativistic elementary particles [8, 9, 12] and the resulting models have been successfully applied in concrete physical problems, like specific external field problems, or the 2-body problem, with an excellent agreement with the experimental data, for example in the 2-body corrections to the H-atom spectrum or in the positronium resonances [13].

In the present paper we make one more step along this line, step which is also at the core of the hypotheses of the above mentioned no-go theorem and which is based on an elementary analysis of the description of, for example, an extended object in the light of the just mentioned approach.

This step can in fact be seen as based on an analysis of the following question: the wave function of a quantal system can be considered as representing either the system, or the description we have of the system. The distinction is not purely academical, both may be physically inequivalent for relativistic systems, as we shall be led to see. In the first case, for example, a spatial extension is a property of a coordinate system; in the second case, it is a property of the physical system *as perceived* by a well defined class of observers. In the first case, to each point of space-time belongs one single intrinsic value for the wave function; in the second case, this value might depend on (and make sense only for) such a class, too, only the measured observables having an objective status. We shall in fact be led to justify the conclusion that the second point of view is the adequate one.

In the first part of this paper we will discuss the relativistic space-time *kinematical* symmetries in the description of the state of an arbitrary extended object. This description is based on the *operational* space-time, in which it shown to be necessary to consider an additional variable related to the notion of simultaneity. This is of course not in contradiction with the final relativistic invariance of the physical predictions. The need for such a notion is now also quite clear in the light of the recent experimental confirmation that quantum correlations are non-local [14], i.e., that they occur in some way "instantaneously". Our main intention is precisely to give a relativistic description of this experimental fact which has in fact motivated and made necessary the whole present reanalysis.

We then apply a recently developed method [9] for the derivation of the corresponding classical and quantal state spaces for single particles of arbitrary spin. Finally we demonstrate a very simple way for deriving explicitly a possible form for the dynamics in this framework for some simple models by generalizing an idea due to Jauch [15] in the non-relativistic case. The resulting state spaces are the expected ones with, in the usual particular cases, some improvements due to the new interpretation, and with the possibility of interacting particles. The framework is also shown to give a meaning to the description of the above mentioned instantaneous wave packet reductions.

2. The operational space and time

If we assume as explained in the introduction that the state of a physical system is given at any stage of its evolution by the properties which are actual, and if actual means that the experimental test would confirm the prediction with certainty, we have to consider first more closely what characterizes space-time measurements. For a given observer, they consist of separate measuring of mass or charge densities or volumes in the *three-dimensional* geometrical space which is located in terms of the rods of this observer, and by a time coordinate, *one-dimensional* valued measure located with devices of an obviously different nature: the clocks. From this operational point of view we should already note that there is an essential difference between, say, the rotations which relate measurements made by devices of the same nature, and the Lorentz boosts which do not. The Poincaré transformations are thus, operationally speaking, not all of the same kind, and are thus not necessarily bound to play all the same role in the theory.

The question is then whereas it is possible to reconciliate this notion of state of an extended object, its status of objectivity (i.e., the fact that a property is actual whether one does test it or not) together with the usual space-time Poincaré symmetries.

The usual point of view, as there exists no absolute three-dimensional space, is to consider the set of successive spaces in time and to characterize together the state and its evolution and thus describe a point system in terms of trajectories, mixing in this way the kinematics with the dynamics, with the above mentioned consequences (no-go theorem). In fact, on a trajectory, nothing moves and it is no longer possible to talk of the evolution, neither to distinguish it from the state. Moreover, it is of course difficult to think of giving a meaning to quantum probabilities in such a scheme.

From our point of view, and coming back to the space-time properties, we have, in view of the above definition of a state, to consider first an *operational space-time*, i.e., an image space containing all possible results of space-time measurements, and then define, *on this description*, the action of the usual equivalence postulates related to the Poincaré transformations.

Let us therefore, apologizing for that, start from the following most elementary discussion. Consider a measured event P in space and time. In order to describe it, a corresponding observer will give four numbers x^μ, which are the results of the measures on his clock and his rulers. For fixing this relation, this observer has first chosen an arbitrary origin of space-time (reflecting the passive symmetry under 4-translations) and an arbitrary set of space axis (reflecting the symmetry under rotations). This given observer is however not free to choose an arbitrary Lorentz frame, the latter being given by the fact that his clock is a "pure clock" in this reference frame. As a consequence, in order to give a complete *operational* meaning to the 4 numbers x^μ, we should say in addition with which clocks and which rods the point has been measured, i.e. give a characterization of the observer to which they refer. Our proposal here is just to explicit this generally tacit part of the information about the measure of P, i.e. to characterize the operational space and time by pairs

$$P = (x, \hat{n}) \tag{2.1}$$

where $x = (x^0, \vec{x})$ refer as usual to the time and space coordinates whereas $\hat{n}$ characterizes the corresponding observer and can be chosen as follows. As for the translations and the rotations the Lorentz boosts act passively and the Lorentz invariance is reflected by the fact that we may characterize $\hat{n}$ with respect to an *arbitrarily* chosen reference observer $\hat{n}_0$. An observer at $\hat{n}$ will therefore be characterized by its relative speed $\vec{v}$ with respect to $\hat{n}_0$, parametrized for convenience by the usual $\vec{\chi} \in \mathbb{R}^3$, in the direction of $\vec{v}$ and of length

$$|\vec{\chi}| = \text{Arc Th} \frac{|\vec{v}|}{c} \tag{2.2}$$

Each space-time measure (and correspondingly each space-time property) will thus be represented in a *seven-dimensional* space of coordinates $(y, \vec{\chi}) \in \mathbb{R}^7$ with the following convention that $(y^0, \vec{y})$ are the coordinates of the operational space and time for the arbitrarily chosen reference observer at $\hat{n}_0$, and $\vec{\chi}$ is as in (2.2). This 7-dimensional space, called the operational space-time will be denoted by Y_q.

Our first proposal is thus just the following: we do not identify *a priori* a space-time point seen by different Lorentz observers as they correspond to *different realities in terms of measurements*, hence, as their relationships with the above given notion of state is different, to *a priori* different properties of the corresponding physical system.

3. Action of the Poincare group on Y_q

Let us first construct on this space Y_q, of coordinates $(y, \vec{\chi})$ the action of the Poincaré group, conformally to the physical interpretation that we have given for this space.

We first choose, $\forall \vec{\chi} \in \mathbb{R}^3$ a coset representative $L_{\vec{\chi}}$ of the Lorentz group $\mathfrak{L}$ modulo the rotations $\mathfrak{J}$, such that

$$L_{\vec{\chi}} : \hat{n}_0 \rightarrow \hat{n}(\vec{\chi}) \tag{3.1}$$

where $L_{\vec{\chi}}$ can be identified with a Lorentz transformation satisfying

$$L_{\vec{\chi}}(1, \vec{0}) = (\sqrt{1+\vec{\chi}^2}, \vec{\chi}) = \hat{n}(\vec{\chi}) \tag{3.2}$$

the set of all observers corresponding then to the upper half hyperboloid

$$H_+ = \{\hat{n} \in \mathfrak{M}(4) \mid \hat{n}^2 = (n^0)^2 - \vec{n}^2 = 1, n^0 > 0\} \tag{3.3}$$

with $\mathfrak{M}(4)$ the Minkovski space. The transformation (3.2) is then unique up to a right multiplication by a rotation $\alpha \in \mathfrak{J} : L'_{\vec{\chi}} = L_{\vec{\chi}} \cdot \alpha$. For each observer, $\hat{n}$ refers to the direction of the rest frame trajectories of his clocks, i.e. the direction of its (operational) time-axis. We had already been led to the introduction of this vector in a slightly different context [12] but with the same physical interpretation.

By definition $(y^0, \vec{y})$ represents the coordinates of time and space for the reference observer at $\hat{n}_0 = (1, \vec{0})$, hence a space-time translation $a \in T_4$ corresponds for this observer $(\vec{\chi} = \vec{0})$ to

$$U(a)(y, \vec{0}) = (y + a, \vec{0}) \tag{3.4}$$

On the other hand a Lorentz boost does not change the interpretation of the y coordinates (in terms of clocks and rods of the reference observer) so that the boosts (3.2), should give, for $\vec{\chi} = \vec{0}$,

$$U(L_{\vec{\chi}})(y^0, \vec{y}, \vec{0}) = (y^0, R(L_{\vec{\chi}})\vec{y}, \vec{\chi}) \tag{3.5}$$

with $R(L_{\vec{\chi}})$ an a priori arbitrary space-rotation. Finally we may ask that for this reference observer a rotation α corresponds to a change of choice of axis of an amount α, i.e.,

$$U(\alpha)(y^0, \vec{y}, \vec{0}) = (y^0, \alpha\vec{y}, \vec{0}), \qquad \forall \alpha \in \mathfrak{J} \tag{3.6}$$

The important point is now that these three assumptions (3.4), (3.5) and (3.6) are compatible and determine uniquely (up to a choice (3.2)), an irreducible (non-linear) representation of the Poincaré group. This representation is explicitly given by

$$U(a)(y, \vec{\chi}) = (y + L_{\vec{\chi}}^{-1}a, \vec{\chi}), \qquad \forall a \in T_4 \tag{3.7}$$

$$U(\Lambda)(y, \vec{\chi}) = (\alpha(\Lambda \cdot L_{\vec{\chi}})y, \vec{\chi}(\Lambda \cdot L_{\vec{\chi}})) \tag{3.8}$$

$\forall \Lambda \in \mathfrak{L}$, the Lorentz group, and with the notation of the coset decomposition

$$\Lambda = L_{\vec{\chi}(\Lambda)} \cdot \alpha(\Lambda) \tag{3.9}$$

and finally, by definition

$$U(a, \Lambda) = U(a)U(\Lambda) \tag{3.10}$$

The irreducibility of this representation is meant in terms of bundle representation (with base $\vec{\chi} \in \mathbf{R}^3$ and fiber $\mathfrak{M}(4)$). In fact this irreducibility is verified in a very strong sense, as the representation is transitive on Y_q, so that this space can be identified with an homogeneous space of the Poincaré group $\mathfrak{P}$. One can check that

$$Y_q \cong \mathfrak{P}/\mathfrak{J} \tag{3.11}$$

with $\mathfrak{J}$ the rotation group in 3-space.

In terms of $\hat{n} \in H_+$, with the relation (3.2), this representation in (3.7)–(3.10) can be rewritten as

$$U(a, \Lambda)(y, \hat{n}) = ((L_{\Lambda\hat{n}}^{-1}\Lambda L_{\hat{n}})y + L_{\Lambda\hat{n}}^{-1}a, \Lambda\hat{n}) \tag{3.12}$$

with $L_{\hat{n}} \equiv L_{\vec{\chi}(\hat{n})}$, so that the rotation in (3.8) is just the Wigner rotation associated to $\hat{n}$ and Λ.

It will be useful to compare this representation in Y_q (in terms of the measuring devices of the reference observer) to the more direct one (in terms of all measuring devices) defined in $\mathfrak{M}(4) \times H_+$ by

$$V(a, \Lambda)(x, \hat{n}) = (\Lambda x + a, \Lambda\hat{n}) \tag{3.13}$$

with the advantage of making a comparison with the usual framework easier, by simple projection $\pi : (x, \hat{n}) \mapsto x$.

In fact one can check that the operator $W : \mathfrak{M}(4) \times H_+ \to Y_q$ defined by

$$W(x, \hat{n}) = (L_{\hat{n}}^{-1}x, \hat{n}) \tag{3.14}$$

interwines (3.12) with (3.13), so that both representations are simply equivalent.

4. Simultaneity and physical states

We have defined as actual a property whose corresponding test measurement would give a positive answer with a probability equal to one. Concerning space-time, such a measurement consists of measures of positions in space (with rods) and with values in the subsets of $\mathbb{R}^3$ and of measures of time with a set of equidistant clocks [16] and with values in $\mathbb{R}$. Necessarily such a measure is related to a well defined class of observers and it is possible to represent it unambigously in Y_q by a set of the form $(x^0, \Delta\vec{x}, \hat{n})$, with $x^0 \in \mathbb{R}$, $\Delta\vec{x} \subseteq \mathbb{R}^3$, $\hat{n} \in H_+$. The point is now that a Lorentz boost does obviously not transform such a set in a similar one, the spatial extension referring only to a well defined $\hat{n}$. A possible property is thus not transformed in a possible property in the above sense, but in a complicated (and non-operational) mixture of space and time measurements. This means that a Lorentz boost does not have to be a symmetry of the state space, as relating different properties and not different equivalent descriptions of the same property. This is of course not in contradiction with the relativistic invariance, the latter referring to the whole description of the system and not to this particular part which is the state. We now want to make all these considerations mathematically precise and therefore first define a notion of simultaneity which corresponds to the natural intuition of 3-dimensional space extension. Our bigger space Y_q makes it precisely possible.

For a given observer, at $\hat{n}$, two events y_1 and y_2 will be called simultaneous if they are related by a pure space translation $\vec{b} \in \mathbb{R}^3$, hence, by (3.12)

$$L_{\hat{n}}(y_1 - y_2) = (0, \vec{b}) \tag{4.1}$$

This definition generalizes in the whole Y_q by the following convention: we shall call two events $(y_1, \hat{n}_1)$ and $(y_2, \hat{n}_2)$ simultaneous, if and only if there is a space translation $\vec{b}$ such that

$$L_{\hat{n}_1} y_1 - L_{\hat{n}_2} y_2 = (0, \vec{b}) \tag{4.2}$$

It is easy to verify that this relation is an equivalence relation, that will be denoted

$$(y_1, \hat{n}_1) \sim (y_2, \hat{n}_2) \tag{4.3}$$

In view of the above interpretation of Y_q, two events $(y_1, \hat{n}_1)$ and $(y_2, \hat{n}_2)$ are thus simultaneous if their time components (for the observers $\hat{n}_1$ and $\hat{n}_2$ respectively) are equal (after of course a general agreement on the choice of a zero of space-time as in the usual Minkovski space).

The relation (4.2) can also be written as

$$y_1 \cdot \tilde{n}_1 = y_2 \cdot \tilde{n}_2 \tag{4.4}$$

with $\tilde{n} = L_{\hat{n}}^{-1} \cdot \hat{n}_0$. We shall call τ the following corresponding function on Y_q

$$\tau(y, \hat{n}) = y \cdot \tilde{n} = y^{\mu} \tilde{n}_{\mu} \tag{4.5}$$

and $d\tau$ the following associated dynamical differential 1-form.

$$d\tau(y, \hat{n}) = \tilde{n}_{\mu}\, dy^{\mu} \tag{4.6}$$

In order to explicit the significance of the above simultaneity relation, let us briefly examine its symmetry properties. It follows from (3.12) and (4.2) that if $(y_1, \hat{n}_1) \sim (y_2, \hat{n}_2)$, then

$$U(a, \Lambda)(y_1, \hat{n}_1) - U(a, \Lambda)(y_2, \hat{n}_2) = \Lambda(0, \vec{b}),$$
$$\forall (a, \Lambda) \in \mathfrak{P}, \quad \text{some} \quad \vec{b} \in T_3 \tag{4.7}$$

so that the simultaneity relation is covariant under $\mathfrak{P}$. But it is obviously not invariant. Its invariance subgroup, defined at a point $z=(y, \hat{n}) \in Y_q$ by the set of all transformations g which map all events simultaneous to z into events simultaneous to gz, i.e.,

$$G(z)=\{g \in \mathfrak{P} \mid gz' \sim gz, \forall z' \text{ with } z' \sim z\} \tag{4.8}$$

is isomorphic $\forall z \in Y_q$, to the semidirect product

$$G = T_4 \circledS \mathfrak{J}, \tag{4.9}$$

with T_4 the 4-translations and $\mathfrak{J}$ the space rotations.

In view of (4.5) we may characterize by τ a surface of simultaneity (and call it τ-surface), the set of all z' which are equivalent to some z, i.e., the equivalence classes defined in Y_q by (4.2). By definition one moves along such a surface by a space translation, or by a change of observer without change of time coordinate.

Let us finally calculate the stabilizer $H(z)$ of a τ-surface through a point z, i.e., the subgroup of $\mathfrak{P}$ (and of $G(z)$) consisting of all elements which map all events simultaneous to z into themselves,

$$H(z)=\{g \in \mathfrak{P} \mid gz' \sim z, \forall z' \sim z\} \tag{4.10}$$

It is easily obtained and is $\forall z \in Y_q$ isomorphic to

$$H = T_3 \circledS \mathfrak{J} \tag{4.11}$$

Let us now interpret these results. In view of the given interpretation of the space Y_q, the group G is the group which maps a set of actual properties into an equivalent description of the same actual properties (passive action), hence a state into a state. It is thus by definition the space-time part of the *kinematical symmetry group* [8]. On the other hand H correspond to the transformations of actual properties into actual properties at the same "stage" of the evolution (same τ-surface), hence, as

$$G/H \cong T_0 \tag{4.12}$$

with T_0 the time translation group, the evolution can be parametrized by the time of any arbitrary observer, all being proportional to τ.

The set of all space-time properties (referring thus to all observers, hence to a whole τ-surface) is of course not independent, (and will in fact even depend on the dynamics). This is somehow similar to the localization of a particle in a region of space, which is not independent of the localization of this particle in another region of space. There might thus be redundances in this description. We shall therefore in the present paper first only consider systems whose knowledge is assumed to be completely determined when determined for a particular $\hat{n}$, and such that the results do not depend on this choice. This means for a point particle for example that knowledge of state and evolution at any fixed $\hat{n}$ is assumed to be sufficient to determine unambiguously the trajectory (and all other desired properties of that particle).

It is important to note firstly that this assumption is not necessary (there might be in principle systems not satisfying it) and secondly that this assumption does not modify the symmetry. The Lorentz boosts do not belong to G in (4.9), they are not kinematical symmetries in the precise sense we have given; they are not bound to be represented by automorphisms of the state space. They will thus play a different role in the theory, in fact as a constraint for the possible forms of the dynamics, as we shall see in Sections 6 and 7.

The main result obtained so far is thus the following: the state space of a relativistic system is given by some $\hat{n} \in H_+$ and a state space $K_{\hat{n}}$, carrier of a representation of the kinematical symmetry group which, as far as space-time is concerned, is given by the stabilizer $G_{\hat{n}}$ of $\hat{n}$, isomorphic $\forall \hat{n}$ to the group G given in (4.9).

There is another direct way of explaining this conclusion: for any observer the state of N classical point particles can be represented in space-time by N equal-time points. The set of all possible states for this observer is given by the set of all possible N equal-time points. By definition it is his state space. The space-time automorphism group of this space is $\mathfrak{P}$ when $N = 1$, else it is G. The description of the state of an extended system implies thus a lowering in the space-time symmetry of the state space.

In quantum mechanics this is even more important, as wave functions will be defined, in function of $\hat{n}$, in 3-dimensional space surfaces perpendicular to these vectors $\hat{n}$ and containing quantum correlations and superpositions (as in a succession of slices along the t-axis, again a picture whose automorphism group is only G). That these correlations need to be described by a simultaneity argument is not only experimentally established [14] but is logically obvious: if one measures, say, the position of an electron on a screen, then the wave packet has to "know" immediately and everywhere in the space that the particle has been localized, so to avoid the possibility of another localization ("after", "elsewhere") where the wave function would "still" be nonzero.

The recent measures of Aspect [14] confirm that this "information" inside of the wave packet indeed occurs at supraluminal speed. This is not in contradiction with relativity, as this "information" cannot be used for the transmission of signals. If the wave function represents the system, hence if given a space-time point the value of the wave function would be fixed for all coordinate systems, then such an instantaneity in the wave packet reduction would make no sense in contradiction with experiment. If however we follow as here the point of view that the wave function represents only the knowledge we have of the system, only the point on the screen has an objective meaning, and different observers may have different wave functions collapsing on different simultaneity planes. We see thus that the wave packet reduction makes the introduction of such a $\hat{n}$ absolutely necessary for consistency.

5. State spaces of elementary systems

We now come to the systematic derivation of the (irreducible) state spaces which correspond to the above analysis, i.e., to the corresponding classification of elementary physical systems. We shall use for that purpose a recently developed group-theoretical formalism [9], that we first very briefly remind and in which it is possible to treat simultaneously and in a common language classical and quantum physics.

This framework is characterized by the fact that we consider as possible state spaces (topological) direct unions of separable Hilbert spaces

$$K = V_S \mathfrak{H}_s, \; s \in S \tag{5.1}$$

each state being thus given by a point $s_0 \in S$ and a ray $\hat{\psi}_{s_0}$ in the corresponding Hilbert space $\mathfrak{H}_{s_0}$. Classical and quantum physics indeed appear as the two extremal cases where each Hilbert space is one dimensional or respectively, where S is a single point. More generally such a choice (5.1) makes it possible to describe systems where some of the observables have a quantal behaviour whereas other ones are of the classical type in the sense that they commute with all other ones and have a purely discrete point spectrum. Because of their interpretation, this latter type of observables has also been called superselection variables. They should not, by the way, be confused with superselection rules, the latter refering to observables which are in addition conserved quantities. In our framework this is in general not true and even then, it can only be asserted when the dynamics are known (see Section 6), whereas the above framework is yet precisely independent of the interactions and of the evolution.

Corresponding to the points of view discussed previously, the physical system is characterized by a set of observables corresponding to all measures that we (possibly) perform on it. The usual physical equivalence postulates in the descriptions, correspond then to the action of the (complete) *kinematical symmetry group* [8] defined by its action on the possible outcomes for all observables, as just seen in the particular case of space-time.

This is sufficient for characterizing the properties of the system and it is, as it should be, independent of the dynamics. Hence the latter does not change the interpretation nor the representation of the observables.

The state spaces (5.1), for a given system, are further specified by the following two main assumptions: firstly they should carry (as automorphisms) an irreducible representation fot the above mentioned kinematical symmetry group. Secondly they should carry a (sufficiently faithful) representation of all properties, hence of all observables corresponding to the physical system. For the first condition we have appropriately generalized in [9] the idea of induced representations [17] on (5.1) and for the second, the idea of systems of imprimitivity [18].

As an automorphism of (5.1) is given [19] by a permutation π of S and a family $\{U_s\}$ of isometries

$$U_{\pi(s)} : \mathfrak{H}_s \to \mathfrak{H}_{\pi(s)} \tag{5.2}$$

one can restrict ourselves for the irreducible representations to the case where the actions induced via π by G on S is transitive so that one can identify S with a quotient space

$$S = G/H, \qquad H = \text{Stab } s_0 \tag{5.3}$$

for some $s_0 \in S$ and one can restrict ourselves to the case where all Hilbert spaces are isomorphic. Corresponding to H let $g = h_g k_{s(g)}$ be the (right) coset decomposition of $g \in G$ for a fixed set of coset representatives $k_{s(g)}$ normalized with $k_{s(e)} = e$, e the unit of G, and with the convention

$$k_{s(g)}^{-1} \cdot s_0 = s(g) = g^{-1} \cdot s_0 \tag{5.4}$$

with the natural action of g on $S = G/H$. Let us now choose $\mathfrak{H}_{s_0}$ to be the carrier space of a unitary projective representation $h \mapsto L(h)$ of H, with arbitrary multiplier $\omega \in Z^2(H, U(1))$, i.e.,

$$L(h_1)L(h_2) = \omega(h_1, h_2)L(h_1 h_2) \tag{5.5}$$

with $\omega: H \times H \to U(1)$. One can now define a unitary projective K-representation of G on (5.1) induced by this representation L of H by

$$U_s(g) = L_s(g) \cdot \xi(g) \tag{5.6}$$

with

$$L_s(g) = L(\nu(s, g)) \tag{5.7}$$

$\nu(s, g)$ being the associated G–S–H-cocycle [20]

$$\nu(s, g) = k_s \cdot g \cdot k^{-1}_{g^{-1}s} \tag{5.8}$$

and $\xi(g)$ corresponding to the action of G on S

$$\xi(g)\phi_s = \phi_{g \cdot s} \tag{5.9}$$

It is important to note that in (5.6) there is no phase factor, unlike in the direct integral case, forcing the projectivity of U to be completely carried by H. This implies e.g. that the Planck constant that turns out to label such an ω is necessarily zero in the classical cases whereas nonzero in the quantal cases [12].

The above construction can also be shown to be exhaustive in the sense that every irreducible projective K-representation of G is equivalent to an induced representation of this type [9].

In view of the following application, we also mention the associated observables in such a K. They are given by a slight generalization of the notion of systems of imprimitivity [18]: let T be any G-space and $\mathfrak{B}(T)$ the Borel sets in T. In physical applications T is nothing but the set of all possible values of a given observable [8]. We consider the mappings

$$P: \Delta \in \mathfrak{B}(T) \to P_\Delta \in \mathfrak{P}(K) \tag{5.10}$$

with $\mathfrak{P}(K)$ the projections in K, i.e., the families $\{P_s, s \in S\}$ of projectors in the corresponding Hilbert spaces $\mathfrak{H}_s$. Assume further that the maps (5.10) satisfy

$$\begin{aligned} &\text{(i)} \quad P_\phi = 0_K, P_T = \mathbb{1}_K \\ &\text{(ii)} \quad P_\Delta \cdot P_{\Delta'} = P_{\Delta \cap \Delta'}, \forall \Delta, \Delta' \in \mathfrak{B}(T) \\ &\text{(iii)} \quad P_{\cup \Delta_i} = \textstyle\sum P_{\Delta_i}, \text{ for } \Delta_i \cap \Delta_j = \phi \text{ if } i \neq j \text{ and } i, j \in I, \text{ a countable set} \end{aligned} \tag{5.11}$$

where sums and multiplications are defined over S term by term.

We have called supersystems of imprimitivity these maps when they satisfy in addition the following covariance conditions

$$(U(g)P_\Delta U(g)^{-1})_s = (P_{\tau(g)\cdot\Delta})_s \tag{5.12}$$

$\forall s \in S$, and with τ the G-action on T.

A particular, very simple but important solution of (5.10), (5.11) and (5.12) is given by $T = S$ and

$$(P_\Delta)_s = \chi_\Delta(s) \cdot \mathbb{1}_s \tag{5.13}$$

with χ_Δ the characteristic function of $\Delta \in \mathfrak{B}(S)$ and $\mathbb{1}_s$ is the identity on $\mathfrak{H}_s$. This observable is the above mentioned super-selection or classical variable.

More generally we have shown in [9] that all supersystems of imprimitivity can be canonically associated to the above induction procedure and to usual restricted Mackey systems of imprimitivity based on $\mathfrak{H}_{s_0}$, for the appropriate subgroups of H. The latter in $\mathfrak{H}_s$ correspond of course to the quantal observables (as being related to self-adjoint operators and their spectral resolutions). We refer to [9] for more details, and for the explicit form of the corresponding imprimitivity theorem.

More important for us here is that it has been possible to deduce from these results a workable method for deriving all irreducible K-representations associated with a given set of observables (i.e., admitting supersystems of imprimitivity for each element of this given set). We also refer to [9] on this point.

Let us now apply this method to our problem and consider a simple massive elementary particle. We assume that the state of this physical system is characterized, at $\hat{n}$, by its observable $q_{\hat{n}} = (q^0_{\hat{n}}, \vec{q}_{\hat{n}})$ in the corresponding space and time and a 3-momentum $\vec{p}_{\hat{n}}$ in the space at $\hat{n}$ (i.e., the plane perpendicular to $\hat{n}$ in the $g^{\mu\nu}$ metric), where $\hat{n} \in H_+$ is arbitrary but fixed.

The kinematical symmetry group $G_{\hat{n}}$ is then obtained from (4.13) for what concerns space-time and from the additional postulate for $\vec{p}_{\hat{n}}$ that there exists no absolute rest-frame so that the choice of the zero of momentum in $\vec{p}_{\hat{n}}$ is arbitrary (else there would be a $\hat{n}$ absolutely at rest). Let us remember once more that we talk about the symmetries of the description and not the symmetries of the state themselves (or of their evolution). We thus have

$$\vec{p}_{\hat{n}} \mapsto \vec{p}_{\hat{n}} + \vec{w}_{\hat{n}}, \qquad \vec{w}_{\hat{n}} \in (T^*_3)_{\hat{n}} \cong \mathbb{R}^3 \tag{5.14}$$

The kinematical symmetry group $G_{\hat{n}}$ is thus generated by elements

$$\{(\vec{w}_{\hat{n}}, a^0_{\hat{n}}, \vec{a}_{\hat{n}}, \alpha_{\hat{n}}), \vec{w}_{\hat{n}} \in (T^*_3)_{\hat{n}}, a^0_{\hat{n}} \in (T_0)_{\hat{n}}, \vec{a}_{\hat{n}} \in (T_3)_{\hat{n}}, \alpha_{\hat{n}} \in \mathfrak{J}_{\hat{n}}\}$$

with the following defining action

$$(\vec{p}_{\hat{n}}, q_{\hat{n}}) \mapsto (\alpha_{\hat{n}}(\vec{p}_{\hat{n}} + \vec{w}_{\hat{n}}), q^0_{\hat{n}} + a^0_{\hat{n}}, \alpha_{\hat{n}}(\vec{q}_{\hat{n}} + \vec{a}_{\hat{n}})) \tag{5.15}$$

If we now apply the above mentioned formalism, then we find in a unified way exactly two families of solutions, i.e. of irreducible projective K-representations of $G_{\hat{n}}$ admitting the corresponding observables $\vec{q}_{\hat{n}}$, $\vec{p}_{\hat{n}}$ and $q^0_{\hat{n}}$. Without entering in the detailed calculations we just list the results:

(a) *The classical particles*

In this solution S is given by the 6-dimensional phase space $\Gamma_{\hat{n}} = \{(\vec{p}_{\hat{n}}, \vec{q}_{\hat{n}})\}$ at $\hat{n}$, times the corresponding time axis $\mathbb{R}_{q^0_{\hat{n}}}$ so that

$$K_{\hat{n}} = V_{\Gamma_{\hat{n}} \times \mathbb{R}_{q^0_{\hat{n}}}}(\mathfrak{H}(D^\sigma))_{\vec{p}_{\hat{n}}, q^0_{\hat{n}}, \vec{q}_{\hat{n}}} \tag{5.16}$$

where $\mathfrak{H}(D^\sigma) \cong \mathbb{C}^{2\sigma+1}$ is the carrier space of the usual $2\sigma+1$-dimensional spin σ representation of $\mathfrak{J}_{\hat{n}}$, the rotation group in $\Gamma_{\hat{n}}$. The representation explicitly reads

$$\begin{aligned} U(a_{\hat{n}})\Psi_{\{\vec{p}_{\hat{n}}, q^0_{\hat{n}}, \vec{q}_{\hat{n}}\}} &= \Psi_{\{\vec{p}_{\hat{n}}, q^0_{\hat{n}} + a^0_{\hat{n}}, \vec{q}_{\hat{n}} + \vec{a}_{\hat{n}}\}} \\ U(\vec{w}_n)\Psi_{\{\vec{p}_{\hat{n}}, q^0_{\hat{n}}, \vec{q}_{\hat{n}}\}} &= \Psi_{\{\vec{p}_{\hat{n}} + \vec{w}_{\hat{n}}, q^0_{\hat{n}}, \vec{q}_{\hat{n}}\}} \\ U(\alpha_{\hat{n}})\Psi_{\{\vec{p}_{\hat{n}}, q^0_{\hat{n}}, \vec{q}_{\hat{n}}\}} &= D^\sigma(\alpha_{\hat{n}})\Psi_{\{\alpha_{\hat{n}}\vec{p}_{\hat{n}}, q^0_{\hat{n}}, \alpha_{\hat{n}}\vec{q}_{\hat{n}}\}} \end{aligned} \tag{5.17}$$

with $\psi \in \mathfrak{H}(D^{(\sigma)})$, whereas the observables are obtained (as in (5.13)) via the characteristic functions

$$\begin{aligned}\vec{q}_{\hat{n}}(\Delta\vec{q}_{\hat{n}})\Psi_{\{\vec{p}_{\hat{n}},q^0_{\hat{n}},\vec{q}_{\hat{n}}\}} &= \chi_{\Delta\vec{q}_{\hat{n}}}(\vec{q}_{\hat{n}})\Psi_{\{\vec{p}_{\hat{n}},q^0_{\hat{n}},\vec{q}_{\hat{n}}\}}\\ \vec{p}_{\hat{n}}(\Delta\vec{p}_{\hat{n}})\Psi_{\{\vec{p}_{\hat{n}},q^0_{\hat{n}},\vec{q}_{\hat{n}}\}} &= \chi_{\Delta\vec{p}_{\hat{n}}}(\vec{p}_{\hat{n}})\Psi_{\{\vec{p}_{\hat{n}},q^0_{\hat{n}},\vec{q}_{\hat{n}}\}}\\ q^0_{\hat{n}}(\Delta q^0_{\hat{n}})\Psi_{\{\vec{p}_{\hat{n}},q^0_{\hat{n}},\vec{q}_{\hat{n}}\}} &= \chi_{\Delta q^0_{\hat{n}}}(q^0_{\hat{n}})\Psi_{\{\vec{p}_{\hat{n}},q^0_{\hat{n}},\vec{q}_{\hat{n}}\}}\end{aligned} \tag{5.18}$$

A state being a ray in the corresponding $\mathfrak{H}_s$, there is a one-to-one correspondence between points in the phase space together with the time axis at $\hat{n}$, $\Gamma_{\hat{n}} \times \mathbb{R}_{q^0_{\hat{n}}}$, and the $\sigma = 0$ states. The above solution can thus obviously be identified, up to the additional possible presence of a (quantal) spin, with the usual framework of classical mechanics.

(b) *The quantal particles*

In the second family of solutions, S can be identified with the time axis at $\hat{n}$, hence with the coset space $G/H_{q^0_{\hat{n}}}$ with

$$H_{q^0_{\hat{n}}} = \{(\vec{w}_{\hat{n}}, 0, \vec{a}_{\hat{n}}, \alpha_{\hat{n}})\} \tag{5.19}$$

The space $K_{\hat{n}}$ is given by

$$K_{\hat{n}} = V_{\mathbb{R}_{q^0_{\hat{n}}}}(\mathfrak{L}^2(\mathbb{R}^3, d^3x_{\hat{n}}) \otimes \mathfrak{H}(D^\sigma))_{q^0_{\hat{n}}},\ q^0_{\hat{n}} \in \mathbb{R}_{q^0_{\hat{n}}} \tag{5.20}$$

with the representation

$$\begin{aligned}U(\vec{w}_{\hat{n}})\Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}}) &= \exp(i\hbar^{-1}\vec{w}_{\hat{n}} \cdot \vec{x}_{\hat{n}})\Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}})\\ U(a^0_{\hat{n}})\Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}}) &= \Psi_{q^0_{\hat{n}}+a^0_{\hat{n}}}(\vec{x}_{\hat{n}})\\ U(\vec{a}_{\hat{n}})\Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}}) &= \Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}} - \vec{a}_{\hat{n}})\\ U(\alpha_{\hat{n}})\Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}}) &= D^\sigma(\alpha_{\hat{n}})\Psi_{q^0_{\hat{n}}}(\alpha_{\hat{n}}^{-1}\vec{x}_{\hat{n}})\end{aligned} \tag{5.21}$$

implying in particular the Weyl commutation relations

$$U(\vec{w}_{\hat{n}})U(\vec{a}_{\hat{n}}) = \exp(i\hbar^{-1}\vec{w}_{\hat{n}} \cdot \vec{a}_{\hat{n}})U(\vec{a}_{\hat{n}})U(\vec{w}_{\hat{n}}) \tag{5.22}$$

which explicits the fact that quantum correlations are given, in this description, in the planes perpendicular to $\hat{n}$ (in the $g^{\mu\nu}$ metric).

The observables position and momentum are given by

$$\begin{aligned}\underline{\vec{q}_{\hat{n}}}\Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}}) &= \vec{x}_{\hat{n}} \cdot \Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}})\\ \underline{\vec{p}_{\hat{n}}}\Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}}) &= -i\hbar\vec{\partial}_{\vec{x}_{\hat{n}}}\Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}})\end{aligned} \tag{5.23}$$

corresponding to the characteristic functions in $\vec{x}_{\hat{n}}$ and in the Fourier transformed space, whereas the time observable is given by

$$\underline{q^0_{\hat{n}}}(\Delta q^0_{\hat{n}})\Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}}) = \chi_{\Delta q^0_{\hat{n}}}(q^0_{\hat{n}})\Psi_{q^0_{\hat{n}}}(\vec{x}_{\hat{n}}) \tag{5.24}$$

We note that here again, in all planes perpendicular to $\hat{n}$ we have, as follows from (5.23), the (equal-time) commutation relations

$$[(\underline{\vec{q}_{\hat{n}}})^i, (\underline{\vec{p}_{\hat{n}}})^j] = i\hbar\delta^{ij} \cdot \mathbb{1}_{\hat{n}} \tag{5.25}$$

with $\mathbb{1}_{\hat{n}}$ the identity in $K_{\hat{n}}$.

Except for the presence of the (arbitrary but fixed) $\hat{n} \in H_+$, the above solution can obviously be identified with the framework of *non-relativistic* quantum mechanics discussed in [8] and which is the usual one up to the presence of the time observable due to the more general choice (5.1) with respect to single Hilbert spaces.

We also note at this point that the representations (5.17) or (5.21) are not characterized by any dynamical type relationship as it was the case for the space-time Poincaré [2] or Galilei [21] groups where we had, respectively,

$$\begin{aligned} &p^2 - m^2 = 0 \\ &\vec{p}^2/2m - E = \text{constant} \end{aligned} \tag{5.26}$$

implying in both cases that the corresponding theories would in fact strictly speaking only describe free particles.

Comparing now the relativistic situation with the non-relativistic one (see e.g. [12]), we observe that the kinematical symmetry groups G are the same, the state spaces are the same, with the same physical interpretation, (and this makes comparison and taking limits easier), just the label $\hat{n}$ has a different meaning: in the non-relativistic case (see [12]), the set of all $\hat{n}$ labels a 3-dimensional vector space, corresponding to the Galilean boosts, so that all correlation relation descriptions like the one in (5.25) are identified with each other. Obviously the set $\{(1, \vec{v}), \vec{v} \in \mathbb{R}^3\}$ is nothing but the non-relativistic limit of our hyperboloïd H_+.

In the here considered cases where the final physical predictions are assumed to be fixed for any choice of $\hat{n}$ there are however so far no difference at all: the characterization of an elementary particle is the same in the relativistic and the non-relativistic theories. In fact, if there is nothing relativistic in the obtained state spaces, there is nothing non-relativistic either, as neither the Lorentz nor the Galilei boosts belong to the corresponding kinematical symmetry groups. The only adopted principle is that there is no absolute rest frame (zero for $\vec{p}$) and this principle is the same in relativistic and in non-relativistic physics.

What will change will thus be in the dynamics, the possible evolution laws, as we shall now see explicitly.

6. Relativistic versus non-relativistic dynamics

In this section we want to show how it is possible to derive the possible dynamics, on the so far obtained state spaces, by expressing the remaining symmetry arguments at our disposal (Lorentz or Galilean boosts) in an appropriate way.

Let us therefore first briefly remind how the dynamics enter into the play in our framework, from a general point of view (refering to [22] for more details on this point).

The evolution corresponds by definition to the changes of the states, hence it is specified (if reversible) by a two parameter family of automorphisms of the state spaces. For specifying the dynamics we thus have to introduce the parameter itself, an evolution parameter that we also call τ. Such a parameter cannot be itself an observable (it is not a property but a label for the changes in the properties [7]) and it cannot thus enter in the dependence of the generator of the evolution so that the two parameter family reduces to a 1-parameter group.

Together with some differentiability conditions we are led, in the space K in (5.1) to the following generalized Schrödinger equations coupled with classical evolution equations in the superselection variable set S:

$$\dot{s} = \mathfrak{X}(s) \tag{6.1}$$

$$i\partial_\tau \Psi_s = H_s \Psi_s \tag{6.2}$$

where the dot means differentiation with respect to τ, $\{H_s, s \in S\}$ is a family of self-adjoint operators and $\mathfrak{X}$ is a vector field on S. In order to find out the allowable dynamics we now can and have to make more assumptions that will lead to explicit (6.1) and (6.2).

In order to do that, we can take advantage of the fact that in our approach we have a unified language, a unified mathematical formalism for the discussion and the interpretation of the physical quantities in classical and in quantum physics. Moreover we have in each of these cases the same state spaces for the relativistic and for the non-relativistic cases. We shall therefore discuss the (easiest) classical non-relativistic dynamical principle first and then generalize to quantum physics on the one side and to relativistic dynamics on the other.

Let us first discuss the spinless case ($\sigma = 0$) and choose for simplicity the arbitrary zero in H_+ so that $\hat{n} = \hat{n}_0 = (1, \vec{0})$. As this index $\hat{n}$ will be fixed in the following, we also omit for simplicity of notation altogether both indices σ and $\hat{n}$ everywhere.

(a) *Non-relativistic dynamics*

We shall here follow an idea due to Jauch [15]: we first suppose that $\mathfrak{X}$ is an Hamiltonian vector field with respect to the usual symplectic form on phase space and with generating function H, so that we have

$$\dot{\underline{\vec{q}}} = \frac{\partial H}{\partial \vec{p}} \tag{6.3}$$

with $\dot{\underline{\vec{q}}}$ a possible function of the only observables, i.e. of $\underline{\vec{q}}$, $\underline{\vec{p}}$ and $\underline{\hat{q}}^0$. Jauch then defines an elementary massive particle by the fact that there exists a constant m such that under $\vec{w}$ in (5.14) $\dot{\underline{\vec{q}}}$ transforms like

$$U(\vec{w})\, \dot{\underline{\vec{q}}}\, U(\vec{w})^{-1} = \dot{\underline{\vec{q}}} + \frac{\vec{w}}{m} \tag{6.4}$$

which reflects in an obvious way the effect of an instantaneous Galilean change of reference frame. It follows from the supersystems of imprimitivity in (5.18) that thus

$$\dot{\underline{\vec{q}}} = \frac{1}{m}(\underline{\vec{p}} - \vec{A}(q^0, \vec{q})) \tag{6.5}$$

with $\vec{A}$ three *arbitrary* functions of q^0 and $\vec{q}$. With (6.5) we can now integrate (6.3) and we immediately find

$$H = \frac{1}{2m}(\underline{\vec{p}} - \vec{A}(q^0, \vec{q}))^2 + V(q^0, \vec{q}) \tag{6.6}$$

with V again an arbitrary function. The dynamical principle (6.4) allows thus to derive very easily the usual general Hamiltonian (6.6) with arbitrary (non-necessarily electromagnetic) external fields $\vec{A}$ and V. It is nice in this respect, as emphasized by Jauch, to remark that the particular form (6.6) is thus not bound to a particular kind of interactions, but really rests on a deeper common symmetry argument as in (6.4).

The reasoning easily generalizes to the quantal case if we write (6.2) for the observables (Heisenberg picture). The space (5.16) and the representation (5.17) are replaced by (5.20) and (5.21) whereas (6.3) now reads for each $q^0 \in \mathbb{R}$

$$\dot{\vec{\underline{q}}} = i[H_{q^0}, \vec{\underline{q}}] \tag{6.7}$$

Let us now use the same dynamical principle (6.4) with correspondingly changed representations for $\dot{\vec{\underline{q}}}$ and $U(\vec{w})$. It directly follows from (5.23) that

$$\dot{\vec{\underline{q}}} = \frac{1}{m}(-i\hbar\vec{\partial}_x - \vec{A}_{q^0}(\vec{x})) \tag{6.8}$$

with again $\vec{A}_{q^0}(\vec{x})$ three arbitrary functions. If we now assume that the operator H_{q^0} in (6.7) is in the image of the Weyl–Wigner transformation of a C_1-function in $\vec{p}$ and $\vec{q}$, we may write formally

$$[H(\underline{p}, \underline{q}), \underline{q}^i] = \frac{\partial}{\partial \underline{p}^i} H(\underline{p}, \underline{q}) \tag{6.9}$$

so that, with (6.7) and (6.8) we obtain directly, together with (5.23)

$$H_{q^0}(-i\hbar\vec{\partial}_x, \vec{x}) = \frac{1}{2m}(-i\hbar\vec{\partial}_x - \vec{A}_{q^0}(\vec{x}))^2 + V_{q^0}(\vec{x}) \tag{6.10}$$

which is the usual most general Hamiltonian for one spinless massive quantal particle. For what concerns (6.1), the superselection variable evolution, we may simply postulate (see [7]) that

$$\dot{q}^0 = 1 \tag{6.11}$$

which means that the time flows in an undisturbed newtonian way, so that the time variable, being linear in τ, can indeed be used as the evolution parameter.

(b) *Relativistic dynamics*

Let us now show how the very simple derivation of Jauch above generalizes to the relativistic framework, first in the classical case. We therefore again and for the same reason write the 3-velocity $\dot{\vec{\underline{q}}}$ (defined, we remind it, with respect to the tacit $\hat{n}$) as (cf. e.g. (2.70) of [23])

$$\dot{\vec{\underline{q}}} = \frac{\partial H}{\partial \vec{p}} \tag{6.12}$$

i.e. we assume Hamiltonian equations in the plane perpendicular to $\hat{n}$. What now changes is the transformation character of this quantity under a $\vec{w}$-translation in momentum space, the relationship being no longer linear as in (6.4). Let us define

$$\vec{f}(\underline{\vec{p}}, \underline{q}) = \frac{\dot{\underline{\vec{q}}}}{\sqrt{1-\dot{\underline{\vec{q}}}^2/c^2}} \tag{6.13}$$

and postulate the following generalization of (6.4) for a single massive spinless particle

$$U(\vec{w})\, \vec{f}(\underline{\vec{p}}, \underline{q})\, U(\vec{w})^{-1} = \vec{f}(\underline{\vec{p}}, \underline{q}) + \frac{\vec{w}}{m} \tag{6.14}$$

In the non relativistic limit (6.14) obviously gives (6.4), and this equation clearly reflect the effect of an instantaneous Lorentz change of reference frame.

It immediately follows from (6.14) and the supersystems of imprimitivity (5.18) with (5.17) that

$$\vec{f}(\underline{\vec{p}}, \underline{q}) = \frac{1}{m}(\vec{p} - \vec{A}(q)) \tag{6.15}$$

with $\vec{A}$ again arbitrary. Inserting (6.15) in (6.13) and resolving for $\dot{\underline{\vec{q}}}$ we easily find that

$$\dot{\underline{\vec{q}}} = \frac{\varepsilon \cdot c(\vec{p} - \vec{A}(q))}{\sqrt{(\vec{p} - \vec{A}(q))^2 + m^2c^2}} \tag{6.16}$$

where the sign $\varepsilon = \pm 1$ comes from the fact that we have in this derivation to take a square root. Using now (6.12) we immediately find as the most general corresponding Hamiltonian

$$H = \varepsilon \cdot c\sqrt{(\vec{p} - \vec{A}(q))^2 + m^2c^2} + V(q) \tag{6.17}$$

with $\vec{A}$ and V arbitrary functions of q_0 and $\vec{q}$. This Hamiltonian determines the evolution of $\vec{p}$ and $\vec{q}$ in the usual way whereas for q^0 we may again simply assume

$$\dot{q}^0 = 1 \tag{6.18}$$

reflecting the fact that for each observer the time flows in an undisturbed newtonian way.

We shall of course call a particle the system for which $\varepsilon = +1$ and an antiparticle the one for which $\varepsilon = -1$.

Let us now generalize this principle to the quantal case given in (5.20), (5.21), (5.23). We assume that $[\vec{q}, H]$ has no spectrum outside $)-c, c($, so that we may write

$$\vec{f}(\underline{\vec{p}}, \underline{q}) = \left[\left(1 - \frac{[\vec{q}, H_{q^0}]^2}{c^2}\right)^{-1}\right]^{1/2} \cdot [\vec{q}, H_{q^0}] \tag{6.19}$$

and assume again, as dynamical principle, that

$$U(\vec{w})\, \vec{f}(\underline{\vec{p}}, \underline{q})\, U(\vec{w})^{-1} = \vec{f}(\underline{\vec{p}}, \underline{q}) + \frac{\vec{w}}{m} \tag{6.20}$$

for some positive constant m, implying, from (5.21) and (5.23) that

$$\vec{f}(\underline{\vec{p}}, \underline{q}) = \frac{1}{m}(\underline{\vec{p}} - \vec{A}_{q^0}(\vec{q})) \tag{6.21}$$

Using then (6.19) and (6.21), squaring and using the fact that $[\vec{q}, H]$ commutes with the operator under the square root and finally taking the left inverse, we obtain

$$[\underline{\vec{q}}, H_{q^0}]^2 = \frac{1}{m^2}\left(1 - \frac{[\vec{q}, H_{q^0}]^2}{c^2}\right)(\underline{\vec{p}} - \vec{A}_{q^0}(\underline{\vec{q}}))^2 \tag{6.22}$$

which can be rewritten as

$$[\underline{\vec{q}}, H_{q^0}]^2(m^2c^2 + (\underline{\vec{p}} - \vec{A}_{q^0}(\underline{\vec{q}}))^2) = c^2(\underline{\vec{p}} - \vec{A}_{q^0}(\underline{\vec{q}}))^2 \tag{6.23}$$

Assuming now that $(m^2c^2 + (\vec{p} - \vec{A})^2)$ is invertible, and using that it is positive, so as the right hand side of (6.23), and that it commutes with this right hand side, we get

$$[\underline{\vec{q}}, H_{q^0}] = \varepsilon \cdot c(\underline{\vec{p}} - \vec{A})[(m^2c^2 + (\underline{\vec{p}} - \vec{A})^2)^{-1}]^{1/2} \tag{6.24}$$

which can be easily integrated, assuming that H_{q^0} is again in the image of a C_1 function in $\vec{p}$ in $\vec{q}$ under the Weyl-Wigner transformation so that (6.9) holds. Using moreover (5.23) explicitly, and introducing the time as before, we finally find the following evolution

$$H_{q^0} = \varepsilon \cdot c(m^2c^2 + (-i\hbar\vec{\partial}_x - \vec{A}_{q^0}(\vec{x}))^2)^{1/2} + V_{q^0}(\vec{x}) \tag{6.25}$$

$$\dot{q}^0 = 1 \tag{6.26}$$

We thus obtain in all cases the usual spinless Hamiltonians, with now arbitrary external fields without using the Lorentz boosts as a symmetry for K, but in the more hidden way of instantaneous Lorentz transformation character in (6.14) respectively (6.20). The Lorentz boost is also not a symmetry for (6.17) nor for (6.25) whereas the theory is well known in the electromagnetic potential case to be, as a whole, invariant [23], exactly as it is gauge invariant whereas the Hamiltonians are not.

Let us also remark here that, although (6.25) looks like the square root of the Klein–Gordon equation, it is now completely defined in the space K given in (5.20), because $\vec{x}$ (see (5.23)) is now a self-adjoint operator on K contrarily to the (even free) Klein–Gordon equation obtained from the representations of the Poincaré group, this fact being at the root of some of the difficulties of this equation. Moreover it has now really the satisfying simultaneous meaning of the space variable and of the space position operator. This new interpretation embodies also the fact that the solutions of (6.2) with (6.25) really contain the spatial features (the form) of an obviously extended object: the wave function.

Let us finally remark that although particles and antiparticles do appear simultaneously as one-particle theories, there is no pair annihilation as interpreted as a transition to negative energy states. This last phenomenon is thus really a pure field theoretical aspect (for which the K-spaces just obtained will be of course meant as the basic building blocks).

7. Generalizations

The method we have just used for the determination of an allowable dynamics can be extended to the classical (5.16) or quantal (5.20) state spaces with $\sigma \neq 0$, and the appropriate representations (5.17) and (5.21) respectively, or to the case where more particles are present. We want here just to briefly sketch how this is possible, leaving for the moment open the question of the various corresponding physical interpretations.

Suppose in the simplest case a classical spin 1/2 non-relativistic particle as in (5.16) with the appropriate $\hat{n}$. We need, in view of (6.2), and in addition to the vector field (6.1) given in (6.6) to determine in the two dimensional underlying Hilbert-spaces a family of self-adjoint operators to express the dynamics. This means that the operator $\dot{\underline{q}}$ has to be given in a 2×2 matrix form, and can thus be written

$$\dot{\underline{q}} = (\dot{\underline{q}})^{\mu} \cdot \sigma_{\mu} \qquad \mu = 0, 1, 2, 3 \tag{7.1}$$

where we have used that $\sigma_0 = \mathbb{1}_2$ and the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ form a basis for the self-adjoint operators in $\mathbb{C}^2$. Let us assume as the simplest possible dynamical principle that the generalization of (6.4) is given by

$$U(\vec{w})\, \dot{\underline{q}}\, U(\vec{w})^{-1} = \dot{\underline{q}} + \frac{\vec{w}}{m} \cdot \sigma_0 \tag{7.2}$$

Integrating as before we immediately obtain that the evolution is the same on the base space $S = \mathbb{R}_{q^0} \times \Gamma \ni \{q^0, \vec{p}, \vec{q}\}$ with the same Hamiltonian (6.6), whereas on $\mathbb{C}^2$ we now have

$$H_{q^0,\vec{p},\vec{q}} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}_{q^0,\vec{p},\vec{q}} = V^{\mu}(q)\sigma_{\mu} \begin{pmatrix} f_+ \\ f_- \end{pmatrix}_{q^0,\vec{p},\vec{q}} \tag{7.3}$$

where $V^{\mu}(q)$ are 4 free integration constants, which may arbitrarily depend on q. Absorbing the term V^0 in the potential V of (6.6) and changing notation, we can thus simply write the spin part of the classical Hamiltonian as

$$H_{q^0,\vec{p},\vec{q}} = \vec{B}(q^0, \vec{q}\,) \cdot \vec{\sigma} \tag{7.4}$$

where $\vec{B}$ is an arbitrary vector field.

In the quantal case the evolution in the space (5.20) with $\sigma = 1/2$ is now given, from (6.2) by a self-adjoint operator in $(\mathcal{L}^2(\mathbb{R}^3) \times \mathbb{C}^2)_{q^0}$ together with the same equation $\dot{q}^0 = 1$ as before, whereas the kinematical symmetry group acts on the spinor

$$\Psi_{q^0}(\vec{x}) = \begin{pmatrix} \Psi_+(\vec{x}) \\ \Psi_-(\vec{x}) \end{pmatrix}_{q^0} \tag{7.5}$$

as

$$U(\vec{w}, a^0, \vec{a}, \alpha)\Psi_{q^0}(\vec{x}) = D^{1/2}(\alpha) \exp\,(i\hbar^{-1}\vec{w}\vec{x})\Psi_{q^0}(g^{-1}\vec{x}) \tag{7.6}$$

If we integrate the dynamical principle (7.2) in the same way as for the spinless case, we immediately obtain, together with the self-adjointness condition

$$(H_{q^0})_{ii} = \frac{1}{2m}(\vec{p} - (\vec{A}_{q^0}(\vec{q}))_i)^2 + (V_{q^0}(\vec{q}))_i \tag{7.7}$$

$$(H_{q^0})_{12} = W_{12}(q) = (\overline{H_{q^0}})_{21} \tag{7.8}$$

with $i = 1, 2$ and where $W_{12}(q)$ is an arbitrary complex function. In the case where

$\vec{A}_1 = \vec{A}_2$ we thus simply get

$$H_{q^0} = \left(\frac{1}{2m}(\vec{p} - \vec{A}_{q^0}(\vec{q}))^2 + V_{q^0}(\vec{q})\right)\sigma_0 + \vec{\sigma} \cdot \vec{B}_{q_0}(\vec{q}) \tag{7.9}$$

which is the usual time-dependent Schrödinger–Pauli Hamiltonian.

The same reasoning immediately generalizes to the relativistic case, and we obtain in the same way, together with (6.17) and (6.18)

$$H_{q^0,\vec{q},\vec{p}} = \vec{\sigma} \cdot \vec{B}_{q^0,\vec{q}} \tag{7.10}$$

in the classical case, and

$$H_{q^0} = [\varepsilon c(m^2c^2 + (\vec{p} - \vec{A}_{q^0}(\vec{q}))^2)^{1/2} + V_{q^0}(\vec{q})]\sigma_0 + \vec{\sigma} \cdot \vec{B}_{q^0}(\vec{q}) \tag{7.11}$$

with (6.26) in the quantal case. These operators can be rewritten in a more covariant way, for example as

$$H_{q^0,\vec{q},\vec{p}} = \Sigma^{\mu\nu} \cdot (F_{q^0,\vec{q}})_{\mu\nu} \tag{7.12}$$

$$H_{q^0} = \varepsilon c[m^2c^2 + (\vec{p} - \vec{A}_{q^0}(\vec{q}))^2]^{1/2} + V_{q^0}(\vec{q}) + \Sigma^{\mu\nu} \cdot (F_{q^0}(\vec{q}))_{\mu\nu} \tag{7.13}$$

with e.g. (in the rest frame of the particle) $\Sigma^{0i} = d \cdot \sigma^0\sigma^i$, $\Sigma^{ij} = i\mu \cdot \sigma^i\sigma^j$, d, $\mu \in \mathbb{R}$ and where F is an arbitrary antisymmetric real tensor field on space-time. We leave for the moment open the question of the quite apparent physical interpretation of these results, of their consequences (e.g. the resulting spin motion [24]) and of their comparison with other recent similar progresses in the description of single relativistic particles (see e.g. [25]). All what we want to point out here is how, unless the very different symmetry hypotheses on the state space K (and on the definition of relativistic states) we find, as a particular case (i.e., for a special form of the dynamical principle) the usual theory of particles in external fields (with, as shown, some improvements due to the new interpretation).

Another illustration is given by the description of N particles: the set of observables is then given, for each tacit $\hat{n}$, by

$$\{\vec{p}_\beta, \vec{q}_\beta, q^0\}, \qquad \beta = 1, 2, \ldots, N \tag{7.14}$$

with for each β the same action of G

$$\{\vec{p}_\beta, \vec{q}_\beta, q^0\} \mapsto \{\alpha(\vec{p}_\beta + \vec{w}), \alpha(\vec{q}_\beta + \vec{a}), q^0 + a^0\} \tag{7.15}$$

(the time being the one of the observer, it is the same $\forall\beta$). The corresponding state spaces are then given by

$$K = V_{q^0,\vec{q}_\beta,\vec{p}_\beta}\left(\bigotimes_{\beta=1}^{N} (\mathbb{C}_\beta^{2\sigma+1})\right)_{q^0,\vec{q}_\beta,\vec{p}_\beta} \tag{7.16}$$

in the classical, and by

$$K = V_{q^0}\left(\mathfrak{L}^2(\mathbb{R}^{3N}) \otimes \left(\bigotimes_{\beta=1}^{N} \mathbb{C}_\beta^{2\sigma+1}\right)\right)_{q^0} \tag{7.17}$$

in the quantal case. Writing the same dynamical principles as before, for each β, we immediately obtain as the simplest solution for the non relativistic spinless

Hamiltonian

$$H_{q^0}=\Sigma_\beta \frac{1}{2m_\beta}(\vec{p}_\beta-(\vec{A}_\beta)_{q^0}(\vec{q}_1,\ldots,\vec{q}_N))^2+V_{q^0}(\vec{q}_1,\ldots,\vec{q}_N) \tag{7.18}$$

and for the relativistic spinless Hamiltonian

$$H_{q^0}=\Sigma_\beta\varepsilon_\beta c\sqrt{(\vec{p}_\beta-(\vec{A}_\beta)_{q^0}(\vec{q}_1,\ldots,\vec{q}_N))^2+m_\beta^2c^2}+V_{q^0}(\vec{q}_1,\ldots,\vec{q}_N) \tag{7.19}$$

Here again, it is not the purpose of the present paper to discuss this particular form of the dynamics, but to show that it represents a realistic example of dynamics compatible with the new definition and covariance of the notion of state. As for the wave functions, the Hamiltonians represent the description, here at $\hat{n}=\hat{n}_0$, of the evolution (but not the evolution itself). This implies that in (7.19) the fields are not necessarily external, but as depending on the N position observables, they are also allowed to describe an action at-a-distance in the planes perpendicular to the associated $\hat{n}$.

Conclusion

The Lorentz (or the Galilean) boosts contain in fact two distinct symmetry principles: the first one asserts in both cases that there exists no frame which is absolutely at rest. The second one expresses the explicit relationship between the space-time coordinates of respectively to each other moving frames. That these aspects are different can be seen in the simple fact that the first principle is identical in the relativistic and non-relativisitic domains whereas the second is correspondingly different. In the present paper we have exploited the fact that the first principle is of kinematical nature and is thus related to a symmetry in the description of the state spaces, whereas the second one is of dynamical nature and is related to the possible forms of the evolution laws. We have shown explicitly that we could in this way derive a framework compatible with the usual relativistic known theories (eliminating by the way some usual difficulties) but allowing also the description of extended objects (in a true 3-dimensional sense), and avoiding, in a simple but argumented way the basic hypotheses of the well known no-go theorems for interacting N particles. In fact explicit examples of such non-trivial interactions compatible with the above scheme and thus giving the same results have already been given in the literature [13] and successfully applied to concrete physical situations as we have mentioned. One of the progresses (and of the motivations) of the present reformulation of the state space aspects is that it now allows for example, as we have seen, a covariant relativistic description (we do not speak of the mechanism, here) of instantaneous wave packet reductions (as in the non-local EPR paradox experiments), something which is quite impossible with the usual interpretation and definition of states. The result of this reformulation is a new one-particle (and antiparticle) and a new N-particle state space theory, natural base for a new field theory.

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