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MAGNETIC PROPERTIES OF SUPERCONDUCTING OR NORMAL NETWORKS AND RANDOM WALKS ON PERCOLATING CLUSTERS

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A scaling theory for the diamagnetic susceptibility and the upper critical field of random mixtures of superconducting and insulating elements is presented . The resulting predictions for the critical field are compared with experiments on InGe films and numerical simulation data. One important critical exponent in the theory is related to the spectral dimensionality of a fractal structure, which governs also the diffusion properties . We suggest that the number S_N of distinct sites visited during an N-step random walk on an infinite cluster at percolation threshold varies asymptotically as : $S_N \sim N^{2/3}$, in any dimension $d \leqslant 6$.
For more regular networks, such as periodic lattices (with a strip geometry) and fractal structures , we have determined the spectrum of Landau levels. The contribution of the edge states to the quantized Hall conductance is explained in a way which clarifies the role of the geometry in the derivations of this effect .

I - Superconducting diamagnetism near the percolation threshold

The percolation problem has played a leading role in the study of disordered systems. During the past 25 years, considerable progress has been achieved, specially concerning the properties around the percolation threshold (for a recent review, see [1]). As in ordinary phase transitions, one defines critical exponents, associated with static and dynamic properties. One important issue, during the last decade, has been to determine the list of independent exponents characterizing the percolation transition and, more precisely, whether the dynamic exponent t of the conductivity is related to static exponents, such as β and ν, associated with the infinite-cluster density and with the correlation length.

Random mixtures of superconducting and insulating elements, such as in Ge films, have been recently investigated in Tel-Aviv [2]. In particular, the upper critical field H_{c2} was measured close to T_c, the superconducting transition temperature, on the metallic side of the percolation threshold. If p is the concentration of In and p_c the critical percolation concentration, the slope dH_{c2}/dT was found to diverge, for $p \to p_c^+$, as :

$$\frac{dH_{c2}}{dT} \simeq \frac{1}{(p-p_c)^k} \quad ,$$

with $k = 0.6 \pm 0.05$. The authors of [2] proposed a theoretical interpretation leading to a relation $k = t + \beta - \beta'$, where β' is the backbone-density exponent [1]. Using presently accepted values for t, β, β', this theory predicts : $k \simeq 0.87$.

On the insulating side of the transition ($p < p_c$), a quantity of interest is the diamagnetic susceptibility χ. It was suggested by de Gennes [3] that this quantity might provide a sensitive test of the topology of percolation clusters (more sensitive than the conductivity). The slope of the susceptibility close to T_c, $d\chi/dT$, was predicted to vary, for $p \to p_c^-$, as some inverse power of $(p_c - p)$ but no definite prediction appeared in the published version of his letter. Bolder was the analysis of M. Stephen [4], predicting $d\chi/dT \sim (p_c - p)^{-b}$, with $b \simeq -0.77$, negative in two dimensions. To our knowledge, no experimental data for the susceptibility are available yet.

We have presented [5] a different approach to these diamagnetic superconducting properties near the percolation threshold. It is based on scaling arguments relating the behaviour of various physical quantities in different regions of the phase diagram (T, p) around the multicritical point at $T = T_c$, $p = p_c$. In dimension two, with some simple-looking assumptions, one obtains, for the exponents introduced above,

$$k = \frac{\nu\bar{\delta}}{2} \qquad \text{and} \qquad b = \nu(2 - \bar{\delta}) \quad .$$

Exponents k and b are expressed in terms of one unknown exponent $\bar{\delta}$. The exponent ν is the correlation-length exponent, which is now generally believed to be exactly 4/3 in two dimensions [6].

In a second stage of the argument, the exponent $\bar{\delta}$ is related to other percolation exponents, namely

$$\nu\bar{\delta} = t - \beta \quad ,$$

where the exponents t and β are those mentioned in the beginning. The argument is based on the recognition that the infinite percolation cluster at threshold is a self-similar object. Using relations derived [4,7] for a model self-similar structure called the Sierpinski gasket, and the value of the fractal dimensionality of the percolating cluster, one obtains the preceding equation, whose validity thus assumes a universality of some sort for self-similar objects.

From the knowledge of the exponents β and t, which are known with increasing accuracy [1,6], one derives

$$\bar{\delta} \simeq 0.85 \quad , \quad k \simeq 0.57 \quad , \quad b \simeq 1.53 \quad .$$

This value for k is in good accord with the published data [2]. Note that the susceptibility slope is predicted to diverge, in contrast with [4].

This theory has been submitted to numerical test [8]. In a bond-percolation model, the diamagnetic susceptibility of finite samples has been computed. With available precision, the scaling form of [5] appears well supported. Quantitatively, a value for exponent b is obtained

$$b = 1.55 \pm 0.04 \quad ,$$

in good accord with the prediction of [5].

However more data, experimental and numerical have to be collected before a compelling picture emerges. Indeed, our expression for exponent b,

$$b = 2\nu - t + \beta$$

is presently questioned. Some good experts [9] suggest a different expression :

$$b = 2\nu - t \quad .$$

Note firstly that this is now a 10% controversy, not any longer a controversy over sign [4,5]. Note also that present numerical data [8], in the absence of experimental data, seem to favor the first expression.

There have been other controversies in the past, concerned with such presence or absence of a β term (de Gennes, Stauffer). Here the physical issue seems to be whether the susceptibility near the percolation threshold is dominated by the contribution of the largest clusters or whether clusters of all sizes contribute significantly.

This controversy should act as a healthy stimulus for experimentalists. In addition, everything remains to be done in three dimensions, where a marginal divergence for χ has been suggested [3].

II - Spectrum of the Schrödinger equation on a self-similar structure

As discussed above, the Sierpinski gaskets [10] are self-similar structures which can be viewed as qualitative models for other fractal objects. They are regular structures with a dilation symmetry instead of a translation symmetry. An infinite percolation cluster is not a regular object, but it is presently believed that for some properties (to be precised by future study) the fractal character dominates and the disorder is not relevant. Anyway, the regular structure of the gaskets lends itself conveniently to renormalization approaches and it is probably possible to derive a wealth of exact results on them. In this respect, they may be compared to Bethe lattices, except that they have a much richer physics because of their multiple-connectedness.

One of us has obtained a complete description of the spectrum and of the eigenmodes of the harmonic vibrations (or tight-binding) problem on Sierpinski gaskets in any Euclidean dimension d [11]. The spectrum consists of two entangled parts : a pure point component, corresponding to local modes, and another pure point component, whose support is a Cantor set of zero measure, corresponding to hierarchical states.

Such a complete picture has not yet been achieved for the Landau levels (Schrödinger equation in the presence of a magnetic field) on a Sierpinski gasket in two dimensions. The difficulty lies in obtaining the spectrum in the asymptotic limit of arbitrary large sizes. However, by studying the iteration of finite-size gaskets, we have been able to observe and derive some remarkable Nesting Properties of the spectrum [12]. These Nesting Properties are reminiscent but different from those observed on translation-invariant lattices [13]. The low-field behaviour of the edges of the spectrum is governed by the exponent $\overline{\delta}$ introduced in Section I.

III - Random walks on fractal structures and percolation clusters

Self-similar spaces with a dilation symmetry, such as Sierpinski gaskets, are characterized by at least three dimensions : d, the dimension of the embedding Euclidean space ; $\overline{d}$, the fractal (Hausdorff) dimension [10] ; $\tilde{d}$, the spectral dimension. A lot of attention has been devoted to the fractal dimensionality in the past. However, it is only recently that the importance of the spectral dimension has been recognized by S. Alexander and R. Orbach [14] (they called it fracton dimension), after some earlier insights [15].

The spectral dimension governs the behaviour of the low-frequency density of states $\rho(\omega)$ on a fractal structure :

$$\rho(\omega) \sim \omega^{\tilde{d}-1} \quad .$$

As a consequence, it governs also many diffusion properties. For instance, the mean-squared displacement from the origin after N steps, during a random walk, behaves asymptotically as

$$\langle R^2 \rangle \sim N^{\tilde{d}/\bar{d}} = N^{2/2+\bar{\delta}} \quad .$$

The previously introduced exponent $\bar{\delta}$ appears as a combination of the fractal and spectral dimensions. This combination enters into the mean-squared displacement because the measure of distances brings with it the fractal dimension.

In order to obtain "pure" quantities, where the spectral dimension enters alone, one has to consider [16] other random walk properties such as the probability of return to the origin after N steps P_0 :

$$P_0 \sim \frac{1}{N^{\tilde{d}/2}} \quad ,$$

or the average number of distinct sites visited during an N-step random walk S_N :

$$S_N \sim N^{\tilde{d}/2} \qquad \text{(provided } \tilde{d} < 2) \quad .$$

This law has been checked numerically [17] on a Sierpinski gasket in dimension d = 2, for which $\tilde{d}$ is easily calculated : $\tilde{d} = 2\,\frac{\ell n 3}{\ell n 5}$. The numerically determined value of the exponent is 0.682 ± 0.005, to be compared with $\tilde{d}/2 = 0.68260$.

For a percolating cluster at threshold, the expression for the spectral dimension can be derived from the previously given expression of $\bar{\delta}$ in Section I, and from $\bar{d} = d - \frac{\beta}{\nu}$. One obtains :

$$\tilde{d} = 2\,\frac{d\nu - \beta}{t - \beta + 2\nu} \quad .$$

It has been observed in [14] that $\tilde{d}$, so determined with known estimates of t, β and ν, appears to be numerically close to 4/3 for all dimensions $1 < d \leqslant 6$. This leads to the remarkably simple prediction

$$S_N \sim N^{2/3}$$

on a percolating cluster. Actually, an argument has been presented [16], which suggests that perhaps the 2/3 exponent might be an exact and not only approximate value. Define the "open frontier" as the number of fresh sites adjacent to the visited sites during an N-step random walk. It is given [16] by $S_N(dS_N/dN)$ and it behaves asymptotically as :

$$\begin{aligned} &\sim 0 && \text{for} \quad p < p_c \quad , \\ &\sim S_N && \text{for} \quad p > p_c \quad . \end{aligned}$$

If one assumes that, at the threshold $p = p_c$, the open frontier is marginally equal to the gaussian fluctuation in the number of accessible sites, due to the random-percolation process, then $S_N(dS_N/dN) \sim \sqrt{S_N}$ and the 2/3 law follows.

The 2/3 law has also been checked numerically [17] on a percolating cluster in two dimensions. The exponent has been determined as :

$$0.65 \pm 0.01 \quad ,$$

to be compared with 2/3.

At stake is the existence of an exact relation between dynamic and static percolation exponents. If $\tilde{d} = 4/3$ is an exact result for the infinite percolation cluster at threshold, then the dynamic exponent t is not independent from the static exponents β and ν.

IV - Landau levels on strips and the quantized Hall effect

We have recently studied the Landau levels on regular two-dimensional networks with a strip geometry, infinite in one direction, finite in the other [18,19]. We have shown that the bulk formula giving the quantized Hall conductance can be explained in terms of a special gauge-invariance property of the edge states. We have thus clarified the role of the geometry in the derivations of the quantized Hall effect. As a bonus, it is possible to generalize and to prove, by a gauge-invariance argument à la Laughlin [20], a conjecture of Wannier [21] for the density of states of Landau levels in the presence of a two-dimensional periodic potential.

References

[1] D. Stauffer, Phys. Rep. 54 (1979) 1.

[2] G. Deutscher, I. Grave, S. Alexander, Phys. Rev. Lett. 48 (1982) 1497.

[3] P.G. de Gennes, C.R. Hebd. Séan. Acad. Sci., Série II, 292 (1981) 9, 279.

[4] M.J. Stephen, Phys. Lett. 87 A (1981) 67.

[5] R. Rammal, T.C. Lubensky, G. Toulouse, J. Physique Lett. 44 (1983) L-65.

[6] B. Derrida, J. Vannimenus, J. Phys. A 15 (1982) L-557.

[7] S. Alexander, Phys. Rev. B 27 (1983) 1541.

[8] R. Rammal, J.C. Angles d'Auriac, to be published.

[9] S. Alexander, P.G. de Gennes, private communications.

[10] B.B. Mandelbrot, Fractals : Form, Chance and Dimension (Freeman, San Francisco) 1977 ; Y. Gefen, A. Aharony, B.B. Mandelbrot, S. Kirkpatrick, Phys. Rev. Lett. 47 (1981) 1771.

[11] R. Rammal, to be published.

[12] R. Rammal, G. Toulouse, Phys. Rev. Letters 49 (1982) 1194.

[13] D.R. Hofstadter, Phys. Rev. B 14 (1976) 2239 ; F.H. Claro, G.H. Wannier, PHys. Rev. B 19 (1979) 6068.

[14] S. Alexander, R. Orbach, J. Physique Lett. 43 (1982) L-625.

[15] E.F. Shender, J. Phys. C 9 (1976) L-309 ; D. Dhar, J. Math. Phys. 18 (1977) 577.

[16] R. Rammal, G. Toulouse, J. Physique Lett. 44 (1983) L-13.

[17] J.C. Angles d'Auriac, A. Benoit, R. Rammal, to be published.

[18] R. Rammal, T.C. Lubensky, G. Toulouse, to be published.

[19] R. Rammal, G. Toulouse, M.T. Jaekel, B.I. Halperin, to be published.

[20] R.B. Laughlin, Phys. Rev. B 23 (1981) 5632.

[21] G.H. Wannier, Phys. Stat. Sol.(b) 88 (1978) 757.