Dispersion relations for Feynman graphs

Helvetica Physica Acta, Vol. 54 (1981), Birkhäuser Verlag, Basel

Dispersion relations for Feynman graphs

by **E. Remiddi**

Département de Physique Théorique, Université de Genève, 1211 Genève 4, Switzerland
and
Istituto di Fisica dell'Università Bologna, INFN, Sezione di Bologna

(1. IV. 1981)

Abstract. Some results of Ref. 1 are reviewed and their application worked out for two, three and four point amplitudes in perturbative Lagrangian field theory, discussing the relation between imaginary parts, cutting equations and discontinuities. A simple way is proposed for obtaining dispersion relations in some invariant variable at fixed values of the others. The argument is based on Fourier transform formulae for real variable functions and on the choice of suitable kinematical configurations, without explicit reference to analyticity properties.

1. Introduction

Dispersion relations and the related language (analyticity properties, discontinuities and the like) provide a by now classical tool for the study of strong interaction physics. In this paper, however, we will be concerned with another way of using them, which is also of the greatest importance, i.e. for the actual evaluation, analytic or numerical, of Feynman graph amplitudes in perturbative Lagrangian field theory.

Dispersion relations for Feynman graphs are usually derived (or justified) in standard textbooks through the following steps: a parametric representation for the amplitude, discussion of its Landau singularities, analytic continuation in some variable and investigation of its analyticity properties, until one eventually obtains the related discontinuities, to be further used in the dispersion relations, when desired. It is of course known and emphasized that analyticity relies on the locality and causality properties of the theory, which are very simple when formulated in ordinary space-time, but a shorter path between first principles and above mentioned results seems still missing in the literature.

It is the main aim of the present paper to propose a way of shortening the path from causality to dispersion relations by means of elementary arguments only (Fourier transforms of real variable functions and *ad hoc* kinematical configurations), hoped to provide the reader with do-it-yourself rules for working out dispersion relations in practical cases. This paper relies heavily on Ref. 1, which shows how to establish the space-time causality properties of Feynman graphs by means of direct inspection on each separate amplitude. As stressed there, that approach is the graph-by-graph version of the order-by-order inductive construction of the renormalized perturbative series of Quantum Field

Theory as proposed in Ref. 2 and as worked out in a complete and mathematically rigorous way in Ref. 3. In Ref. 1 causality properties are used to derive an equation for the imaginary parts, essentially equivalent to the Cutkosky rule [4], and a wide family of dispersion relations in the energy, however of non-invariant type. It will be shown here how to extract from them dispersion relations of the usual invariant form, i.e. in some scalar variable at fixed values of the others, at least in certain kinematical configurations. That is achieved by suitably taking the $P\to\infty$ limit of Ref. 5; the way in which it is used here is closer to a later work [6].

As this paper intends to pursue also pedagogical purposes, any effort was done to make it self-contained and clear. Due to the first reason, the relevant results of Ref. 1, widely circulated but not in printed form, are reproduced; as a consequence of the second, the mathematically oriented reader will probably dislike the poor precision of the used language.

The plan of the paper follows.

Section 1 is the introduction (this section).

Section 2 repeats the relevant arguments and results of Ref. 1.

Section 3 works them out in momentum space and discusses the imaginary parts of the two, three and four-point amplitudes.

Section 4 deals with the dispersion relations for those amplitudes.

Section 5 discusses the limits of the validity of previous sections results.

For the sake of simplicity, the fully scalar case is taken, where all the propagators have the same non vanishing mass m and the interaction vertices involve 3 particles and no derivative couplings; as examples, only the two, three and four point functions are considered. The extension of the arguments to more general cases should be straightforward in principle, although complicated in practice.

2. The causality equations

Let us start by recalling some standard formulae. The Feynman propagator for a particle of mass m is

$$\Delta(x)=\int\frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m^2-i\varepsilon}e^{ipx}, \tag{1}$$

with $px=\vec{p}\cdot\vec{x}-p_0x_0$. By contour integration in p_0 one has

$$\Delta(x)=\theta(x_0)\Delta^+(x)+\theta(-x_0)\Delta^-(x) \tag{2}$$

with

$$\Delta^{\pm}(x)=\int\frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2+m^2)\theta(\pm p_0)e^{ipx}. \tag{3}$$

The following properties are seen to hold (by letting $p\to-p$ in the above integrals, for instance)

$$\Delta(x)=\Delta(-x), \tag{4}$$

$$\Delta^+(-x)=\Delta^-(x), \tag{5}$$

$$(\Delta^+(x))^*=\Delta^-(x). \tag{6}$$

One has further

$$\Delta^*(x)=\Delta^*(-x)=\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2+m^2+i\varepsilon}\,e^{ipx}, \tag{7a}$$

$$\Delta^*(x)=\theta(x_0)\Delta^-(x)+\theta(-x_0)\Delta^+(x). \tag{7b}$$

Use will also be made of the representation

$$\theta(x_0)=\int\frac{dk_0}{2\pi}\frac{-i}{k_0-i\varepsilon}\,e^{ik_0x_0}=\int\frac{d^4k}{(2\pi)^4}\frac{-i}{k_0-i\varepsilon}(2\pi)^3\delta(\vec{k})e^{-ikx}, \tag{8}$$

which is easily verified, again, by means of contour integration in k_0.

A Feynman graph in coordinate space is represented by N vertex points $x_1, x_2\cdots x_N$ suitably joined by lines. The corresponding amplitude $F(x_i)$ is (omitting coupling constants) the product of a factor i for each of the N vertices and a propagator $\Delta(x_i-x_j)$ for each line joining any two points i, j. With such definition, $F(x_i)$ does not imply any integration and there is no distinction between internal and external points of the graph.

Given any N-point amplitude $F(x_i)$, consider all the related amplitudes obtained as follows: take the graph representing $F(x_i)$; duplicate it as many times as needed adding circles around some of its vertices in all possible ways (the total number of such drawings is 2^N, including the original); correspondingly to each graph with some circles, define its amplitude as the product of the following factors:

i for each uncircled, $(-i)$ for each circled vertex;
$\Delta(x_i-x_j)$ for each line joining two uncircled vertices x_i, x_j;
$\Delta^+(x_i-x_j)$ for each line joining a circled x_i to an uncircled x_j;
$\Delta^*(x_i-x_j)$ for each line joining two circled vertices x_i-x_j.

The operation of putting circles in a graph is closely related to complex conjugation. Take indeed any graph with some circles and consider the new graph obtained by removing the existing circles and putting new circles in all the other vertices. From the above rules and equation (6) it is immediately seen that the corresponding amplitudes are the complex conjugate of each other. The graph with circles on all the vertices, in particular, is the complex conjugate $F^*(x_i)$ of the original Feynman amplitude $F(x_i)$.

The largest time equation asserts that the sum of 2^N 'underlined' amplitudes, corresponding to the 2^N graphs circled in all possible ways, vanishes (the original amplitude $F(x_i)$ is included in the sum). The proof is simple. Assume that x_l^0 is the largest of the time components of the N points x_i; group the circled graphs in pairs, consisting of a graph, where x_l is not circled and the other points are in any circle configuration, and the corresponding graph in which x_l is circled and the other points are in the same configuration. Call s any of the points joined to l; if s is not circled, there is a factor $\Delta(x_l-x_s)$ in the first amplitude and $\Delta^+(x_l-x_s)$ in the second; but, from equation (2)

$$\Delta(x_l-x_s)=\Delta^+(x_l-x_s),\quad \text{if}\quad x_l^0>x_s^0.$$

Similarly, if s is circled, there is $\Delta^+(x_s-x_l)$ in the first amplitude and $\Delta^*(x_l-x_s)$ in the second, but, from equations (7) and (5)

$$\Delta^*(x_l-x_s)=\Delta^+(x_s-x_l),\quad \text{if}\quad x_s^0<x_l^0.$$

The factors corresponding to the lines, therefore, do not change; the only change is the factor $(-i)$ for the graph with the circle in x_l as compared to the factor i for the graph without the circle in x_l. The two amplitudes of each pair therefore differ only in an overall sign, and their sum vanishes.

Picking out the amplitude of the graph without circles, $F(x)$, and the amplitude of the graph with all the possible circles, $F^*(x)$, the largest time equation can be written as

$$F(x_i)+F^*(x_i)\equiv 2\,\mathrm{Re}\,F(x_i)=-\mathbf{F}(x_i), \tag{9}$$

where $\mathbf{F}(x_i)$ stands for the sum of all the amplitudes corresponding to the (2^N-2) graphs having from 1 to $(N-1)$ circles. (Let us recall here that the argument leading to equation (9) applies as far as one of the times x_i^0 is larger than all the others).

Suppose now that $x_m^0<x_n^0$; in this case x_m^0 is not the largest time and the largest time equation applies separately to the graphs with and without the circle around x_m. In particular, one can write

$$\theta(x_n^0-x_m^0)(F(x_i)+\mathbf{F}(m,x_i)=0, \tag{10}$$

where $F(x_i)$ is the original amplitude, $\mathbf{F}(m,x_i)$ the sum of all the amplitudes with circles not in x_m (equation (10) refers to half of all the 2^N graphs). Similarly, one has

$$\theta(x_m^0-x_n^0)(F(x_i)+\mathbf{F}(n,x_i))=0. \tag{11}$$

We will now sum equations (10) and (11). The amplitudes in which neither x_m nor x_n are circled (their number is $\frac{1}{4}2^N$) appear in both equation (10) and equation (11); in the sum they are multiplied by

$$\theta(x_n^0-x_m^0)+\theta(x_m^0-x_n^0)=1.$$

The sum of equations (10) and (11) therefore can be written as

$$F(x_i)=-\mathbf{F}(m,n,x_i)-\theta(x_n^0-x_m^0)\mathbf{F}(m,\mathbf{n};x_i)-\theta(x_m^0-x_n^0)\mathbf{F}(\mathbf{m},n;x_i),$$

where: $\mathbf{F}(m,n;x_i)$ is the sum of all the amplitudes with x_m, x_n not circled ($\frac{1}{4}2^N-1$ terms); $\mathbf{F}(m,\mathbf{n};x_i)$ the sum of all the amplitudes having x_n circled and x_m not circled ($\frac{1}{4}2^N$ terms); $\mathbf{F}(\mathbf{m},n;x_i)$, similarly is the sum of the $\frac{1}{4}2^N$ amplitudes having x_m circled and x_n not circled.

As an illustration of the above discussion let us look to Fig. 1. The lines in 1 and 2 as well as the arrows are to be disregarded for the moment. Graph 1 of Fig. 1 represents the Feynman amplitude in coordinate space:

$$F(x_1,x_2,x_3,x_4)=i^4\Delta(x_1-x_3)\Delta(x_1-x_4)\Delta(x_3-x_4)\Delta(x_2-x_3)\Delta(x_2-x_4).$$

Graph (9), for instance, corresponds to

$$i^2(-i)^2\Delta^-(x_1-x_3)\Delta^-(x_1-x_4)\Delta^*(x_3-x_4)\Delta^+(x_3-x_2)\Delta^+(x_4-x_2)$$

and graph (16) is $(F(x_1,x_2,x_3,x_4))^*$.

If, say, x_3^0 is the largest, the pairwise compensations leading to equation (9) occur within the pairs (1, 3), (2, 6), (4, 9) etc. In the notation of equation (10), $\mathbf{F}(2;x_i)$, for instance, corresponds to graphs (3, 4, 5, 9, 10, 11, 15); with respect to equation (12), $\mathbf{F}(1,2;x_i)$ consists of graphs (3, 4, 9), $\mathbf{F}(1,\mathbf{2};x_i)$ of (5, 10, 11, 15) and $\mathbf{F}(\mathbf{1},2;x_i)$ of (2, 6, 7, 12).

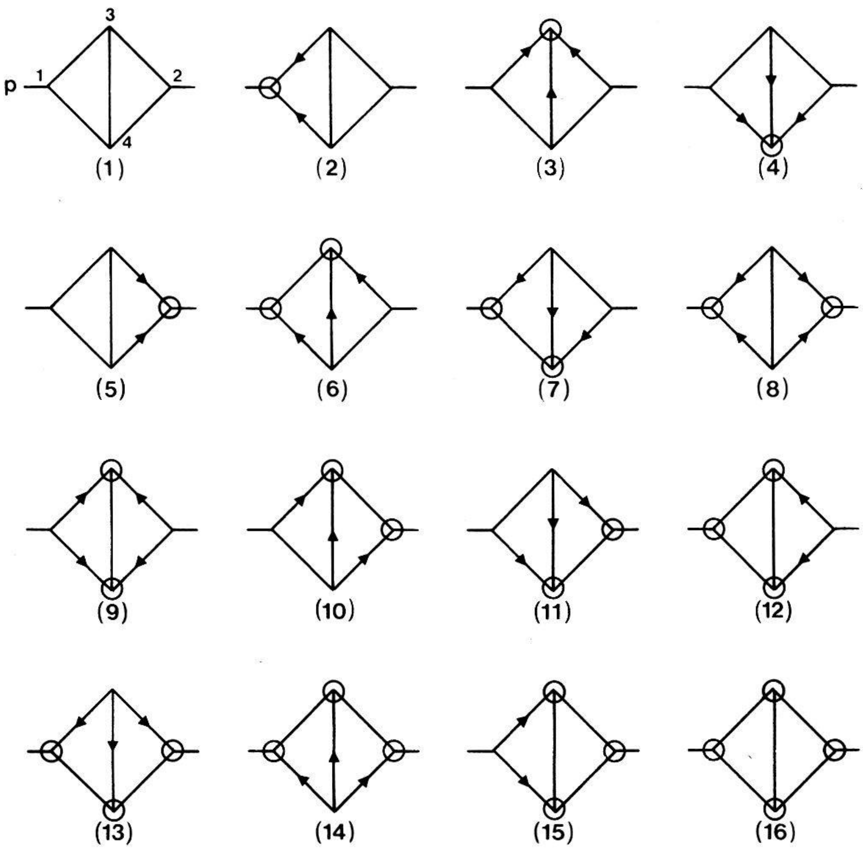

Figure 1
A Feynman graph and its related 'circled' graphs.

3. Momentum space. The imaginary part

We take now the Fourier transform of the equations established in the previous section, in order to discuss their content in the momentum representation.

Let us define the Fourier transform $G(p_n)$ of $F(x_i)$ through

$$G(p_n) = \frac{1}{i} \int (Dx) F(x_i), \tag{13}$$

where (Dx) stands for

$$(Dx) \equiv \left(\prod_i dx_i\right)\left(\prod_j e^{ip_jx_j}\right)\left(\prod_k e^{-ip_kx_k}\right) \Big/ \left((2\pi)^4 \delta^4\left(\sum_j p_j - \sum_k p_k\right)\right) \tag{14}$$

(we 'factorize out' conservation of four-momentum). In above formulae, i runs on all the points of the graph, j on the incoming (external) lines bringing the momenta p_j, k on the outcoming lines leaving the graph with momenta p_k and n on all the in and out external lines. The conventional factor $1/i$ in equation (13), reminds the factor i between S and T matrix ($S = 1 + iT$).

Before proceeding, let us make an obvious remark: if

$$f(x) = f(-x) \tag{15}$$

and

$$\tilde{f}(p) \equiv \int dx e^{-ipx} f(x) \tag{16}$$

by letting $x \to -x$ in equation (16), one has

$$\tilde{f}(p) = \int dx \cos(px) f(x). \tag{17}$$

If $f(x)$ is complex, equations (17) can be written as

$$\begin{aligned} \operatorname{Re} \tilde{f}(p) &= \int dx e^{-ipx} (\operatorname{Re} f(x)), \\ \operatorname{Im} \tilde{f}(p) &= \int dx e^{-ipx} (\operatorname{Im} f(x)). \end{aligned} \tag{18}$$

The extension to more variables is obvious.

For any Feynman amplitude

$$F(x_i) = F(-x_i), \tag{19}$$

because $F(x_i)$ is a product of propagators, even under the inversion of all the coordinates, equation (4). Therefore, in particular

$$\operatorname{Im} G(p_n) = \frac{1}{i} \int (Dx)(\operatorname{Re} F(x_i)). \tag{20}$$

Due to equation (9), equation (20) becomes

$$\operatorname{Im} G(p_n) = \frac{i}{2} \int (Dx) \mathbf{F}(x_i). \tag{21}$$

Strictly speaking equation (9) was established only if one of the times x_i^0 is the largest and not, for instance, if two times are equal and larger than the others. We have therefore to assume (or to verify in a practical case) that the considered Feynman graph is sufficiently regular in coordinate space so that such equal time regions do not give finite contributions to equation (21).

One can now insert in the r.h.s. of equation (21) the representations (1, 3, 7a) and integrate on all the points x_i, obtaining the usual four-momentum conservation at each vertex etc. Recalling that an exponential $e^{ip(x-y)} = e^{i[\vec{p}(\vec{x}-\vec{y})-p_0(x_0-y_0)]}$ gives a momentum $p = (p_0, \vec{p})$ flowing from y to x, one finds that a line joining a non circled to a circled vertex and carrying a momentum p towards the circle corresponds to $(2\pi)\theta(p_0)\delta(p^2+m^2)$, while lines without or with two circles carry energy in both directions. The rules are summarized in Fig. 2.

Lines of the type (2) of Fig. 2 are called cut lines and a graph containing cut lines is a cut graph. Equation (21), which holds for arbitrary values of the momenta p_n, gives the imaginary part of a graph as a sum of cut graphs (Cutkosky rule [4]). If one goes from the graphs to the S-matrix, by putting the external lines on the mass shell etc., equation (21) can be used as a starting point for establishing unitarity equations.

We will now discuss equation (21) in some particular cases.

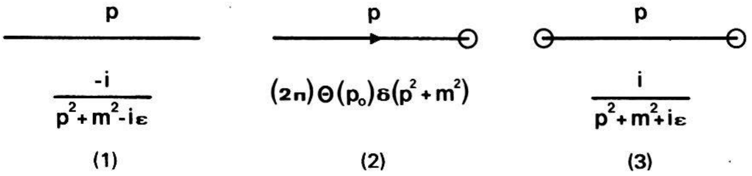

Figure 2
Graphical rules in momentum space.

A two point (self-mass) amplitude

Let us take again the graphs of Fig. 1, considering this time also the momenta and the arrows, which are drawn according to the rules of Fig. 2. By working out the θ-function constraints, i.e. by looking at the arrows, one sees immediately that the cut graphs 3, 4, 8, 9, 13, 14 vanish for any value of the momentum p. In the cut graph No. 3, for instance, all the three lines bring positive non-zero energy to the vertex 3, while no line carries out energy, incompatibly with energy conservation (note, in this respect, that energy can flow in both directions along lines joining two vertices both without or with circles). Always as a consequence of the θ-functions, one further finds that for a given $\vec{p}$ the cut graphs (5, 15) and (10, 11) do not vanish only if $p_0 > \sqrt{\vec{p}^2+4m^2}$ and $p_0 > \sqrt{\vec{p}^2+9m^2}$ respectively, while the cut graphs (2, 6) and (7, 12) do not vanish only for negative values of p_0, $p_0 < -\sqrt{\vec{p}^2+4m^2}$ and $p_0 < -\sqrt{\vec{p}^2+9m^2}$ respectively. In all cases, the value of the vanishing cut graphs depends, on invariance grounds, on $p^2 = \vec{p}^2 - p_0^2$ and not on p_0, $\vec{p}$ separately.

If $G(p)$ is the scalar Feynman amplitude for the graph No. 1 of Fig. 1, one can therefore write

$$\begin{aligned}\operatorname{Im} G(p) = {} & [\theta(p_0 - \sqrt{\vec{p}^2+4m^2}) + \theta(p_0 + \sqrt{\vec{p}^2+4m^2})] h_2(p_0^2 - \vec{p}^2) \\ & + [\theta(p_0 - \sqrt{\vec{p}^2+9m^2}) + \theta(p_0 + \sqrt{\vec{p}^2+9m^2})] h_3(p_0^2 - \vec{p}^2). \end{aligned} \tag{22}$$

It is convenient to define a new scalar amplitude $H(t)$ depending only on the invariant $p_0^2 - \vec{p}^2$

$$H(p_0^2 - \vec{p}^2) = G(p); \tag{23}$$

equation (22) then reads

$$\operatorname{Im} H(t) = \theta(t - 4m^2) h_2(t) + \theta(t - 9m^2) h_3(t). \tag{24}$$

The real functions $h_2(t)$, $h_3(t)$ are referred to as the two- and three-particle cut contributions to the imaginary part of $H(t)$, with thresholds $4m^2$ and $9m^2$. The cut graphs contributing to them can be drawn in the simplified form of Fig. 3. A solid line drawn across all the cut lines cut the graphs of Fig. 3 in two regions, one

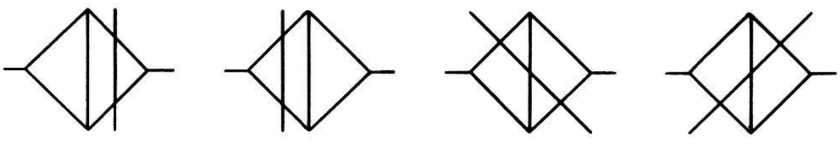

Figure 3
The independent non vanishing cut graphs of Fig. 1.

with circles and the other without the circles, and the circles are not more shown (the always vanishing cut graphs are omitted for simplicity). From the previous discussion, one is free to choose the direction of the energy flow across the cutting line and the cut graphs of Fig. 3 are then seen to be equal to the cut graphs No. 5, 15, 10, 11 or 2, 6, 7, 12 of Fig. 1, depending on the choice.

The vertex amplitude

We consider now the scalar vertex amplitude No. 1 of Fig. 4 (to be interpreted as a vertex graph in general, not as a particular one). By carrying out the same detailed analysis bringing from Fig. 1 to Fig. 3 one finds that the total momentum flowing across the various cutting lines in Fig. 4 can be different in different cut graphs. It is natural to classify them according to the invariant square mass of that momentum. More precisely, write the vertex amplitude No. 1 of Fig. 4 as

$$G(p_1, p_2, p_3) = V(-p_1^2, -p_2^2, -p_3^2), \qquad p_1 + p_2 = p_3, \tag{25}$$

where in the r.h.s. explicit reference is made to the dependance on scalar variables. One then has, in general,

$$\mathrm{Im}\, V(a, b, c) = \theta(a - a_0)v(\mathbf{a}, b, c) + \theta(b - b_0)v(a, \mathbf{b}, c) + \theta(c - c_0)v(a, b, \mathbf{c}). \tag{26}$$

The first term in the r.h.s. stands for a possible contribution to Im $V(a, b, c)$ for a above its threshold a_0 and arbitrary b, c; it corresponds to the cut graph No. 2 of Fig. 4 and the underlining of the variable a in $V(\mathbf{a}, b, c)$ refers to the square invariant mass of the cut lines. The interpretation of the other terms of equation (26) is analogous. (In the actual case of a complicated vertex graph, each term can consist in turn of several cut graphs). For large enough values of a, b, c any combination of the three terms in the r.h.s. can contribute at the same time to equation (26).

Consider again the cut graphs of Fig. 3 and take the energy flowing from left to right. The contribution of the first cut graph of Fig. 3 to equation (24) is the two-particle phase space at square energy t, times a vertex amplitude whose arguments are the square invariant masses of the cut lines and t, say $V(m^2, m^2, t)$. As $t > 4m^2$, i.e. larger than the t threshold for that vertex graph, $V(m^2, m^2, t)$ develops an imaginary part and becomes complex. That imaginary part is however compensated by the second cut graph of Fig. 3, which is the complex conjugate of the first, and the total contribution to equation (24) is therefore real, as expected.

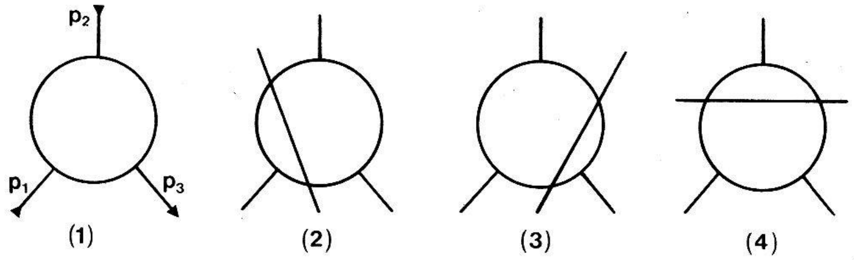

Figure 4
The vertex graph and its cut graphs.

The four-point amplitude

Let us consider, finally, the four point graph, No. 1 of Fig. 5. Let the kinematical conventions and definitions be

$$
\begin{aligned}
&p_1+p_2=p_3+p_4,\\
&\qquad a_i=-p_i^2, \qquad i=1,\dots,4,\\
&\qquad s=-(p_1+p_2)^2,\\
&\qquad t=-(p_1-p_4)^2,\\
&\qquad u=-(p_1-p_3)^2.
\end{aligned}
\tag{27a}
$$

The 7 invariants a_i, s, t, u are not independent, but satisfy the familiar relation

$$s+t+u=a_1+a_2+a_3+a_4. \tag{27b}$$

Finally, the four-point scalar amplitude can be written as

$$G(p_1, p_2, p_3, p_4)=B(a_1, a_2, a_3, a_4, s, t, u). \tag{28}$$

One can have, in general, seven different cuts on all the seven invariant variables, although they are not independent; they are shown by the graphs No. 2–8 of Fig. 5. The contribution to the imaginary part of the box amplitude from, say, the s-cut graph No. 6 will be indicated, by underlining the cut variable, as $b(a_1, a_2, a_3, a_4, \mathbf{s}, t, u)$; s_0 will be the corresponding threshold and a similar notation will be used for the other contributions.

In order to illustrate the structure of the (s, t, u) cuts, it may be convenient to look at the simple box graphs of Fig. 6. Graph 1 of Fig. 6 represents an 's, t box' and graphs No. 2, 3, 4 its s, t and u cuts. Note that the u-cut has a four particle intermediate state (to help the eye, the actually cutting of a line is evidenced by a dot) and splits the original graph into four disconnected pieces. Its value can be different from zero only if, besides $u>16m^2$, one has also $a_i>4m^2$ for all the external invariant masses.

Graphs 5–8 show the similar case of a 't, u box' in which the role of s and u are exchanged.

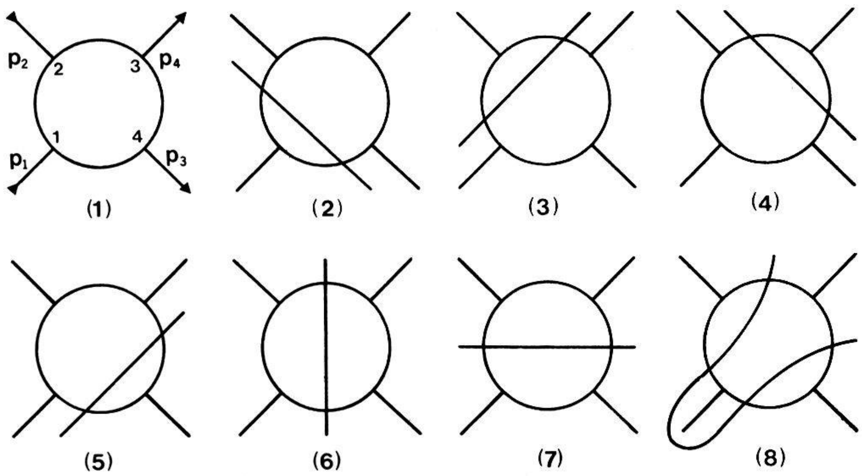

Figure 5
The four-point graph and its cut graphs.

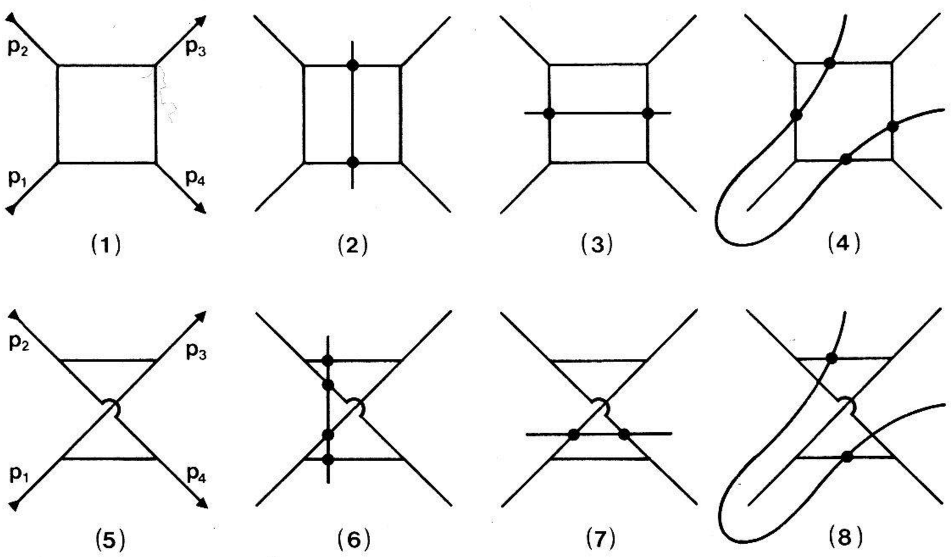

Figure 6
Two box graphs with their s, t and u cuts.

To complete this discussion, let us observe that in the case of the vertex, for instance, there are three different kinds of cuts and three independent variables; one can then keep two of them, say a and b, fixed and below threshold and let the third, c, vary above its threshold, so that the imaginary part of the vertex is entirely given by the c-cut. In the case of the box, as already seen, there are 7 cuts but only 6 independent variables. One can keep constant 5 of them at most; when the sixth is varied independently, the seventh is also varied according to equation (27b). In the usual situation of, say, s channel scattering at fixed t, the four external masses and t are fixed and below threshold, while s varies above threshold; the imaginary part of the amplitude is then entirely given by

$$b(a_1, a_2, a_3, a_4, \mathbf{s}, t, u = a_1 + a_2 + a_3 + a_4 - s - t),$$

with the dependent variable u also below threshold.

In next section we will consider another, less usual kinematical configuration, in which a_1, a_3, a_4, s and t are constant; if a_2 is above threshold and large enough, u is also brought above threshold by equation (27b) and the imaginary part of the amplitude is given in principle by the a_2-cut and the u-cut at the same time (see however Fig. 6) and the accompaning discussion). More in general, for large enough values of the invariant variables, satisfying of course equation (27b), almost any combination of the 7 cut graphs of Fig. 5 can be simultaneously different from zero.

4. The real part in momentum space

Let us now turn our attention to the real part of the graph amplitude, as given by equation (12) (as matter of fact, equation (12) refers to both the real and the imaginary part, but we will restrict almost always the discussion to kinematical configurations in which the imaginary part vanishes). By taking the Fourier transform of equation (12) one obtains

$$G(p_i)=i\int(Dx)[\mathbf{F}(m,n;x_i)+\theta(x_n^0-x_m^0)\mathbf{F}(m,\mathbf{n};x_i)+\theta(x_m^0-x_n^0)\mathbf{F}(\mathbf{m},n;x_i)],\tag{29}$$

with $G(p_i)$, (Dx) defined by equations (13) and (14). As shown in Ref. 1, on account of the integral representation equation (8), in momentum representation equation (29) takes the form of a (non-invariant) dispersion relation in an energy. We will show, in addition, how to rewrite it in the more familiar form of a dispersion relation in some invariant variable.

The two-point (self-mass) amplitude

We find convenient to illustrate the procedure in various cases, starting as before from the two-point amplitude represented by the graph No. 1 of Fig. 1. Let us choose the two points m, n appearing in equation (29) to be the points 1, 2 of the concerned graph; the first term $\mathbf{F}(m,n;x_i)$ in the r.h.s. will then drop out. We further take p entering from the left and spacelike, for simplicity, with

$$p^2=\vec{p}^2-p_0^2>0.\tag{30}$$

The Fourier transform of the second and third terms of equation (29) can then be depicted by the graphs No. 1 and 2 of Fig. 7. The dotted lines stand for the Fourier transform of the θ-function equation (8); they carry formally a four-momentum k, which has however only the fourth component different from zero

$$k=(k_0,\vec{0}).$$

The remaining part of graph 1 of Fig. 7 represents all the cut graphs having a circle at the vertex 2 and no circle at 1; the arrows at the extremities of the cutting line remind the direction of ths energy flow within the cut graph. The dotted line is not cut; note that the orientation of its momentum k, which is determined of course by equation (8), is different in the two graphs of Fig. 7. Recalling equation (21), equation (29) takes the already referred to form of a non-covariant dispersion relation in the energy k_0:

$$G(p)=\frac{1}{\pi}\int\frac{dk_0}{k_0-i\varepsilon}\Big[\theta(p_0+k_0)\,\mathrm{Im}\,G(p+k)+\theta(k_0-p_0)\,\mathrm{Im}\,G(p-k)],\tag{31}$$

the θ-functions corresponding to the arrows of Fig. 7.

We can now use equation (22), keeping for simplicity only one term in the

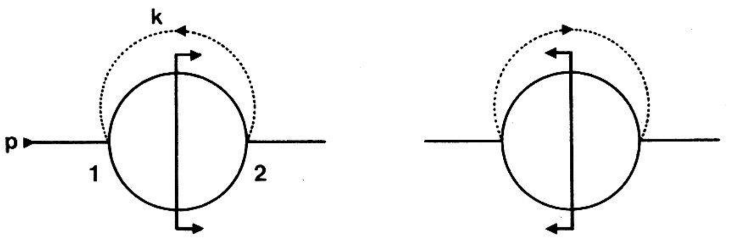

Figure 7
The contributions to equation (29) for the two-points amplitude.

r.h.s., with threshold t_0, instead of the two terms with thresholds $4m^2$ and $9m^2$; equation (31) then becomes

$$G(p)=\frac{1}{\pi}\int\frac{dk_0}{k_0-i\varepsilon}[\theta(k_0+p_0-\sqrt{\vec{p}^2+t_0})h((p_0+k_0)^2-\vec{p}^2)$$
$$+\theta(k_0-p_0-\sqrt{\vec{p}^2+t_0})h((p_0-k_0)^2-\vec{p}^2)]. \tag{32}$$

In order to rewrite it in covariant form, in the first term of the r.h.s. of equation (32), change the integration variable from k_0 to

$$t=(p_0+k_0)^2-\vec{p}^2,$$

by putting, according to the first of the θ-functions,

$$k_0=-p_0+\sqrt{\vec{p}^2+t}; \tag{33}$$

similarly, put

$$k_0=p_0+\sqrt{\vec{p}^2+t} \tag{34}$$

in the second term. By further using equation (23), equation (32) becomes

$$H(-p^2)=\frac{1}{\pi}\int\frac{dt}{2\sqrt{\vec{p}^2+t}}\left[\frac{1}{-p_0+\sqrt{\vec{p}^2+t}-i\varepsilon}+\frac{1}{p_0+\sqrt{\vec{p}^2+t}-i\varepsilon}\right]h(t), \tag{35}$$

i.e., unifying the denominators,

$$H(-p^2)=\frac{1}{\pi}\int\frac{dt}{t+p^2-i\varepsilon}h(t). \tag{36}$$

Equation (36) has the familiar form of a dispersion relation for $H(t)$ in the invariant variable $t=(-p^2)$; it has been obtained, however, as an integral relation, for spacelike p, between the two real functions of real variable $H(-p^2)$ and $h(t)$, without any reference to analyticity considerations. When continued to time-like p, (as matter of fact the above derivation applies to time-like p as well) for $(-p^2)>t_0$ it develops a discontinuity which is seen to be $2ih(t)$, i.e. twice the imaginary part of $H(t)$, as defined in equation (24) – in full agreement, of course, with the usual relation between discontinuity and imaginary part.

We show now another method for obtaining equation (36) from equation (35). To that aim, take the components of p to be, for instance,

$$p_0=P, \qquad p_x, p_y, p_z=P. \tag{37}$$

One has

$$p^2=\vec{p}^2-p_0^2=p_x^2+p_y^2, \tag{38}$$

i.e. p^2 is independent of the value of P. Take now the limit [5, 6]

$$P\to\infty \tag{39}$$

in equation (35). From equation (33) one has

$$k_0=(-p_0+\sqrt{\vec{p}^2+t})\xrightarrow[P\to\infty]{}\frac{t+p^2}{2P}\to 0 \tag{40}$$

and from equation 34)

$$k_0=(p_0+\sqrt{\vec{p}^2+t})\xrightarrow[P\to\infty]{}2P+\frac{t+p^2}{2P}. \tag{41}$$

Correspondingly, for the two terms of equation (35) one obtains

$$\lim_{P\to\infty}\frac{1}{2\sqrt{\vec{p}^2+t}}\frac{1}{-p_0+\sqrt{\vec{p}^2+t}-i\varepsilon}=\frac{1}{t+p^2+i\varepsilon} \tag{42}$$

$$\lim_{P\to\infty}\frac{1}{2\sqrt{\vec{p}^2+t}}\frac{1}{p_0+\sqrt{\vec{p}^2+t}-i\varepsilon}\leqslant\lim_{P\to\infty}\frac{1}{2(P^2+t)}=0; \tag{43}$$

equation (36) is again recovered, provided that the limit can be exchanged with the t-integration. That is the case if $h(t)$ vanishes fast enough for large t (which is anyhow required for equations (35) and (36) to make sense). Such asymptotic behaviour is easily checked for the graph 1 of Fig. 1 (of which $h(t)$ is the imaginary part).

The discussion of non-convergent graph amplitudes, requiring renormalization, can be done according to the following lines (we sketch only the argument, which can be found in Ref. 1 and in Ref. 2, 3 where it is exploited for the rigorous inductive construction of the renormalized perturbation series of quantum field theory). Observe that the cut amplitude is finite (because renormalization in lower orders, when needed, has been already carried out), so that the divergence is due to the k_0-integration only. Use some regularization prescription (for instance the continuous dimension [1] one), obtain a meaningful dispersion relation, write it in a suitably subtracted form and take the 'non-regularization' limit. Alternatively write from the start a suitably subtracted dispersion relation in k_0, taking it to be the very definition of the amplitude (and forget about other forms of regularization).

The vertex amplitude

Let us consider the vertex amplitude equation (25), represented by the first graph of Fig. 4; the contributions to the r.h.s. of equation (29), with m, n equal to the points 2 and 3 of the graph, are shown in Fig. 8.

The first graph of Fig. 8, corresponding to the first term in the r.h.s. of equation (29), is the a-cut of the vertex, graph No. 2 of Fig. 4 (to be obtained for negative p_{10}, as it corresponds to one circle in 1, no circles in 2 and 3). The second term in the r.h.s. of equation (29) consists of all the cut graphs with one circle at the point 3, no circle at 2. They are either b-cuts (circle at 1) or c-cuts (no circle at 1) and are represented by graphs No. 2, 3 of Fig. 7, where the dotted line represents as in Fig. 6 the θ-function. The remaining graphs No. 4, 5 of Fig. 8, similarly, represent the third term in the r.h.s. of equation (29), which then becomes, in the notation of equations (25) and (26),

$$\begin{aligned}
&V(-p_1^2,-p_2^2,-p_3^2)\\
&\quad= i\theta(p_{10}+\sqrt{\vec{p}_1^2+a_0})v(\underline{-p_1^2},-p_2^2,-p_3^2)\\
&\qquad+\frac{1}{\pi}\int\frac{dk_0}{k_0-i\varepsilon}[\theta(k_0+p_{20}-\sqrt{\vec{p}_2^2+b_0})v(-p_1^2,\underline{-(p_2+k)^2},-(p_3+k)^2)\\
&\qquad+\theta(k_0+p_{30}-\sqrt{\vec{p}_3^2+c_0})v(-p_1^2,-(p_2+k)^2,\underline{-(p_3+k)^2})\\
&\qquad+\theta(k_0-p_{20}-\sqrt{\vec{p}_2^2+b_0})v(-p_1^2,\underline{-(p_2-k)^2},-(p_3-k)^2)\\
&\qquad+\theta(k_0-p_{30}-\sqrt{\vec{p}_3^2+c_0})v(-p_1^2,-(p_2-k)^2,\underline{-(p_3-k)^2})].
\end{aligned} \tag{44}$$

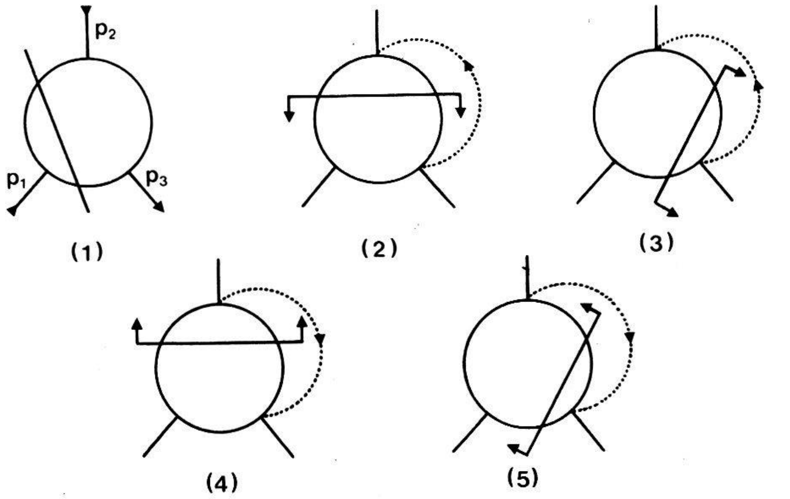

Figure 8
The contributions to equation (29) for the vertex amplitude.

Accordingly to previous conventions, $k=(k_0, \vec{k}=0)$ and the underlining of a variable refers to the cut in that variable.

Equation (44) can be regarded as a kind of dispersion relation in k_0, of non-invariant type. The first term in the square bracket, for instance, involves the b-cut of the vertex amplitude, where the first variable $a=-p_1^2$ is fixed, but the second and the third variables b, c vary together with k_0. A similar interpretation holds for the other terms.

We will now show how to convert equation (44) in an invariant dispersion relation say in b at fixed a, c, at least in suitable kinematical configurations. To that aim, let us take the following explicit values for the components of the momenta (the first component being the energy)

$$\begin{aligned} p_1 &= (P, p_{1x}, p_{1y}, P), \\ p_2 &= (-P, p_{2x}, p_{2y}, -P), \\ p_3 &= (0, p_{1x}+p_{2x}, p_{1y}+p_{2y}, 0). \end{aligned} \tag{45}$$

They satisfy the conservation requirement $p_1+p_2=p_3$; they are all spacelike and give the following negative values for the invariants, independent of P

$$\begin{aligned} a &= -p_1^2 = -(p_{1x}^2+p_{1y}^2)<0 \\ b &= -p_2^2 = -(p_{2x}^2+p_{2y}^2)<0 \\ c &= -p_3^2 = -(p_{1x}+p_{2x})^2-(p_{1y}+p_{2y})^2<0. \end{aligned} \tag{46}$$

Consider then the first term in the square bracket of equation (44) and introduce the new integration variable b' defined as

$$b' = -(p_2+k)^2. \tag{47}$$

Due to the θ-function in equation (44) and to equation (45), one has

$$k_0 = P+\sqrt{P^2+b'-b}, \qquad dk_0 = \frac{db'}{2\sqrt{P^2+b'-b}}.$$

In the $P \to \infty$ limit

$$k_0 \simeq 2P, \qquad c' = -(p_3 + k) \simeq 4P^2. \tag{49}$$

If

$$v(a, \underline{b}', c' \simeq 4P^2) \xrightarrow[P \to \infty]{} 0, \tag{50}$$

fast enough to allow the exchange of the $P \to \infty$ limit with the b'-integration, in that limit the contribution of the concerned term to equation (44) vanishes.

Consider now the next term of equation (44) and introduce the new integration variable c' through

$$c' = -(p_3 + k)^2 = c + k_0^2, \tag{51}$$

so that

$$k_0 = \sqrt{c' - c}, \qquad dk_0 = \frac{dc'}{2\sqrt{c' - c}} \tag{52}$$

and

$$b' = -(p_2 + k)^2 = b + c' - c - 2P\sqrt{c' - c}. \tag{53}$$

If, again

$$v(a, b' \simeq -2P\sqrt{c' - c}, \underline{c}') \xrightarrow[P \to \infty]{} 0, \tag{54}$$

also this contribution vanishes. The same argument applies to the last term of equation (44), corresponding to the graph No. 5 of Fig. 8.

Consider finally the remaining term, corresponding to the graph No. 4 of Fig. 8. (As it will be seen, it will play the role of the graph 1 of Fig. 6 for the two point amplitude). Introduce

$$b' = -(p_2 - k)^2 \tag{55}$$

so that

$$k_0 = -P + \sqrt{P^2 + b' - b}, \qquad dk_0 = \frac{db'}{2\sqrt{P^2 + b' - b}}. \tag{56}$$

In the $P \to \infty$ limit

$$\begin{aligned} k_0 &\simeq \frac{b' - b}{2P} \to 0, \\ \frac{dk_0}{k_0 - i\varepsilon} &\simeq \frac{db'}{b' - b - i\varepsilon}, \\ c' &= -(p_3 - k)^2 \simeq c. \end{aligned} \tag{57}$$

Equation (44) then becomes

$$V(a, b, c) = \frac{1}{\pi} \int \frac{db'}{b' - b - i\varepsilon} v(a, \underline{b}', c); \tag{58}$$

equation (58) is the required integral relation between the vertex amplitude and one of its cuts, established here in the kinematical configuration of equation (46) and with the assumptions (50), (54) on the asymptotic behaviour of the cuts; it has the form of a dispersion relation in the invariant variable b, for fixed values of the other variables a and c – see in this respect the remarks following equation (36) and equation (43).

The four-point amplitude

The approach of the previous paragraphs can be easily extended to the four-point amplitude, apart from the obvious increase in complexity due to its much richer structure.

Let us consider (without writing it explicitly, for simplicity) equation (29) for the four-point amplitude equation (28) corresponding to the graph No. 1 of Fig. 5, choosing n, m to be to points 2 and 3, and limiting ourselves, from the very beginning, to spacelike moment only (see that all the scalar variables of equation (27) take negative values). The first term of the r.h.s. of equation (29) drops out, and one is left with the contributions to the non-covariant dispersion relation depicted in Fig. 9. With respect to the similar previously discussed Figs 6 and 7, note that each of the graphs of Fig. 9 corresponds to two countributions, differing in the direction of the energy flow across the cutting line and the dotted line representing the θ-function. One has a total of 8 contributions to the dispersion relation; each of them is the covolution of the dotted line, the θ-function, with a cut four-point graph. Following as much as possible the notation of equations (27), (28), the external momenta of the four-point graphs of Fig. 9 are p_1, p_4 and p_2', p_3'. The values of the last two are

$$p_2' = p_2 \pm k, \qquad p_3' = p_3 \pm k, \tag{58}$$

the two possibilities corresponding to the two different contributions represented by each single graph in Fig. 8. As a consequence, the scalar variables

$$\begin{aligned} a_2' &= -(p_2')^2, \\ a_3' &= -(p_3')^2, \\ s' &= -(p_1 + p_2')^2, \\ u' &= -(p_1 - p_3')^2 \end{aligned} \tag{59}$$

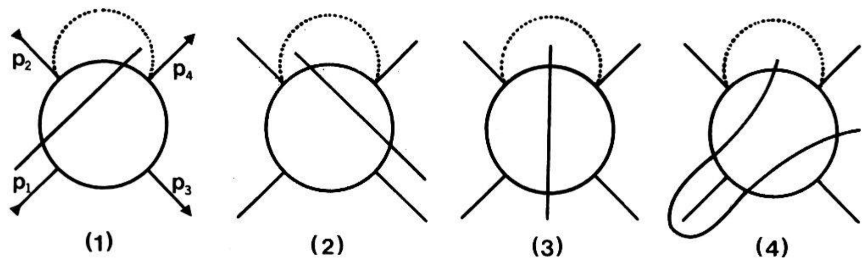

Figure 9
Contributions to equation (29) for the four-point amplitude.

depend all on k and vary along with k_0 in the non-invariant k_0-dispersion relation. On the contrary, the scalar variables a_1, a_4 and t remain constant, i.e. independent of k, in all the cut graphs of Fig. 9.

We discuss now the k_0-dispersion relation in the kinematical configuration

$$\begin{aligned} p_1 &= (P, p_{1x}, p_{1y}, P), \\ p_2 &= (-P, p_{2x}, p_{2y}, -P), \\ p_3 &= (0, p_{3x}, p_{3y}, 0), \\ p_4 &= (0, p_{4x}, p_{4y}, 0). \end{aligned} \tag{60}$$

It is understood that the unspecified x, y components of the momenta satisfy equation (27). It is straightforward to extend the analysis already done for the vertex to the eight contributions from the graphs of Fig. 9. For each of them, we replace k_0 by the appropriate integration variable (i.e. the variable in which the graph is cut) and express in terms of it k_0 and all the other variables in $P\to\infty$ limit. In most cases some of those variables diverge with P and we drop the corresponding contributions under the usual assumptions on the asymptotic behaviour of the Feynman graph and of its cuts. Only one of the two contributions from each of the graphs 1 and 4 of Fig. 8 are then found to survive in the $P\to\infty$ limit. The result can be written as

$$\begin{aligned} B(a_1, a_2, a_3, a_4, s, t, u) &= \frac{1}{\pi}\int \frac{da_2'}{a_2' - a_2 - i\varepsilon}\, b(a_1, \underline{a_2'}, a_3, a_4, s, t, u') \\ &+ \frac{1}{\pi}\int \frac{du'}{u' - u - i\varepsilon}\, b(a_1, a_2', a_3, a_4, s, t, \underline{u}'), \end{aligned} \tag{61}$$

where, besides the identity equation (28), one has

$$s + t + u' = a_1 + a_2' + a_3 + a_4. \tag{62}$$

Equation (61) has the required form of an invariant dispersion relation in a_2 and u at fixed a_1, a_3, a_4, s and t.

Equation (60), of course, do not provide the only possible kinematical configuration; in general, each different configuration is expected to give a different kind of dispersion relations. We will limit ourselves to illustrate one more case, with the kinematics of equation (63):

$$\begin{aligned} p_1 &= (P, p_{1x}, p_{1y}, P), \\ p_2 &= (0, p_{2x}, p_{2y}, 0), \\ p_3 &= (0, p_{3x}, p_{3y}, 0), \\ p_4 &= (P, p_{4x}, p_{4y}, P). \end{aligned} \tag{63}$$

As for equation (60), all momenta are spacelike and momentum conservation is understood. By carrying out the same analysis as for the previous case, in the $P\to\infty$ limit and under the usual assumptions on the asymptotic behaviour, only one of the two contributions from each of the graphs 3 and 4 of Fig. 8 is found to survive; the result is

$$B(a_1, a_2, a_3, a_4, s, t, u) = \frac{1}{\pi} \int \frac{ds'}{s' - s - i\varepsilon} b(a_1, a_2, a_3, a_4, \underline{s}', t, u') + \frac{1}{\pi} \int \frac{du'}{u' - u - i\varepsilon} b(a_1, a_2, a_3, a_4, s', t, \underline{u}'), \tag{64}$$

with

$$s' + t + u' = a_1 + a_2 + a_3 + a_4. \tag{65}$$

Equation (64) can be regarded as an invariant dispersion relation in s and u at fixed a_1, a_2, a_3, a_4 and t.

As a consequence of equation (27b), both equations (61) and (64) involve two terms. Concerning the u-cuts, however, see the comments to the simple graphs of Fig. 5. In the case of the first graph of that figure, the u-cut contributions drop out from equations (61), (64) for spacelike a_1, a_3, a_4. Equation (61) then becomes an obvious generalization of equation (57), i.e. a dispersion relation in one of the external square invariant masses at fixed values of all the other variables. In the case of the box graph No. 5 of Fig. 6, the u-contribution to equation (1) remains: this fact might be seen as a consequence of requiring a constant value of s in a graph whose 'natural' variables are t and u. Similarly, for that same graph the u-contribution remains while the s-contribution drops out from the fixed t dispersion relation equation (64).

5. Conclusions

Formulae for the imaginary parts of Feynman graphs in momentum space and for their real parts have been established. The first, valid without restrictions for arbitrary real values of the momenta, express the imaginary part of a graph as the sum of all its cut graphs, as illustrated in Section 3.

The equations for the real parts presented in Section 4 have the form of dispersion relations in some invariant variable, at fixed values of the others. They are obtained within the realm of real functions, by taking the Fourier transform of a local causality equation in suitably chosen kinematical configurations; in particular, all the momenta must be spacelike and all the invariant variables negative, i.e. outside the physical region (so that the $-i\varepsilon$, nicely present in equation (58), (61), (69), is of no effect there).

In the case of the two-point amplitude, the derivation applies to arbitrary timelike momentum as well; for positive and above threshold values of the only existing invariant, the amplitude becomes complex and the dispersion relation shows that the discontinuity is equal to twice the imaginary part, as obtained by the cutting formulae. A similar, suitably generalized result is expected to hold for the other amplitudes too, but the argument of Section 4, at least in its present form, does not apply to timelike vectors. It is not clear whether such physically interesting values can be reached with arguments based on real functions techniques only or whether analytic continuation arguments are needed (let us recall here that the usual expression of a graph amplitude as the integral on internal loop momenta of Feynman propagators does define directly the Feynman amplitude as a complex number depending on real parameters, namely the external momenta, allowed to take arbitrary real values). The formulae of Section 4 require also assumptions on the asymptotic behaviour of the amplitudes, to prevent appearance of subtraction terms and the like; but in our opinion this problem may be dealt with more easily than the previous one.

Finally, it might be interesting to look for an extension of the formulae towards two-variable dispersion relations.

Acknowledgements

It is the author's pleasure to acknowledge clarifying discussions with P. Menotti and V. Glaser and the kind hospitality at the Theoretical Physics Department of the Geneva University, where this work was completed.

REFERENCES

[1] G. t'Hooft and M. Veltman, *Diagrammar*, yellow Cern Report 73–9, 1973), see also M. Veltman, Physica 29, 186 (1963).
[2] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, New York, (1959).
[3] H. Epstein and V. Glaser, Ann. Inst. Henri Poincaré Section A, Vol. XIX, 211 (1973).
[4] R. E. Cutkosky, J. of Math. Physics *1*, 429 (1960).
[5] S. Fubini and G. Furlan, Physics *1*, 229 (1965).
[6] D. Amati, R. Jengo and E. Remiddi, Nuovo Cimento *51A*, 999 (1967).