# Three particle bound states in even P()_2 models

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# Three particle bound states in even $\lambda P(\phi)_2$ models

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*Abstract.* We discuss the existence of three particle bound states in even weakly coupled $\lambda P(\phi)_2$ models. It is shown that, in models possessing a two particle bound state, a three particle bound state may also occur, depending on certain properties of the three-body Bethe–Salpeter kernel. We consider a specific class of models in which a three particle bound state does occur.

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## Introduction

In this work we shall be concerned with the spectrum of the mass operator in weakly coupled even $\lambda P(\phi)_2$ models, where $P(x)=\sum_{k=0}^{N} C_{2k}x^{2k}$, $C_{2N}>0$.

These models are defined by a set of distributions, known as the Schwinger functions, satisfying a number of axioms, the Osterwalder–Schrader axioms. It is by now well known (see [OS]) that from a set of Schwinger functions satisfying these axioms one can construct, by analytic continuation, a set of tempered distributions satisfying the so called Wightman axioms. The reconstruction theorem [SW] then provides us with a Hilbert space $\mathscr{H}$, a representation $U(a, \Lambda)$ of the Poincaré group ($a$ is a translation, $\Lambda$ is a Lorentz rotation); a local and covariant field $\varphi(x)$ and a unique state $\Omega$ invariant under $U(a, \Lambda)$, called the vacuum state which is cyclic for the field $\varphi$.

The infinitesimal generators of $U(a, 1)$, with $a=(a_0, \vec{a})$ are denoted $P_0$, $\vec{P}$,

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the energy and momentum operators. The mass operator is defined by $M=(P_0^2-\vec{P}^2)^{1/2}$.

The Schwinger functions associated to a given $P(\phi)_2$ model are constructed by means of the following limiting procedure.

Let $d\mu_0$ denote the Gaussian measure on $\mathscr{S}'(\mathbf{R}^2)$ with mean zero and covariance $C(x-y)=(-\Delta+m_0^2)^{-1}(x, y)$ $(x=(x^0, x^1)\in\mathbf{R}^2)$. Then we have:

$$S_n(\lambda, x_1, \ldots, x_n)=\lim_{h\to 1}\frac{\int \phi(x_1)\cdots\phi(x_n)\exp\left[-\lambda\int d^2x :P(\phi(x)): h(x)\right]d\mu_0}{\int \exp\left[-\lambda\int d^2x :P(\phi(x)): h(x)\right]d\mu_0}$$

where Wick order is defined with respect to $C(x-y)$.

More generally, with $A_j(x)=:\phi^j:(x)$, one has:

$$\langle A_{j_1}(x_1)\cdots A_{j_n}(x_n)\rangle=\lim_{h\to 1}\frac{\int A_{j_1}(x_1)\cdots A_{j_n}(x_n)\exp\left[-\lambda\int d^2x :P(\phi(x)): h(x)\right]d\mu_0}{\int \exp\left[-\lambda\int d^2x :P(\phi(x)): h(x)\right]d\mu_0}$$

These functions are constructed by the cluster expansion of Glimm, Jaffe and Spencer and satisfy the Osterwalder–Schrader axioms [GJS1, 2].

Concerning these models, the results which are known about the spectrum of the mass operator are, among others:

i) there is an isolated point $m(\lambda)$ corresponding to the mass of the lightest particle described by the field $\varphi$, $m(\lambda)>0$ [GJS1, 2].
ii) for even models, the mass spectrum is discrete and of finite multiplicity below $2m(\lambda)$ [SZ].
iii) again for even models, the coefficient $C_4$ of the $:\phi^4:$ term decides on the presence or absence of two-particle bound states: if $C_4>0$, there is no spectrum below $2m(\lambda)$ in the even subspace of $\mathscr{H}$ and if $C_4<0$ there is exactly one point $m_B(\lambda)$ in the interval $(m(\lambda), 2m(\lambda))$ corresponding to a two-particle bound state. Furthermore, there is no other spectrum up to $3m(\lambda)-\varepsilon$ [DE, DE2].
iv) for general $P(\phi)_2$ models, a similar result has been obtained in [K], where conditions are given that enable one to decide on the presence or absence of two-particle bound state (see also [GJ3]).

Our purpose is to carry further the analysis of the mass spectrum in even models, in such a way that one could study the existence of *three-particle bound ststes*. These bound states show up as points in the spectrum of the mass operator restricted to the odd subspace of $\mathscr{H}$ and for weak coupling they should lie very close to (and below) $3m(\lambda)$.

The method we use to undertake such an analysis relies heavily on the work of Spencer and Zirilli [SZ], Dimock and Eckmann [DE, DE2] and Koch [K]. This method makes abundant use of analyticity properties of the functions involved.

Most important for our purposes is the three-body Bethe–Salpeter kernel $K_3(\lambda, k, p_1, p_2, q_1, q_2)$ (for a definition see Chapter I; for a longer motivation, see [GJ1, 2]). Let also $\tilde{S}_4^C(\lambda, k, p_1, p_2)$ denote the connected four point Schwinger function (in the $1 \to 3$ channel, which means that in $x$-space we choose as variables $\tau = x_1 - \frac{1}{3}(x_2 + x_3 + x_4)$, $\xi_1 = x_2 - x_3$, $\xi_2 = x_3 - x_4$, and $k$, $p_1$, $p_2$ are the associated momenta). Further, $R_3(\lambda, k, p_1, p_2, q_1, q_2)$ is the 2-particle irreducible (2p.i.) six point function (in an even model, we can consider the 1p.i. six point function which is automatically 2p.i.), and let

$$L_3(\lambda, k, p_1, p_2) = \tilde{S}_2(\lambda, k)^{-1} \int dp_1' \, dp_2' \tilde{S}_4^C(\lambda, k, p_1', p_2') R_3^{-1}(\lambda, k, -p_1', -p_2', p_1, p_2).$$

Before stating our main result, we shall consider the two-body problem. Let $K_2(\lambda, x_1, x_2, x_3, x_4)$ denote the two-body Bethe–Salpeter kernel and define

$$R_{02}(\lambda, x_1, x_2, x_3, x_4) = S_2(\lambda, x_1 - x_3) S_2(\lambda, x_2 - x_4) + S_2(\lambda, x_1 - x_4) S_2(\lambda, x_2 - x_3).$$

Consider also $R(\lambda, x_1, x_2, x_3, x_4)$ defined by the equation $R^{-1} = R_{02}^{-1} + 3K_2$ or, alternatively, by $R = R_{02} - 3R_{02}K_2R$, and let $R(\lambda, k', p, q)$ denote its Fourier transform in the variables $\sigma = \frac{1}{2}(x_1 + x_2 - x_3 - x_4)$, $\xi = x_1 - x_2$, $\eta = x_3 - x_4$. We know from [DE] that, in a model possessing a two particle bound state, $R(\lambda, k', p, q)$ has a pole at $k' = (im_B(\lambda), 0)$.

*Remark.* The function analysed in [DE] coincides with the function $R_2$ to be defined in Chapter I. For this function one writes $R_2^{-1} = R_{02}^{-1} + K_2$, the two-body Bethe–Salpeter equation. Nevertheless, the combinatorics of the three body problem is such that the relevant kernel for us turn out to be $R$, defined with the factor 3 multiplying $K_2$. One can however easily convince oneself that poles of $R_2(\lambda, k')$ are in one-to-one correspondence to poles of $R(\lambda, k')$. In particular, if $m_B'(\lambda)$ is the two particle bound state pole of $R_2(\lambda, k')$ then the corresponding pole of $R(\lambda, k')$ is at $m_B(\lambda)$ and we have

$$m_B'(\lambda) - m_B(\lambda) = 2m\alpha_1^2\pi^2\left(\frac{\lambda}{m^2}\right)^2 + \mathcal{O}(\lambda^3)$$

with

$$\alpha_1 = \partial_\lambda K_2(\lambda = 0, k' = (2im, 0), 0, 0).$$

Let now

$$\mu = (\mu_0, \mu_1) \in \mathbf{C}^2,$$

$$\mu_0 = \frac{-i(m_B^2 - m^2 - \frac{1}{3}(m - m_B)^2)}{2(m + m_B)}, \qquad \mu_1 = 0.$$

The main result to be proven in our work is the following

**Theorem.** *Consider a weakly coupled even $\lambda P(\phi)_2$ model having a two-particle bound state. Assume that:*

i) $\partial_\lambda K_3(\lambda = 0, k^0 = i(m + m_B), \mu, \mu/2, \mu, \mu/2) \equiv \alpha_2 \neq 0$

ii) $\partial_\lambda L_3(\lambda = 0, k^0 = i(m + m_B), \mu, \mu/2) \neq 0$

Then we have that $\alpha_2 > 0$ implies that there is no three-particle bound state, and $\alpha_2 < 0$ implies that there is exactly one three-particle bound state. Furthermore, there is no other spectrum in the odd subspace of $\mathcal{H}$ in the interval $(m(\lambda), 3m(\lambda))$.

This result is proven in two steps. The first one is to show that the 2p.i. six-point function $R_3(\lambda, k, p_i, q_i)$ has a singularity that will be physical or unphysical depending on the sign of $\alpha_2$ as explained in the theorem. The second one is to analyse the two-point function by means of $R_3(\lambda, k)$ and $L_3(\lambda, k)$. Given the conditions of the theorem, we can show that a physical singularity of $R_3(\lambda, k)$ always induces a singularity of the two-point function. This singularity is what we call a three-particle bound state because, as we shall see, it lies just below the three-particle threshold $3m$. Since it is a pole of the two-point function, we see that the field itself makes a transition from the vacuum to states consisting of one such a bound state.

In fact, the situation is as follows: if $k_1^0(\lambda)=i\kappa_1(\lambda)$ is the singularity of $R_3(\lambda, k)$, it turns out that $k_1^0(\lambda)$ is a zero of $\tilde{S}_2(\lambda, k)$. The pole of $\tilde{S}_2(\lambda, k)$ is very close to this zero, say at $k_2^0(\lambda)=im_3(\lambda)$ and we have that $m_3(\lambda)>\kappa_1(\lambda)$. One gets for $\tilde{S}_2(\lambda, k)$ the following Lehmann representation:

$$\tilde{S}_2(\lambda, k)=\frac{Z(\lambda)^2}{k^2+m(\lambda)^2}+\frac{Z_3(\lambda)^2}{k^2+m_3(\lambda)^2}+\int_{m+m_B}^{\infty} d\rho_\lambda(a)\frac{1}{k^2+a^2}$$

We can draw the graph of $\tilde{S}_2(\lambda, \kappa)\equiv\tilde{S}_2(\lambda, k=(i\kappa, 0))$ as follows, according to this formula

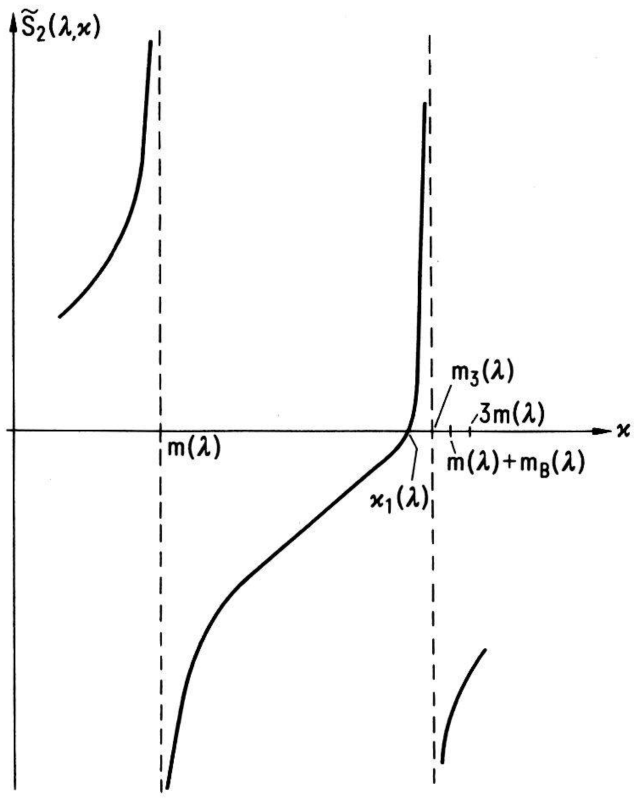



Figure 1
$\kappa_1(\lambda)$: the pole of $R_3(\lambda, \kappa)$, $m_3(\lambda)$: the three-particle bound state, $m(\lambda)+m_B(\lambda)$: the threshold of one $m$-particle plus one $m_B$-particle, $3m(\lambda)$: the three-particle threshold.

We sketch briefly the reasons why $R_3(\lambda, k)$ can have a singularity provided the model under consideration has a two-particle bound state.

We shall use the notation $X(\lambda, \kappa)=X(\lambda, k=(i\kappa, 0))$ for any kernel $X$. We come back for a while to the two-body problem and the Bethe–Salpeter equation:

$$R_2(\lambda, \kappa, p, q)$$
$$= R_{02}(\lambda, \kappa, p, q) - \int dp'\, dq' R_{02}(\lambda, \kappa, p, p')K_2(\lambda, \kappa, -p', -q')R_2(\lambda, \kappa, q', q).$$

In the approximation sometimes called 'the ladder approximation', which has been extensively used on a large number of problems, one substitutes for $K_2$ its lowest order contribution in $\lambda$. Let us consider a polynomial having a term of the form $C_4 :\phi^4:$, in such a way that there is a first order contribution $\alpha_1$ to $K_2$. It is given by $\alpha_1 = \lambda \cdot C_4 \cdot \text{const.}$, where const. stands for a positive constant.

In this case, the above equation can be explicitly solved and one gets, with $r_{00}(\lambda, \kappa) = \int dp\, dq R_{02}(\lambda, \kappa, p, q)$:

$$f_2(\lambda, \kappa) = \int dp\, dq R_2(\lambda, \kappa, p, q) = r_{00}(\lambda, \kappa)(1 - \alpha_1 r_{00} + \alpha_1^2 r_{00}^2 - \cdots)$$
$$= r_{00}(\lambda, \kappa)(1 + \alpha_1 r_{00}(\lambda, \kappa))^{-1}.$$

One can also use that

$$R_{02}(\lambda, \kappa, p, q) = 4\pi \tilde{S}_2\left(\lambda, p + \frac{i\kappa}{2}\right)\tilde{S}_2\left(\lambda, p - \frac{i\kappa}{2}\right)\delta(p+q)$$

and the Lehmann spectral formula for $\tilde{S}_2$ to obtain:

$$r_{00}(\lambda, \kappa) = Z(\lambda)^4 \cdot \frac{1}{\pi}\int dp\left[\left(p - \frac{i\kappa}{2}\right)^2 + m^2\right]^{-1}\left[\left(p + \frac{i\kappa}{2}\right)^2 + m^2\right]^{-1} + \rho_2(\lambda, \kappa)$$
$$= 4Z(\lambda)^4(4m^2 - \kappa^2)^{-1/2} \cdot \frac{1}{\kappa} \arcsin \frac{\kappa}{2m} + \rho_2(\lambda, \kappa)$$

where $\rho_2(\lambda, \kappa)$ is holomorphic and bounded for $0 < \kappa < 3m$. *So* $r_{00}(\lambda, \kappa)$ *is singular as* $\kappa \to 2m$. Since $\alpha_1 \sim 0(\lambda)$, we see that singularities of $f_2(\lambda, \kappa)$ for small $\lambda$ can only occur for $\kappa$ near $2m$. Furthermore, such a singularity can only occur if $\alpha_1 < 0$ (and so $C_4 < 0$), otherwise $1 + \alpha_1 r_{00}(\lambda, \kappa)$ is always bounded away from zero. But if $\alpha_1 < 0$, there always exists a solution $\kappa_B(\lambda)$ of $1 + \alpha_1 r_{00}(\lambda, \kappa) = 0$ which corresponds to the two-particle bound state.

A rigorous analysis of this problem faces the question of knowing whether this result remains valid when the full kernel $K_2(\lambda, \kappa, p, q)$ is considered. The answer to this question, which turns out to be positive, is essentially the first part of the paper by Dimock and Eckmann [DE].

The corresponding situation in the *three-body problem* is a bit more complicated. The three-body Bethe–Salpeter kernel $K_3$ is defined, in analogy with $K_2$, as the connected part of $R_3^{-1} : K_3 = R_3^{-1} - R_{2B}^{-1}$ where the kernel $R_{2B}$ plays here the role of $R_{02}$ in the two-body problem. The analysis of the analytic structure of $R_{2B}(\lambda, \kappa)$ is the subject of Chapter II below, and for the purposes of this introduction we only need to know that it describes the two-particle rescattering processes, that is, processes described by graphs in which at most two of the particles interact at any given vertex. Among the relevant graphs, two of them will interest us for the moment:

i) the graph describing three free particles, call it $R_{03}$. It is given by $R_{03}(\lambda, x_1, x_2, x_3, y_1, y_2, y_3) \sim S_2(\lambda, x_1 - y_1) S_2(\lambda, x_2 - y_2) S_2(\lambda, x_3 - y_3)$ and it is drawn $R_{03} = \equiv$, each line —— corresponding to a propagator $S_2$.

ii) the graphs describing a process where only two of the particles interact, call it $b(\lambda, x_1, x_2, x_3, y_1, y_2, y_3) \sim S_2(\lambda, x_1 - y_1) R(\lambda, x_2, x_3, y_2, y_3)$ where in this example particle 2 and 3 interact (the $R$ factor) and particle 1 remains free (the $S_2$ factor). It is drawn $b =$ where corresponds to $R$.

Let $s_{00}(\lambda, \kappa) = \int dp_1\, dp_2\, dq_1\, dq_2 R_{2B}(\lambda, \kappa, p_1, p_2, q_1, q_2)$. Correspondingly we define: $s_{00}^{(1)}(\lambda, \kappa) = \int dp_i\, dq_i R_{03}(\lambda, \kappa, p_i, q_i)$ and $s_{00}^{(2)}(\lambda, \kappa) = \int dp_i\, dq_i b(\lambda, \kappa, p_i, q_i)$.

The result concerning the analyticity properties of $s_{00}^{(1)}(\lambda, \kappa)$ is quite standard. It corresponds to a graph like

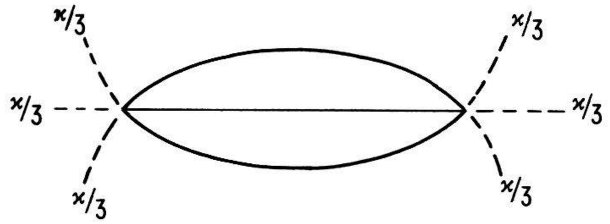



Figure 2

where dashed lines correspond to amputated lines. This graph has a kinematical singularity at the threshold $\kappa = 3m$ in the form of a square root branch point. But unlike the analogous case in the two-particle problem, the function is bounded on a neighbourhood of $\kappa = 3m$. Thus the mechanism described above for the formation of two particle bound states does not seem to apply.

Concerning $s_{00}^{(2)}(\lambda, \kappa)$, we must use the result of [DE] on the two-particle problem. There it is proven that, in case there is no two-particle bound state, $R(\lambda, \kappa', p, q)$ is analytic and uniformly bounded in $\kappa'$ for fixed $\lambda$ on a neighbourhood of $\kappa' = 2m$. This implies that $s_{00}^{(2)}(\lambda, \kappa)$ in this case is also analytic and bounded on a neighbourhood of $\kappa = 3m$.

Refering to the two-body problem, we note that a bound state could occur because $r_{00}(\lambda, \kappa)$ was not bounded on a neighbourhood of the threshold $\kappa = 3m$. We will now see that, *provided there exists a two-particle bound state*, the function $s_{00}^{(2)}(\lambda, \kappa)$ becomes unbounded on a neighbourhood of the threshold $m + m_B(\lambda)$. This implies that $s_{00}(\lambda, \kappa)$ is also unbounded as $\kappa \to m + m_B(\lambda)$ and we will see that this singularity is sufficient to generate a pole of $R_3(\lambda, \kappa)$ by a mechanism analogous to the one responsible for a two-particle bound state.

Consider then a model possessing a two-particle bound state. In such a model, the function $R(\lambda, \kappa')$ has the form

$$R(\lambda, \kappa', p, q) = \frac{Z_B(\lambda)^2}{-\kappa'^2 + m_B^2} h(\lambda, p) h(\lambda, q) + \rho_3(\lambda, \kappa', p, q)$$

with $\rho_3(\lambda, \kappa')$ analytic and bounded around $\kappa' = 2m$. As before, the term $\rho_3(\lambda, \kappa')$ makes a contribution to $s_{00}^{(2)}(\lambda, \kappa)$ which is bounded around $\kappa = 3m$. The term of interest to us is the other one, describing the propagator of the bound state. We show (see Appendix A) that

$$b(\lambda, \kappa, p_1, p_2, q_1, q_2) \sim \tilde{S}_2\left(\lambda, p_1+\frac{i\kappa}{3}\right) R\left(\lambda, p_1-\frac{2i\kappa}{3}, \frac{-p_1}{2}+p_2, \frac{p_1}{2}+q_2\right) \delta(p_1+q_1)$$

so that the contribution of the bound state term of $R$ is

$$s_{00}^{(3)}(\lambda, \kappa) = Z(\lambda)^2 Z_B(\lambda)^2 \int dp_1\, dp_2\, dq_1\, dq_2 \frac{1}{\left(p_1+\frac{i\kappa}{3}\right)^2+m^2} \cdot \frac{1}{\left(p_1-\frac{2i\kappa}{3}\right)^2+m_B^2}$$

$$\times h\left(\lambda, \frac{-p_1}{2}+p_2\right) h\left(\lambda, \frac{p_1}{2}+q_2\right) \delta(p_1+q_1)$$

$$= Z(\lambda)^2 Z_B(\lambda)^2 A^2 \int dp_1 \frac{1}{\left(p_1+\frac{i\kappa}{3}\right)^2+m^2} \cdot \frac{1}{\left(p_1-\frac{2i\kappa}{3}\right)^2+m_B^2}$$

with

$$A = \int dp h(\lambda, p).$$

But this last integral shows the characteristic behaviour of a two-particle threshold (in two space-time dimensions), namely

$$\int dp \frac{1}{\left(p+\frac{i\kappa}{3}\right)^2+m^2} \cdot \frac{1}{\left(p-\frac{2i\kappa}{3}\right)^2+m_B^2} = (-\kappa^2+(m+m_B)^2)^{-1/2} t_0(\kappa)$$

with $t_0(\kappa)$ bounded away from zero and analytic near $m+m_B(\lambda)$. We see that in this case the threshold $(m+m_B)$ corresponding to a free particle of mass $m$ and a free particle of mass $m_B$ behaves in such a way as to make $s_{00}^{(2)}(\lambda, \kappa)$ unbounded as $\kappa \to (m+m_B)$. Now consider again the three-body Bethe–Salpeter equation in the ladder approximation. Assume also that the interaction polynomial has a $C_6 :\phi^6:$ term, so that the first order contribution to $K_3(\lambda, \kappa)$ can be written as $\alpha_2 = \lambda \cdot C_6 \cdot \text{const.}$ where const. stands again for a positive constant. We can solve explicitly the equation to get:

$$f_3(\lambda, \kappa) \equiv \int dp_i\, dq_i R_3(\lambda, \kappa, p_i, q_i) = s_{00}(\lambda, \kappa)(1+\alpha_2 s_{00}(\lambda, \kappa))^{-1}.$$

Our previous discussion shows that in models possessing a two-particle bound state the function $s_{00}(\lambda, \kappa)$ is unbounded as $\kappa \to (m+m_B)$. This implies that, for small $\lambda$, there is a solution of $1+\alpha_2 s_{00}(\lambda, \kappa)=0$ near (and below) $\kappa = m+m_B$ provided $\alpha_2<0$ (and so $C_6<0$). This solution ultimately corresponds to the three-particle bound state of the model. The preceding discussion permits us to say that in some sense this three-particle bound state is really a bound state of two-particles, one of them being the particle of mass $m$, the other one the two-particle bound state of mass $m_B$.

The rigorous proof of this result has to deal with two things: that the main contribution to $R_{2B}(\lambda, \kappa)$ is indeed the one picked above (namely, the propagator of the two-particle bound state) and that the consideration of the full kernel $K_3$

does not change things too much. This is, essentially, the content of the following pages.

## I. Preliminaries

The particles described by a quantum field model correspond to the point spectrum of the mass operator $M=(P_0^2-P^2)^{1/2}$. This spectrum can in turn be analysed by looking at the singularities (in momentum space) of the Schwinger functions defining the model (we briefly sketch this connection in Chapter IV). On the other hand, the way to carry out such an analysis of Schwinger functions is to study irreducible functions and then tracing back their analytical properties to the Schwinger functions. The equations defining functions which are suitably irreducible in all channels are equations of the Bethe–Salpeter type. We shall be concerned with these equations in the case of the two and the three-body problem. The basic objects of the analysis are defined recursively as follows ($A$ is any product of Euclidean fields):

$$\mathbf{P}_0 A=\langle A\rangle$$

$$R_n(\lambda, x_1,\dots,x_n, y_1,\dots,y_n)$$
$$=\langle\phi(x_1)\cdots\phi(x_n)(1-\mathbf{P}_0-\mathbf{P}_1-\cdots-\mathbf{P}_{n-1})\phi(y_1)\cdots\phi(y_n)\rangle$$

$$\mathbf{P}_n A=\int dx_1\cdots dx_n\,dy_1\cdots dy_n(1-\mathbf{P}_0-\cdots-\mathbf{P}_{n-1})\phi(x_1)\cdots\phi(x_n)$$
$$\times R_n^{-1}(\lambda, x_1,\dots,y_n)\langle\phi(y_1)\cdots\phi(y_n)(1-\mathbf{P}_0-\cdots-\mathbf{P}_{n-1})A\rangle.$$

The kernels $R_n(\lambda,\mathbf{x},\mathbf{y})$ ($\mathbf{x}=\{x_1,\dots,x_n\}$, $\mathbf{y}=\{y_1,\dots,y_n\}$) are expected to be $(n-1)$-particle irreducible in the channel $\mathbf{x}\to\mathbf{y}$ (this in fact can be proven for weakly coupled $\lambda P(\phi)_2$ by the method of Spencer, see [CD]), which amounts to say, loosely speaking, that we have a bound of the form

$$|R_n(\lambda,\mathbf{x},\mathbf{y})|\le \text{const. exp}\left[-n(m-\varepsilon)\left|\frac{\sum_i x_i}{n}-\frac{\sum_i y_i}{n}\right|\right]$$

where $m=m(\lambda)$ is the mass of the lightest particle created by the field $\phi$ ($\sim$ the decay rate of the two-point Schwinger function) and $\varepsilon=\varepsilon(\lambda)$ goes to zero as $\lambda\to 0$. The above bound implies analyticity in the energy-variable $k$ up to $n(m-\varepsilon)$. We can increase the degree of irreducibility by one unit if we take the inverse:

$$|R_n^{-1}(\lambda,\mathbf{x},\mathbf{y})|\le \text{const. exp}\left[-(n+1)(m-\varepsilon)\left|\frac{\sum_i x_i}{n}-\frac{\sum_i y_i}{n}\right|\right]$$

where $R_n^{-1}(\lambda,\mathbf{x},\mathbf{y})$ is defined by

$$\int d\mathbf{x}' R_n^{-1}(\lambda,\mathbf{x},\mathbf{x}')R_n(\lambda,\mathbf{x}',\mathbf{y})=\prod_{i=1}^n \delta(x_i-y_i)$$

For weak coupling, this decay of $R_n^{-1}$ is also established in [CD]. Nevertheless, $R_n^{-1}$ is not connected. Its connected part, $K_n(\lambda,\mathbf{x},\mathbf{y})$, is known as the *n-body Bethe–Salpeter kernel.*

From now on, we specialize our discussion to the cases $n=2, 3$. The case $n=2$ is well-known, so we mostly discuss the case $n=3$. In this case, let $K_3(\lambda, \mathbf{x}, \mathbf{y})$ be the connected part of $R_3^{-1}(\lambda, \mathbf{x}, \mathbf{y})$. In other words, $K_3 = R_3^{-1} - R_{2B}^{-1}$ where $R_{2B}^{-1}$, which will be soon defined, is precisely the non-connected part of $R_3^{-1}$. We will see that $R_{2B}^{-1}$ can be expressed in terms of $K_2$ and $K_1$ ($\equiv R_1^{-1}$), the Bethe–Salpeter kernels corresponding to the two and one-particle problem, respectively. Its inverse, $R_{2B}(\lambda, \mathbf{x}, \mathbf{y})$, describes the so-called two-particle rescattering processes. The equation

$$K_3 = R_3^{-1} - R_{2B}^{-1} \tag{I.1}$$

is the three-body Bethe–Salpeter equation. It can be equivalently written as

$$R_3 = R_{2B} - R_{2B}K_3R_3 \tag{I.2}$$

with a formal solution

$$R_3 = (1 + R_{2B}K_3)^{-1}R_{2B} \tag{I.3}$$

We will be mostly dealing with the equation in this last form (see Chapter III). For the moment, note that we can analyse the analytical structure of $R_3$ from (I.3) once the corresponding analytical structure of $R_{2B}$ and $K_3$ is known. According to its definition, one expects $K_3$ to have nice analytical properties (in weakly coupled $\lambda P(\phi)_2$ these results on $K_3$ can be proven by the method of Spencer, see [CD] and Chapter III below). The study of $R_{2B}$ is the subject of Chapter II. Once the analytical structure of $R_3$ is known, we are able to face the analysis of the Schwinger function, in particular the two-point function. These are more directly connected to the mass spectrum, as shown in Chapter IV.

In the remainder of this chapter, we define the kernel $R_{2B}(\lambda, \mathbf{x}, \mathbf{y})$ and state the equations in momentum space. At the end we review some general facts about the two-body problem.

To define $R_{2B}$ we have to introduce the following functions:

$$\begin{aligned}
&R_2(\lambda, x_1, x_2, y_1, y_2) = \langle \phi(x_1)\phi(x_2)(1-\mathbf{P}_0-\mathbf{P}_1)\phi(y_1)\phi(y_2)\rangle \\
&R_{02}(\lambda, x_1, x_2, y_1, y_2) = R_1(\lambda, x_1-y_1)R_1(\lambda, x_2-y_2) \\
&\qquad\qquad + R_1(\lambda, x_1-y_2)R_1(\lambda, x_2-y_1)\ ^{2)} \\
&R_3(\lambda, x_1, x_2, x_3, y_1, y_2, y_3) \\
&\quad = \langle \phi(x_1)\phi(x_2)\phi(x_3)(1-\mathbf{P}_0-\mathbf{P}_1-\mathbf{P}_2)\phi(y_1)\phi(y_2)\phi(y_3)\rangle \\
&R_{03}(\lambda, x_1, x_2, x_3, y_1, y_2, y_3) \\
&\quad = \sum_{\pi} R_1(\lambda, x_1-y_i)R_1(\lambda, x_2-y_j)R_1(\lambda, x_3-y_k)
\end{aligned} \tag{I.4}$$

where $\pi$ ranges over the six permutations $(i, j, k)$ of $(1, 2, 3)$,

$$K_2(\lambda, x_1, x_2, y_1, y_2) = R_2^{-1}(\lambda, x_1, x_2, y_1, y_2) - R_{02}^{-1}(\lambda, x_1, x_2, y_1, y_2)$$

which is the two-body Bethe–Salpeter kernel. Let $\alpha = (1, 2), (1, 3), (2, 3)$ label

2) In an even theory, $R_1(\lambda, x-y) = S_2(\lambda, x-y)$. We will use indistinctly both notations since we shall be dealing with even models.

pairs of initial or final particles. We define:

$$M_{\alpha\beta}(\lambda, x_1, x_2, x_3, y_1, y_2, y_3) = R_1^{-1}(\lambda, x_i - y_j) K_2(\lambda, x_{\alpha_1}, x_{\alpha_2}, y_{\beta_1}, y_{\beta_2}), \qquad i \notin \alpha, \qquad j \notin \beta, \tag{I.5}$$

$$M(\lambda, \mathbf{x}, \mathbf{y}) = \frac{1}{9} \sum_{\alpha,\beta} M_{\alpha\beta}(\lambda, \mathbf{x}, \mathbf{y})$$

We have now all the ingredients to define:

$$R_{2B}^{-1}(\lambda, \mathbf{x}, \mathbf{y}) = R_{03}^{-1}(\lambda, \mathbf{x}, \mathbf{y}) + M(\lambda, \mathbf{x}, \mathbf{y}). \tag{I.6}$$

Note that (I.6) is equivalent to (using operator notation):

$$R_{2B} = R_{03} - R_{03} M R_{2B} \tag{I.7}$$

which has a formal solution:

$$R_{2B} = (1 + R_{03} M)^{-1} R_{03}. \tag{I.8}$$

We will not prove here that the kernel $R_{2B}^{-1}$ so defined is the non-connected part of $R_3^{-1}$. A discussion and a proof of this point can be found in [CD], [GJ1], [GJ2] to which we refer the reader.

We note that the three-body Bethe–Salpeter equation (I.2) can be realized on a space of functions $f(x_1, x_2, x_3)$ symmetric under permutation of variables. In this case, we have the following simplification:

$$\begin{aligned} R_{03}(\lambda, \mathbf{x}, \mathbf{y}) &= 6 R_1(\lambda, x_1 - y_1) R_1(\lambda, x_2 - y_2) R_1(\lambda, x_3 - y_3) \\ M(\lambda, \mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{3} M_i(\lambda, \mathbf{x}, \mathbf{y}) \\ M_i(\lambda, \mathbf{x}, \mathbf{y}) &= \tfrac{1}{3} R_1^{-1}(\lambda, x_i - y_i) K_2(\lambda, x_{\alpha_1}, x_{\alpha_2}, y_{\alpha_1}, y_{\alpha_2}), \qquad i \notin \alpha, \\ \alpha &= (1, 2), (1, 3), (2, 3). \end{aligned} \tag{I.5a}$$

We introduce the variables:

$$\begin{aligned} \xi_1 &= x_1 - x_2 & \eta_1 &= y_1 - y_2 \\ \xi_2 &= x_2 - x_3 & \eta_2 &= y_2 - y_3 \end{aligned} \qquad \tau = \tfrac{1}{3}(x_1 + x_2 + x_3 - y_1 - y_2 - y_3)$$

and consider again the equation (I.2):

$$R_3(\lambda, \mathbf{x}, \mathbf{y}) = R_{2B}(\lambda, \mathbf{x}, \mathbf{y}) - \int d\mathbf{x}' d\mathbf{y}' R_{2B}(\lambda, \mathbf{x}, \mathbf{x}') K_3(\lambda, \mathbf{x}', \mathbf{y}') R_3(\lambda, \mathbf{y}', \mathbf{y})$$

Introduce also:

$$\begin{aligned} \xi_1' &= x_1' - x_2' & \eta_1' &= y_1' - y_2' & \tau' &= \tfrac{1}{3}(x_1' + x_2' + x_3' - y_1' - y_2' - y_3') \\ \xi_2' &= x_2' - x_3' & \eta_2' &= y_2' - y_3' & \tau'' &= \tfrac{1}{3}(y_1' + y_2' + y_3' - y_1 - y_2 - y_3) \end{aligned}$$

Note that the transformation $\mathbf{x}', \mathbf{y}' \to \xi_i', \eta_i', \tau', \tau''$ has Jacobian one.
We use the translation invariance of the kernels to write:

$$\begin{aligned} R_3(\lambda, \tau, \xi_i, \eta_i) = R_{2B}(\lambda, \tau, \xi_i, \eta_i) - \int d\xi_i' \, d\eta_i' \, d\tau' \, d\tau'' R_{2B}(\lambda, \tau - \tau' - \tau'', \xi_i, \xi_i') \\ \times K_3(\lambda, \tau', \xi_i', \eta_i') R_3(\lambda, \tau'', \eta_i', \eta_i) \end{aligned} \tag{I.9}$$

Let $p_1, p_2, q_1, q_2, k$ be the momenta conjugate to $\xi_1, \xi_2, \eta_1, \eta_2, \tau$ respectively.[3]) We take the Fourier transform of each of the above kernels.[4]) Let:

$$R_3(\lambda, k, p_i, q_i) = (2\pi)^{-5}\int d\xi_i\, d\eta_i\, d\tau$$
$$\times \exp\,[i(k\tau + p_1\xi_1 + p_2\xi_2 + q_1\eta_1 + q_2\eta_2]R_3(\lambda, \tau, \xi_i, \eta_i)$$
$$K_3(\lambda, k, p_i, q_i) = (2\pi)^{-3}\int d\xi_i\, d\eta_i\, d\tau$$
$$\times \exp\,[i(k\tau + p_1\xi_1 + p_2\xi_2 + q_1\eta_1 + q_2\eta_2)]K_3(\lambda, \tau, \xi_i, \eta_i)$$
$$R_{2B}(\lambda, k, p_i, q_i) = (2\pi)^{-5}\int d\xi_i\, d\eta_i\, d\tau$$
$$\times \exp\,[i(k\tau + p_1\xi_1 + p_2\xi_2 + q_1\eta_1 + q_2\eta_2)]R_{2B}(\lambda, \tau, \xi_i, \eta_i)$$

Using the fact that the integral (I.9) is a convolution in the $\tau$-variables, and the $\tau \to -\tau$ invariance, we can write the equation in momentum space:

$$R_3(\lambda, k, p_i, q_i) = R_{2B}(\lambda, k, p_i, q_i) - \int dp_i'\, dq_i' R_{2B}(\lambda, k, p_i, p_i')$$
$$\times K_3(\lambda, k, -p_i', q_i')R_3(\lambda, k, -q_i', q_i)$$

or, as operators:

$$R_3(\lambda, k) = R_{2B}(\lambda, k) - R_{2B}(\lambda, k)K_3(\lambda, k)R_3(\lambda, k). \tag{I.10}$$

We consider this as an equation for operators acting on a space of functions $f(q_1, q_2)$ invariant under the transformations

$$\begin{cases} q_1 \to -q_1 + q_2 \\ q_2 \to q_2 \end{cases}, \qquad q_1 \Leftrightarrow -q_2, \qquad \begin{cases} q_1 \to q_1 \\ q_2 \to q_1 - q_2 \end{cases} \tag{I.11}$$

which express the symmetry under permutations of the functions in $x$-space.

Following [SZ] and [DE], we realize equation (I.10) on a space of analytic functions. For $p_1 = (p_1^{(0)}, p_1^{(1)})$, $p_2 = (p_2^{(0)}, p_2^{(1)})$, consider the Hardy space $A_3$ of functions $f(p_1, p_2)$ invariant under (I.11) and analytic in the domain $|\mathrm{Im}\, p_1^{(0)}|$, $|\mathrm{Im}\, p_2^{(0)}| < \frac{3}{4}m(\lambda) - \varepsilon = \delta_0^{(3)}$, $|\mathrm{Im}\, p_1^{(1)}|$, $|\mathrm{Im}\, p_2^{(1)}| < \frac{1}{4}m(\lambda) - \varepsilon = \delta_1^{(3)}$, with norm given by:

$$\|f\|_{A_3}^2 = \sup_{\substack{\alpha_1^{(i)} < \delta_i^{(3)} \\ \alpha_2^{(i)} < \delta_i^{(3)}}} \int dp_1\, dp_2\, |w(p_1 + i\alpha_1)w(p_2 + i\alpha_2)f(p_1 + i\alpha_1, p_2 + i\alpha_2)|^2$$

where

$$w(p) = (p^2 + 16m^2)^{-2/3}.$$

We realize thus equation (I.10) as an equation in $\mathscr{L}(A_3, A_3^*)$, with $A_3^*$ the dual of $A_3$ and $\mathscr{L}(A_3, A_3^*)$ standing for the set of bounded maps from $A_3$ to $A_3^*$.

---

3) Note that $p_1$ and $p_2$ are not the relative momenta between particles 1 and 2, 2 and 3, respectively. Instead, letting $r_1$, $r_2$, $r_3$ denote the momenta of particles 1, 2, 3, respectively, we have: $k = r_1 + r_2 + r_3$, $2(k/3) - p_1 = r_2 + r_3$, $k/3 - p_2 = r_3$.

4) We shall NOT distinguish the kernels in momentum space from the kernels in position space. The argument of the function should make clear this point.

*Remark* I.1. The index 3 in $A_3$ indicates that this is the adequate space for studying three-particle processes. In $\delta_i^{(3)}$ we use the index to stress its connection to $A_3$ (we will soon have analogous definitions with an index 2).

We begin our review of the two-body problem by considering again the equation defining $R(\lambda, x_1, x_2, y_1, y_2)$, namely $R^{-1} = R_{02}^{-1} + 3K_2$. Changing to the variables $\sigma = \frac{1}{2}(x_1 + x_2 - y_1 - y_2)$, $\xi = x_1 - x_2$, $\eta = y_1 - y_2$, we take the Fourier transform:

$$R(\lambda, k', p, q) = (2\pi)^{-3} \int d\sigma\, d\xi\, d\eta e^{i(k'\sigma + \xi p + \eta q)} R(\lambda, \sigma, \xi, \eta)$$

$$K_2(\lambda, k', p, q) = (2\pi)^{-1} \int d\sigma\, d\xi\, d\eta e^{i(k'\sigma + p\xi + q\eta)} K_2(\lambda, \sigma, \xi, \eta) \tag{I.12}$$

$$R_{02}(\lambda, k', p, q) = (2\pi)^{-3} \int d\sigma\, d\xi\, d\eta e^{i(k'\sigma + p\xi + q\eta)} R_{02}(\lambda, \sigma, \xi, \eta).$$

In momentum space, the equation defining $R$ reads:

$$R(\lambda, k', p, q)$$
$$= R_{02}(\lambda, k', p, q) - 3 \int dp'\, dq' R_{02}(\lambda, k', p, p') K_2(\lambda, k', -p', q') R(\lambda, k', -q', q)$$

or, as operators,

$$R(\lambda, k') = R_{02}(\lambda, k') - 3R_{02}(\lambda, k') K_2(\lambda, k') R(\lambda, k'). \tag{I.13}$$

The kernel $R(\lambda, k')$ will arise in the next chapter when considering $R_{2B}$. We will see that an essential point in the analysis of $R_{2B}$ is the study of a process where one of the particles, say particle 1, does not interact with the other two. Such a process is described by a kernel

$$b_1(\lambda, \mathbf{x}, \mathbf{y}) = R_1(\lambda, x_1, y_1) R(\lambda, x_2, x_3, y_2, y_3).$$

In momentum space, we have

$$b_1(\lambda, k, p_1, p_2, q_1, q_2) \sim \delta(p_1 + q_1) R_1\left(\lambda, \frac{k}{3} + p_1\right) R\left(\lambda, \frac{2k}{3} - p_1, \frac{-p_1}{2} + p_2, \frac{p_1}{2} + q_2\right).$$

Hence, matrix elements of $b_1(\lambda, k)$ are given by integrals of the form

$$\langle \psi_1, b_1(\lambda, k) \psi_2 \rangle_{A_3}$$
$$\sim \int dp_1 R_1\left(\lambda, \frac{k}{3} + p_1\right) \int dp_2\, dq_2 \psi_1\left(p_1, \frac{p_1}{2} + p_2\right) R\left(\lambda, \frac{2k}{3} - p_1, p_2, q_2\right) \psi_2\left(p_1, \frac{p_1}{2} - q_2\right)$$

Since $\psi_i \in A_3$, we have that, for fixed $p_1$, $\psi_{i,p_1}(q) \equiv \psi_i(p_1, p_1/2 + q)$ are functions in $A_2$, where $A_2$ is the Hardy space of functions $f(q) = f(-q)$ analytic in $|\text{Im}\, q^{(0)}| < \frac{1}{2}(\frac{3}{4}m - \varepsilon) = \delta_0^{(2)}$, $|\text{Im}\, q^{(1)}| < \frac{1}{2}(\frac{1}{4}m - \varepsilon) = \delta_1^{(2)}$ with norm given by

$$\|f\|_{A_2}^2 = \sup_{\alpha_i < \delta_i^{(2)}} \int d^2q\, |w(q + i\alpha) f(q + i\alpha)|^2$$

This is the motivation to realize equation (I.12) in the space $A_2$ (which is defined with $\delta_i^{(2)} = \frac{1}{2}\delta_i^{(3)}$, $i = 0, 1$). This fact, together with the factor 3 multiplying $K_2$ in the definition of $R$, introduces slight but harmless changes in the results of [DE] and [K].

*Remark* I.2. With respect to the norms on $A_3$ and $A_2$, we have the following result: let $\psi(p_1, p_2) \in A_3$. Define $\psi_{p_1}(p_2) \in A_2$ by $\psi_{p_1}(p_2) = \psi(p_1, p_1/2 + p_2)$. Then

$$\sup_{|\mathrm{Im}\, p_1^{(i)}| < \delta_i^{(3)}} |w(p_1)\, \|\psi_{p_1}\|_{A_2}| \leq \mathcal{O}(1)\, \|\psi\|_{A_3}.$$

This follows from Cauchy's theorem.

We next present some results about $R(\lambda, k')$. These results are best expressed in terms of a new variables, $\zeta$, which we now introduce. For $k_1'$ real, let $\zeta = \sqrt{4m(\lambda)^2 + k_1'^2 + k_0'^2}$. We think of this transformation as a conformal map of the $k_0'$-complex plane. The physical sheet of the $k_0'$-plane is mapped onto Re $\zeta > 0$. Functions analytically continued across the imaginary axis to Re $\zeta < 0$ correspond, in the $k_0'$-variable, to functions continued to a second sheet of the energy-plane. One defines:

$$\begin{aligned}
&\mathcal{D} = \{k_0' : 0 < \mathrm{Im}\, k_0' < \tfrac{8}{3}m,\ \mathrm{Im}\, k_0' \notin [\sqrt{4m^2 + k_1'^2}, \tfrac{8}{3}m)\}\\
&\hat{\mathcal{D}} = \{\zeta : \zeta = \sqrt{4m^2 + k_1'^2 + k_0'^2},\ k_0' \in \mathcal{D}\}\\
&\hat{\mathcal{D}}' = \hat{\mathcal{D}} \cup -\hat{\mathcal{D}} \cup (\text{connecting line})\\
&\hat{\mathcal{D}}(\beta) = \{\zeta : \zeta \in \hat{\mathcal{D}}',\ \mathrm{Re}\, \zeta > -\beta\}\\
&\mathcal{D}(\beta) = \{k_0' : \sqrt{4m^2 + k_1'^2 + k_0'^2} = \zeta \in \hat{\mathcal{D}}(\beta)\}.
\end{aligned} \tag{I.14}$$

We use the notation $\hat{f}(\zeta) \equiv f(k')$ when $\zeta = \sqrt{4m^2 + k_1'^2 + k_0'^2}$.

We have also the following definitions:

$$R_1(\lambda, p) = \tilde{S}_2(\lambda, p) = \frac{1}{2\pi}\left(\frac{Z(\lambda)^2}{p^2 + m(\lambda)^2} + \int_{3m-\varepsilon}^{\infty} d\rho_\lambda(a)\, \frac{1}{p^2 + a^2}\right)$$

$$\begin{aligned}
R_{02}(\lambda, k', p, q) &= R_{02}(\lambda, k', p)\, \delta(p + q)\\
&= 4\pi \tilde{S}_2\left(\lambda, \frac{k'}{2} + p\right) \tilde{S}_2\left(\lambda, \frac{k'}{2} - p\right) \delta(p + q) = \rho_{01}(\lambda, k') + \rho_{02}(\lambda, k').
\end{aligned}$$

With $\varepsilon_p \in A_2^*$ given by $\langle \psi, \varepsilon_p \rangle_{A_2} = \psi(p)$,

$$\rho_{01}(\lambda, k') = r_{00}(k') Z(\lambda)^4 \varepsilon_0 \langle \cdot, \varepsilon_0 \rangle_{A_2}$$

and

$$\begin{aligned}
r_{00}(k') &= \frac{1}{\pi} \int d^2p \left[\left(\frac{k'}{2} + p\right)^2 + m^2\right]^{-1} \left[\left(\frac{k'}{2} - p\right)^2 + m^2\right]^{-1}\\
&= (4m^2 + k'^2)^{-1/2} r_0(k')
\end{aligned}$$

with $r_0(k')$ holomorphic and bounded for $k_0' \in \mathcal{D}$,

$$r_0(k' = (2im, 0)) = \frac{\pi}{m}.$$

According to this decomposition of $R_{02}(\lambda, k')$, we have:

$$T(\lambda, k') = 3K_2(\lambda, k') R_{02}(\lambda, k') = T_1(\lambda, k') + T_2(\lambda, k').$$

We state the following lemma:

**Lemma I.1.** *$\hat{R}(\lambda, \zeta) = \hat{\rho}_1(\lambda, \zeta) + \hat{\rho}_2(\lambda, \zeta)$ and there exists a $C^\infty$ function $\zeta_1(\lambda)$ with values in $\hat{\mathscr{D}}(\delta_1^{(2)} - 2\varepsilon)$ such that $(\zeta - \zeta_1(\lambda))\hat{\rho}_1(\lambda, \zeta)$ and $\hat{\rho}_2(\lambda, \zeta) \in \mathscr{L}(A_2, A_2^*)$ are $C^\infty$ in $\lambda \geq 0$ small and holomorphic in $\hat{\mathscr{D}}(\delta_1^{(2)} - 2\varepsilon)$ together with their $\lambda$-derivatives. Furthermore,*

$$\hat{\rho}_2(\lambda, \zeta) = \hat{\rho}_{02}(\lambda, \zeta)(1 + \hat{T}_2(\lambda, \zeta))^{-1}$$

*and $\hat{\rho}_1$ is the rank-one operator given by*

$$\hat{\rho}_1(\lambda, \zeta) = \frac{\hat{r}(\lambda, \zeta)}{\zeta - \zeta_1(\lambda)}(1 + \hat{T}_2^*(\lambda, \zeta))^{-1}\varepsilon_0\langle \cdot, (1 + \hat{T}_2^*(\lambda, \zeta))^{-1}\varepsilon_0\rangle_{A_2}$$

*and $\zeta_1(\lambda)$ is the solution of*

$$\zeta_1(\lambda) + \hat{r}_0(\lambda, \zeta_1(\lambda))\langle(1 + \hat{T}_2(\lambda, \zeta_1(\lambda)))^{-1} \cdot 3\hat{K}_2(\lambda, \zeta_1(\lambda))\varepsilon_0, \varepsilon_0\rangle_{A_2} = 0$$

*and we have*

$$\hat{r}(\lambda, \zeta) = \frac{(\zeta - \zeta_1(\lambda))\hat{r}_0(\lambda, \zeta)}{\zeta + \hat{r}_0(\lambda, \zeta)\langle(1 + \hat{T}_2(\lambda, \zeta))^{-1} \cdot 3\hat{K}_2(\lambda, \zeta)\varepsilon_0, \varepsilon_0\rangle_{A_2}}$$

*We note that this is Lemma 27 in* [K], *with the slight modifications caused by our choice of $A_2$ and the factor 3 multiplying $K_2$. The modifications brought about by $A_2$ are:*

i) *in the definition of the domain $\mathscr{D}$, we have $0 < \operatorname{Im} k_0' < \frac{8}{3}m$ instead of $0 < \operatorname{Im} k_0' < 3(m - \varepsilon)$*

ii) *the region of holomorphy in Lemma* I.1 *is $\hat{\mathscr{D}}(\delta_1^{(2)} - 2\varepsilon)$ with $\delta_1^{(2)} = \frac{1}{2} \cdot \frac{1}{4}(m - \varepsilon)$ instead of $\delta_1 = \frac{1}{4}(m - \varepsilon)$ in Lemma* 27 *of* [K].

*These are harmless changes for what follows.* □

*Remark* I.3. The pole $\zeta_1(\lambda)$ of $\hat{R}(\lambda, \zeta)$ is close to $\zeta = 0$ and it is always real. Negative values of $\zeta_1(\lambda)$ do not correspond to poles on the physical sheet of the $k_0'$-plane. In what follows, we will be always considering models for which $\zeta_1(\lambda) > 0$. These are the models that have a two-particle bound state.

Coming back to $\hat{\rho}_1(\lambda, \zeta)$, we shall need a further decomposition of it. Let

$$\hat{H}(\lambda, \zeta, \cdot) \equiv (1 + \hat{T}_2(\lambda, \zeta))^{-1}(0, \cdot) \tag{I.15}$$

We can write

$$\hat{\rho}_1(\lambda, \zeta, p, q) = \frac{\hat{r}(\lambda, \zeta)}{\zeta - \zeta_1(\lambda)}\hat{H}(\lambda, \zeta, p)\hat{H}(\lambda, \zeta, q). \tag{I.16}$$

We define

$$\hat{\rho}(\lambda, \zeta, p, q) = \frac{\hat{r}(\lambda, \zeta_1(\lambda))}{\zeta - \zeta_1(\lambda)}\hat{H}(\lambda, \zeta_1(\lambda), p)\hat{H}(\lambda, \zeta_1(\lambda), q) \tag{I.17}$$

and

$$\hat{\rho}_3(\lambda, \zeta) = \hat{\rho}_1(\lambda, \zeta) - \hat{\rho}(\lambda, \zeta). \tag{I.18}$$

We have:

**Lemma I.2.** *The operator $\hat{\rho}_3(\lambda, \zeta) \in \mathscr{L}(A_2, A_2^*)$ is holomorphic and bounded in $\hat{\mathscr{D}}(\delta_1^{(2)} - 2\varepsilon)$. It is $C^\infty$ in $\lambda \geq 0$ small and the $\lambda$-derivatives are also holomorphic and bounded in $\hat{\mathscr{D}}(\delta_1^{(2)} - 2\varepsilon)$.*

*Proof.* Given the properties of $\hat{\rho}_1(\lambda, \zeta)$ in Lemma I.1, the result follows from the fact that, given a function $f(\zeta)$ holomorphic and bounded uniformly in a domain $\hat{\mathscr{D}}$, and given a point $\zeta_1$ of $\hat{\mathscr{D}}$, then

$$f(\zeta) = f(\zeta_1) + (\zeta - \zeta_1)\bar{f}(\zeta)$$

with $\bar{f}(\zeta)$ holomorphic and bounded uniformly in $\hat{\mathscr{D}}$. The bound on $\bar{f}(\zeta)$ depends, in general, on $\zeta_1$ if $\zeta_1$ can approach a singularity of $f(\zeta)$. This is not the case here, since $\zeta_1(\lambda)$ is close to zero and $\hat{\rho}_1(\lambda, \zeta)$ is holomorphic on a disc (of radius $\delta_1^{(2)} - 2\varepsilon$) around the origin. □

We come back to the $k'$-variable and close this chapter stating some results about the operators $\rho_2(\lambda, k')$ and $\rho_3(\lambda, k')$.

**Lemma I.3.** *Let $\phi \in A_{2,\infty}(= \{\phi \in A_2 \colon \|\phi\|_{2,\infty} = \sup_{|\mathrm{Im}\, p^{(i)}| < \delta_i^{(2)}} |\phi(p)| < \infty\})$. Let*

$$g_2(\phi, \psi, \lambda, k') = \langle \phi, \rho_2(\lambda, k')\psi \rangle_{A_2}$$

$$\left( = \int d^2p \, d^2q \phi(p)\psi(q)\rho_2(\lambda, k', p, q) \right)$$

*and*

$$g_3(\phi, \psi, \lambda, k') = \langle \phi, \rho_3(\lambda, k')\psi \rangle_{A_2}.$$

*Let also $k_1'$ be real and $0 \leq \mathrm{Im}\, k_0' < \frac{5}{3}m$. (The factor 5/3 is arbitrary. All we need is that $\mathrm{Im}\, k_0'$ be bounded away from the threshold $2m$).*
*Then*

i) $|g_2(\phi, \psi, \lambda, k')| \leq \mathcal{O}(1)((\mathrm{Re}\, k_0')^2 + k_1'^2 + 1)^{-1} \|\phi\|_{2,\infty} \|\psi\|_{A_2}$
$|g_3(\phi, \psi, \lambda, k')| \leq \mathcal{O}(1)((\mathrm{Re}\, k_0')^2 + k_1'^2 + 1)^{-1} \|\phi\|_{2,\infty} \|\psi\|_{A_2}$.

*If $k_0'$ is in $\mathscr{D}(\delta_1^{(2)} - 2\varepsilon)$ (see (I.14)), then:*

ii) $$\left| \frac{dg_i}{dk_0'}(\phi, \psi, \lambda, k') \right| \leq \mathcal{O}(1)(4m^2 + k_0'^2 + k_1'^2)^{-1/2} \|\phi\|_{2,\infty} \|\psi\|_{A_2}, \qquad i = 2, 3.$$

*Proof.* The proof of ii) follows from the fact that $\hat{g}_i(\phi, \psi, \lambda, \zeta)$ is holomorphic in $\hat{\mathscr{D}}(\delta_1^{(2)} - 2\varepsilon)$. Using that

$$\frac{dg_i}{dk_0'} = \frac{d\hat{g}_i}{d\zeta} \cdot \frac{d\zeta}{dk_0'} = \frac{d\hat{g}_i}{d\zeta} \cdot k_0'(4m^2 + k_0'^2 + k_1'^2)^{-1/2}$$

and that $(d\hat{g}_i/d\zeta)$ is bounded uniformly in $\hat{\mathscr{D}}(\delta_1^{(2)} - 2\varepsilon)$, we have the result.

The proof of i) for $g_3$ is immediate from its definition. The proof for $g_2$ follows from:

$$\langle \phi, \rho_{02}(\lambda, k')\psi \rangle_{A_2} \leq \mathcal{O}(1)((\mathrm{Re}\, k_0')^2 + k_1'^2 + 1)^{-1} \|\phi\|_{2,\infty} \|\psi\|_{A_2} \quad \text{for} \quad 0 < \mathrm{Im}\, k_0' < \tfrac{5}{3}m.$$

□

## II. The two-body rescattering kernel

Our starting point is the equation defining $R_{2B}$ (see (I.7) and (I.8)):

$$R_{2B} = R_{03}(1+MR_{03})^{-1}. \tag{II.1}$$

We define

$$a(\lambda, \mathbf{x}, \mathbf{y}) = MR_{03}(\lambda, \mathbf{x}, \mathbf{y}) = \sum_{i,j=1}^{3} a_{ij}(\lambda, \mathbf{x}, \mathbf{y}) \tag{II.2}$$

with

$$a_{ij}(\lambda, \mathbf{x}, \mathbf{y}) = \tfrac{1}{3}\delta(x_i - y_j)K_2 R_{02}(\lambda, x_{\alpha_1}, x_{\alpha_2}, y_{\beta_1}, y_{\beta_2}),$$
$$\alpha = \{1, 2, 3\}\backslash\{i\}, \qquad \beta = \{1, 2, 3\}\backslash\{j\}.$$

Let also

$$b_{ij}(\lambda, \mathbf{x}, \mathbf{y}) = R_{03}(\tfrac{1}{9} + a_{ij})^{-1}(\lambda, \mathbf{x}, \mathbf{y}) \tag{II.3}$$

Recalling that

$$R^{-1} = R_{02}^{-1} + 3K_2$$

we can write

$$b_{ij}(\lambda, \mathbf{x}, \mathbf{y}) = 27 R_1(\lambda, x_i - y_j) R(\lambda, x_{\alpha_1}, x_{\alpha_2}, y_{\beta_1}, y_{\beta_2}) \tag{II.4}$$

$\alpha$ and $\beta$ as above. With these definitions, we have

$$R_{2B}(\lambda, \mathbf{x}, \mathbf{y}) = \left(\sum_{i,j=1}^{3} b_{ij}^{-1}(\lambda, \mathbf{x}, \mathbf{y})\right)^{-1}. \tag{II.5}$$

We now introduce the $\tau$, $\xi_i$, $\eta_i$ variables and consider equation (I.7) in momentum space:

$$R_{2B}(\lambda, k, p_i, q_i) = R_{03}(\lambda, k, p_i, q_1) - \int dp_i'\, dq_i' R_{03}(\lambda, k, p_i, p_i')$$
$$\times M(\lambda, k, -p_i', p_i') R_{2B}(\lambda, k, -q_i', q_i)$$

with

$$R_{03}(\lambda, k, p_i, q_i) = (2\pi)^{-5} \int d\tau\, d\xi_i\, d\eta_i$$
$$\times \exp\left[i(k\tau + p_1\xi_1 + p_2\xi_2 + q_1\eta_1 + q_2\eta_2)\right] R_{03}(\lambda, \tau, \xi_i, \eta_i)$$

and

$$M(\lambda, k, p_i, q_i) = (2\pi)^{-3} \int d\tau\, d\xi_i\, d\eta_i$$
$$\times \exp\left[i(k\tau + p_1\xi_1 + p_2\xi_2 + q_1\eta_1 + q_2\eta_2) M(\lambda, \tau, \xi_i, \eta_i).\right.$$

One can also verify that

$$R_{03}(\lambda, k, p_i, q_i) = 6 \cdot 4\pi^2 R_1\left(\lambda, \frac{k}{3} + p_1\right) R_1\left(\lambda, \frac{k}{3} - p_1 + p_2\right) R_1\left(\lambda, \frac{k}{3} - p_2\right)$$
$$\times \delta(p_1 + q_1)\, \delta(p_2 + q_2), \tag{II.6}$$

$$M_{11}(\lambda, k, p_i, q_i) = \frac{2\pi}{9} R_1^{-1}\left(\lambda, \frac{k}{3}+p_1\right) K_2\left(\lambda, \frac{2k}{3}-p_1, \frac{-p_1}{2}+p_2, \frac{p_1}{2}+q_2\right) \times \delta(p_1+q_1).\ ^{5)} \tag{II.7}$$

See the Appendix A for a derivation of these formulas.

Using the fact that $R_1^{-1}(\lambda, k/3+p_1) = (4\pi^2 R_1(\lambda, k/3+p_1))^{-1}$, one gets for $a_{11}$:

$$a_{11}(\lambda, k, p_i, q_i) = \frac{4\pi}{3} \delta(p_1+q_1) R_1\left(\lambda, \frac{k}{3}-p_1+p_2\right) R_1\left(\lambda, \frac{k}{3}-p_2\right) \times K_2\left(\lambda, \frac{2k}{3}-p_1, \frac{-p_1}{2}+p_2, \frac{p_1}{2}+q_2\right). \tag{II.8}$$

The analogous expression for $b_{11}$ is:

$$b_{11}(\lambda, k, p_i, q_i) = 27 \cdot 2\pi\, \delta(p_1+q_1) R_1\left(\lambda, \frac{k}{3}+p_1\right) R\left(\lambda, \frac{2k}{3}-p_1, \frac{-p_1}{2}+p_2, \frac{p_1}{2}+q_2\right) \tag{II.9}$$

with $R(\lambda, k', p, q)$ defined in (I.12).

The expressions for general $a_{ij}$, $b_{ij}$ are obtained by the following rule, which corresponds to symmetrization in $\mathbf{x}-\mathbf{y}$-space. We make the convention that the first index correspond to the $p$-variables, the second to the $q$-variables.

$$\begin{array}{ll} \text{index 2: replace } p_1 \to -p_1+p_2, p_2 \to p_2 \quad \text{or} \quad q_1 \to -q_1+q_2, q_2 \to q_2 \\ \text{index 3: replace } p_1 \leftrightarrow -p_2 \quad \text{or} \quad q_1 \leftrightarrow -q_2 \end{array} \tag{II.10}$$

in the expressions (II.8), (II.9) for $a_{11}$ and $b_{11}$.

In what follows, we will be mainly concerned with $b_{11}$ since a general $b_{ij}$ is obtained through the rule (II.10). We take also $k^1 = 0$ (this corresponds to analyse the problem in the center of mass system) so that $k$-dependence becomes $k^0$-dependence.

The form in which we will be dealing with equation (II.1) is the following. The variable $k^0$ plays the role of a parameter and we write:

$$R_{2B}(\lambda, k^0) = \left(\sum_{i,j=1}^{3} b_{ij}^{-1}(\lambda, k^0)\right)^{-1} \tag{II.11}$$

considering $R_{2B}(\lambda, k^0)$ as an operator in $\mathscr{L}(A_3, A_3^*)$.

We note that the individual $b_{ij}(\lambda, k^0)$ are not operators on $\mathscr{L}(A_3, A_3^*)$ because they do not preserve the invariance properties of functions in $A_3$. This is the reason why we introduce a space of functions $A_3'$ which is the same as $A_3$ except that functions in $A_3'$ need not be invariant under the transformations (I.11).

We now proceed to the analysis of $b_{11}(\lambda, k^0)$. Let

$$A'_{3,\infty} = \{\phi \in A_3' : \|\phi\|_{A_{3,\infty}} = \sup_{|\mathrm{Im}\, p_j^{(i)}| < \delta_i^{(3)}} |\phi(p_1, p_2)| < \infty, j = 1, 2, i = 0, 1\}$$

and let $\phi, \psi \in A_3'$, $\phi \in A'_{3,\infty}$.

---

5) We use $M_{11} = M_{\alpha\beta}$ with $\alpha = \beta = \{2, 3\}$.

The objects under analysis will be matrix elements of $b_{11}(\lambda, k^0)$:

$$\langle\phi, b_{11}(\lambda, k^0)\psi\rangle = \int dp_1\, dp_2\, dq_1\, dq_2 \phi(p_1, p_2) \times b_{11}(\lambda, k^0, p_1, p_2, -q_1, -q_2)\psi(q_1, q_2).$$

Using (II.9), we have:

$$\langle\phi, b_{11}(\lambda, k^0)\psi\rangle = 54\pi \int dp_1 R_1\left(\lambda, \frac{k}{3}+p_1\right) \int dp_2\, dq_2 \phi(p_1, p_2) \times R\left(\lambda, \frac{2k}{3}-p_1, \frac{-p_1}{2}+p_2, \frac{p_1}{2}-q_2\right)\psi(p_1, q_2)$$

or, equivalently,

$$\langle\phi, b_{11}(\lambda, k^0)\psi\rangle = 54\pi \int dp_1 R_1\left(\lambda, \frac{k}{3}+p_1\right) \int dp_2\, dq_2 \phi_{p_1}(p_2) \times R\left(\lambda, \frac{2k}{3}-p_1, p_2, q_2\right)\psi_{p_1}(-q_2) \tag{II.12}$$

where we have defined

$$\phi_{p_1}(p_2) = \phi\left(p_1, \frac{p_1}{2}+p_2\right), \qquad \psi_{p_1}(q_2) = \psi\left(p_1, \frac{p_1}{2}+q_2\right). \tag{II.13}$$

*Remark* II.1. If $\phi, \psi \in A_3$, we can check that

$$\phi_{p_1}(-p_2) = \phi_{p_1}(p_2), \qquad \psi_{p_1}(-q_2) = \psi_{p_1}(q_2) \quad \text{so that} \quad \phi_{p_1}, \psi_{p_1} \in A_2.$$

Our strategy in analysing $b_{11}(\lambda, k^0)$ is the following: we break it into several pieces, all of them regular (= bounded) near the thresholds $3m$ or $m+m_B$ with the exception of the last one, which is a rank-one operator singular at the threshold $m+m_B$. This one (which is analogous to $s_{00}^{(2)}(\lambda, k^0)$ defined in the Introduction) enables us to pursue the analysis of $R_3$ in Chapter III along the lines of a modified ladder approximation as explained in the Introduction.

The first two pieces are defined according to

$$R_1(\lambda, p) = \frac{1}{2\pi}\left(\frac{Z(\lambda)^2}{p^2+m^2} + \int_{3m-\varepsilon}^{\infty} d\rho_\lambda(a) \frac{1}{p^2+a^2}\right).$$

Let $J(\lambda, k^0) = \langle\phi, b_{11}(\lambda, k^0)\psi\rangle = J_1(\lambda, k^0) + J_2(\lambda, k^0)$ with

$$\begin{aligned} J_1(\lambda, k^0) &= 27 \int dp_1 \int_{3m-\varepsilon}^{\infty} d\rho_\lambda(a) \frac{1}{\left(\frac{k}{3}+p_1\right)^2 + a^2} \\ &\quad \times \int dp_2\, dq_2 \phi_{p_1}(p_2)\psi_{p_1}(-q_2) R\left(\lambda, \frac{2k}{3}-p_1, p_2, q_2\right) \\ J_2(\lambda, k^0) &= 27 Z(\lambda)^2 \int dp_1 \frac{1}{\left(\frac{k}{3}+p_1\right)^2 + m^2} \\ &\quad \times \int dp_2\, dq_2 \phi_{p_1}(p_2)\psi_{p_1}(-q_2) R\left(\lambda, \frac{2k}{3}-p_1, p_2, q_2\right). \end{aligned} \tag{II.4}$$

The kind of result we are interested in is illustrated by our first two lemmas.

**Lemma II.1.** *Let $\phi, \psi \in A_3'$, $\phi \in A_{3,\infty}'$. Then $J_1(\lambda, k^0)$ defined in (II.14) is holomorphic in $0 < \operatorname{Im} k^0 < \frac{7}{2}m$ and in this region it satisfies, uniformly in $\lambda \geq 0$ small and $k^0$:*

$$|J_1(\lambda, k^0)| \leq \mathcal{O}(1)\, \|\phi\|_{A_{3,\infty}} \|\psi\|_{A_3}.$$

**Lemma II.2.** *Let $\phi, \psi \in A_3'$, $\phi \in A_{3,\infty}'$. Then $J_2(\lambda, k^0)$ defined in (II.14) is holomorphic in $k^0 \in \mathscr{C} = \{k^0: |\operatorname{Re} k^0| \geq 4m,\ 0 < \operatorname{Im} k^0 < \frac{7}{2}m\} \cup \{k^0: 0 < \operatorname{Im} k^0 < 2m\}$. In addition, in this region we have, uniformly in $k^0$ and $\lambda \geq 0$ small:*

$$|J_2(\lambda, k^0)| \leq \mathcal{O}(1)\, \|\phi\|_{A_{3,\infty}} \|\psi\|_{A_3}.$$

*Proof of* II.1. We show that the integral in (II.14) can be analytically continued in $0 < \operatorname{Im} k^0 < \frac{7}{2}m$, satisfying the bound. To see this, note that $[(k/3 + p_1)^2 + a^2]^{-1}$ is holomorphic in $|\operatorname{Im}(k^0/3 + p_1^{(0)})| < 3(m - \varepsilon)$ (i) and $h_1(k^0, p_1) \equiv \int dp_2\, dq_2 \phi_{p_1}(p_2) \psi_{p_1}(-q_2) R(\lambda, 2k/3 - p_1, p_2, q_2)$ is holomorphic in $0 < \operatorname{Im}(2k^0/3 - p_1^{(0)}) < 2m - \varepsilon$. (ii)

We can thus deform continuously the contour of the $p_1^{(0)}$-integration (for instance up to $\operatorname{Im} p_1^{(0)} = \frac{2}{3}m$) in such a way that (i) and (ii) above always hold for any $k^0$ such that $0 < \operatorname{Im} k^0 < \frac{7}{2}m$.

To show the uniform bound, note that we can choose the contour in such a way that, for $0 < \operatorname{Im} k^0 < \frac{7}{2}m$:

$$\left|\operatorname{Im}\left(\frac{k^0}{3} + p_1^{(0)}\right)\right| < 2m \qquad 0 < \operatorname{Im}\left(\frac{2k^0}{3} - p_1^{(0)}\right) < \tfrac{5}{3}m.$$

so that

$$\left[\left(\frac{k}{3} + p_1\right)^2 + a^2\right]^{-1} \leq \mathcal{O}(1)\left[\left(\operatorname{Re}\left(\frac{k^0}{3} + p_1^{(0)}\right)\right)^2 + p_1^{(1)2} + 1\right]^{-1}, \quad \text{for} \quad a > 3m - \varepsilon,$$

$$|h_1(k^0, p_1)| \leq \mathcal{O}(1)\left[\left(\operatorname{Re}\left(\frac{2k^0}{3} - p_1^{(0)}\right)\right)^2 + p_1^{(1)2} + 1\right]^{-1} \|\phi_{p_1}\|_{A_{2,\infty}} \|\psi_{p_1}\|_{A_2}.$$

The last inequality follows from Lemma I.3.

Hence

$$|J_1(\lambda, k^0)| \leq \mathcal{O}(1)\, \|\phi\|_{A_{3,\infty}} \|\psi\|_{A_3} \times \int dp_1 \frac{((\operatorname{Re} p_1^{(0)})^2 + p_1^{(1)2} + 1)^{2/3}}{\left[\left(\operatorname{Re}\left(\frac{k^0}{3} + p_1^{(0)}\right)\right)^2 + p_1^{(1)2} + 1\right]\left[\left(\operatorname{Re}\left(\frac{2k^0}{3} - p_1^{(0)}\right)\right)^2 + p_1^{(1)2} + 1\right]}$$

where we have used that

$$\sup_{|\operatorname{Im} p_1^{(i)}| < \delta_i^{(3)}} |w(p_1)\, \|\psi_{p_1}\|_{a_2}| \leq \mathcal{O}(1)\, \|\psi\|_{a_3}, \quad \text{see Remark I.1}$$

and that $\int d\rho_\lambda(a) < 1$.

Since the integral above is bounded uniformly in $\operatorname{Re} k^0$, we get the result. □

*Proof of* II.2. The idea is the same as in I.1. We can deform the contour of the $p_1^{(0)}$-integration in order to avoid the singularities of the integrand. This is possible because, if $|\operatorname{Re} k^0| \geq 4m$ or if $0 < \operatorname{Im} k^0 < 2m$, the singularities never pinch together.

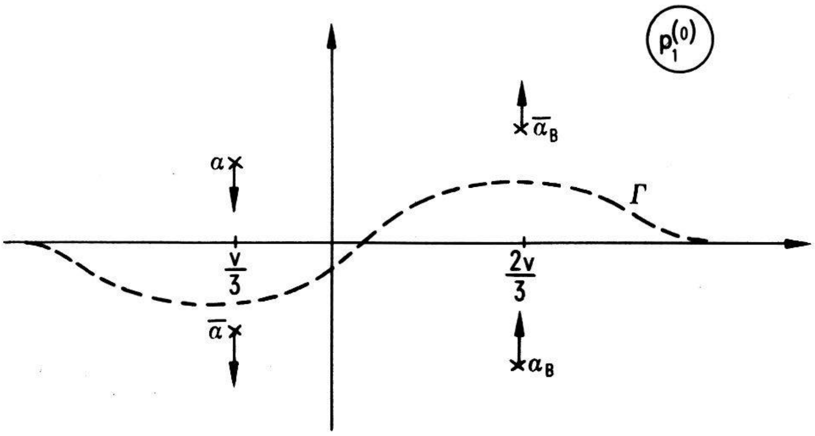



Figure 3
$|v| = |\mathrm{Re}\, k^0|$

This situation is pictured in the above figure. $\alpha$ and $\bar{\alpha}$ are the singularities of $[(k/3+p_1)^2+m^2]$, which move as indicated when we go from $\mathrm{Im}\, k^0 = 0$ to $\mathrm{Im}\, k^0 = \frac{7}{2}m$. $\alpha_B$ and $\bar{\alpha}_B$ are the singularities of $R(\lambda, k^0)$, which also move as indicated. $\Gamma$ is the deformed contour. Note that this is possible since $|v| = (\mathrm{Re}\, k^0)$ is bounded away from zero. $\square$

*Remark* II.2. We have obtained a region of holomorphy going up to $\frac{7}{2}m$ but we remark that we could obtain $4m-\varepsilon$ by refining our proof. Since $\frac{7}{2}m$ is sufficient for our purposes, we do not pursue this point further.

We are left with analysing $J_2(\lambda, k^0)$ in the region around the interval $(2m, \frac{7}{2}m)$. Next we break again $J_2$ in two pieces:

$$J_2(\lambda, k^0) = J_{21}(\lambda, k^0) + J_{22}(\lambda, k^0),$$

$$J_{21}(\lambda, k^0) = 27Z(\lambda)^2 \int_{|p_1^{(1)}| \geq 4m} dp_1 \frac{1}{\left(\frac{k^0}{3}+p_1\right)^2+m^2}$$
$$\times \int dp_2\, dq_2 \phi_{p_1}(p_2) \psi_{p_1}(-q_2) R\left(\lambda, \frac{2k}{3}-p_1, p_2, q_2\right) \tag{II.15}$$
$$J_{22}(\lambda, k^0) = 27Z(\lambda)^2 \int_{|p_1^{(1)}| < 4m} dp_1 \frac{1}{\left(\frac{k}{3}+p_1\right)^2+m^2}$$
$$\times \int dp_2\, dq_2 \phi_{p_1}(p_2) \psi_{p_1}(-q_2) R\left(\lambda, \frac{2k}{3}-p_1, p_2, q_2\right).$$

We have the following result:

**Lemma II.3.** *Let* $\phi, \psi \in A_3'$, $\phi \in A_{3,\infty}'$. *Then* $J_{21}(\lambda, k^0)$ *defined in* (II.15) *is holomorphic in* $0 < \mathrm{Im}\, k^0 < \frac{7}{2}m$ *and it satisfies in this region the uniform* $\lambda, k^0$ *bound:*

$$|J_{21}(\lambda, k^0)| \leq \mathcal{O}(1)\, \|\phi\|_{A_{3,\infty}} \|\psi\|_{A_3}.$$

*Proof.* This can be proven in exactly the same way as Lemma II.1 (note that both $[(k^0/3+p_1^{(0)})^2+p_1^{(1)2}+m^2]^{-1}$ and $R(\lambda, (2k^0/3-p_1^{(0)}, p_1^{(1)}), p_2, q_2)$ are holomorphic for $0<\operatorname{Im} k^0<\frac{7}{2}m$ and $|p_1^{(1)}|\geq 4m$). □

We further split $J_{22}(\lambda, k^0)$, this time according to:

$$R(\lambda, k', p, q)=\rho(\lambda, k', p, q)+\rho_2(\lambda, k', p, q)+\rho_3(\lambda, k', p, q)$$

(see Lemma I.1 and (I.17), (I.18)). Define

$$J_{22}(\lambda, k^0)=J_{221}(\lambda, k^0)+J_{222}(\lambda, k^0),$$

$$J_{221}(\lambda, k^0)=27Z(\lambda)^2\int_{|p_1^{(1)}|\leq 4m} dp_1 \frac{1}{\left(\frac{k}{3}+p_1\right)^2+m^2}$$

$$\times \int dp_2\, dq_2 \phi_{p_1}(-q_2)\psi_{p_1}(p_2)(\rho_2+\rho_3)\left(\lambda, \frac{2k}{3}-p_1, p_2, q_2\right) \tag{II.16}$$

$$J_{222}(\lambda, k^0)=27Z(\lambda)^2\int_{|p_1^{(1)}|\leq 4m} dp_1 \frac{1}{\left(\frac{k}{3}+p_1\right)^2+m^2}$$

$$\times \int dp_2\, dq_2 \phi_{p_1}(p_2)\psi_{p_1}(-q_2)\rho\left(\lambda, \frac{2k}{3}-p_1, p_2, q_2\right).$$

Let

$$\mathscr{C}=\{k^0: 0<\operatorname{Im} k^0<\tfrac{7}{2}m, \operatorname{Im} k^0\notin[3m, \tfrac{7}{2}m), |k^0|>2m\} \text{ }^{6)} \tag{II.17}$$

The restriction $\operatorname{Im} k^0\notin[3m, \frac{7}{2}m)$ is due to the fact that $J_{221}(\lambda, k^0)$ has a branch point at $k^0=3im$. Nevertheless, it remains bounded as $k^0\to 3im$. This is the result of our next lemma.

**Lemma II.4.** *Let $\phi, \psi\in A_3'$, $\phi\in A_{3,\infty}'$. Then $J_{221}(\lambda, k^0)$ defined in* (II.16) *is holomorphic for $k^0$ in $\mathscr{C}$. In this region it also satisfies, uniformly in $k^0$ and $\lambda\geq 0$ small:*

$$|J_{221}(\lambda, k^0)|\leq \mathcal{O}(1)\, \|\phi\|_{A_{3,\infty}}\|\psi\|_{A_3}.$$

*Proof.* By Lemma II.2, we only consider the region $|\operatorname{Re} k^0|\leq 4m$. In this case, we decompose the $p_1^{(0)}$-integral in two parts:

$$J_{221}=I_1+I_2, \quad \text{with} \quad I_1=\int_{|p_1^{(0)}|<6m} dp_1\cdots, I_2=\int_{|p_1^{(0)}|>6m} dp_1\cdots.$$

It is clear that, in $I_2$, both $[(k/3+p_1)^2+m^2]^{-1}$ and $(\rho_2+\rho_3)(\lambda, 2k/3-p_1, p_2, q_2)$ are holomorphic for $|\operatorname{Im} k^0|<\frac{7}{2}m$. Using Lemma II.3, i), we have

$$\left|\int dp_2\, dq_2 \phi_{p_1}(p_2)\psi_{p_1}(-q_2)(\rho_2+\rho_3)\left(\lambda, \frac{2k}{3}-p_1, p_2, q_2\right)\right|$$

$$\leq \mathcal{O}(1)\left[\left(\operatorname{Re}\left(\frac{2k^0}{3}-p_1^{(0)}\right)\right)^2+p_1^{(1)2}+1\right]^{-1}\|\phi_{p_1}\|_{A_{2,\infty}}\|\psi_{p_1}\|_{A_2}.$$

---

6) The restriction $|k^0|>2m$ will be needed later, when defining the rank-one operator $\varepsilon_1(\lambda, k^0)$, see (II.20) and the comments following it.

then

$$|I_2| \le \mathcal{O}(1)\,\|\phi\|_{A_{3,\infty}}\,\|\psi\|_{A_3}$$
$$\times \int_{\substack{|p_1^{(1)}|\le 4m\\ |p_1^{(0)}|>6m}} dp_1 \frac{(p_1^{(0)2}+p_1^{(1)2}+1)^{2/3}}{\left[\left(\mathrm{Re}\,\frac{k^0}{3}+p_1^{(0)}\right)^2+p_1^{(1)2}+1\right]\left[\left(\mathrm{Re}\,\frac{2k^0}{3}-p_1^{(0)}\right)^2+p_1^{(1)2}+1\right]}$$

where we have used that $\sup_{|\mathrm{Im}\,p_1^{(i)}|<\delta^{(3)}} |w(p_1)\,\|\psi_{p_1}\|_{A_2}| \le O(1)\,\|\psi\|_{A_3}$. Since the integral is finite, the uniform bound follows. Consider now $I_1$. Keeping the $p_1^{(0)}$-integration real, we see that the integrand is an analytic function of $k^0 \in \mathscr{C}$. It suffices then to show the bound. But this follows from

$$\langle \phi_{p_1}, (\rho_2+\rho_3)\psi_{p_1}\rangle_{A_2} \le \mathcal{O}(1)\,\|\phi_{p_1}\|_{A_{2,\infty}}\,\|\psi_{p_1}\|_{A_2},$$
$$\sup_{|\mathrm{Im}\,p_1^{(i)}|<\delta_i^{(3)}} |w(p_1)\,\|\psi_{p_1}\|_{A_2}| \le \mathcal{O}(1)\,\|\psi\|_{A_3}$$

and

$$\int_{\substack{|p_1^{(1)}|\le 4m\\ |p_1^{(0)}|\le 6m}} dp_1 \left[\left(\frac{k}{3}+p_1\right)^2+m^2\right]^{-1} |w(p_1)|^{-1} < \infty \quad \square$$

It remains to analyse $J_{222}(\lambda, k^0)$ given in (II.16). We remind the reader that (see (I.15), (I.17) and Remark I.3):

$$\hat{\rho}(\lambda, \zeta, p, q) = \frac{\hat{r}(\lambda, \zeta_1(\lambda))}{\zeta-\zeta_1(\lambda)} \hat{H}(\lambda, \zeta_1(\lambda), p)\hat{H}(\lambda, \zeta_1(\lambda), q)$$

where $\zeta_1(\lambda)$ is the pole of $\hat{R}(\lambda, \zeta)$. Let

$$m_B(\lambda) = \sqrt{4m(\lambda)^2 - \zeta_1(\lambda)^2}. \tag{II.18}$$

We will see that $J_{222}(\lambda, k^0)$ is not bounded as $k^0 \to i(m+m_B)$ but one can show that *its singular part is contained in a rank-one operator* which we now turn to define. For $\hat{H}(\lambda, \zeta_1(\lambda, \cdot)$ defined in (I.15), let $\varepsilon_1(\lambda, k^0) \in A_3'^*$ be defined by:

$$\langle \phi, \varepsilon_1(\lambda, k^0)\rangle = \widehat{\phi\varepsilon_1}(\mu(k^0)),$$
$$\widehat{\phi\varepsilon_1}(p_1) = \int dp_2\, \phi\left(p_1, \frac{p_1}{2}+p_2\right)\hat{H}(\lambda, \zeta_1(\lambda), p_2) \tag{II.19}$$

for

$$\phi(p_1, p_2) \in A_3'(\text{or } A_3) \quad \text{and} \quad \mu(k^0) = (\mu_0(k^0), 0) \in \mathbf{C}^2,$$

with

$$\mu_0(k^0) = \frac{1}{2k^0}\left(m_B^2 - m^2 + \frac{k^{02}}{3}\right)\chi_{\ge 2m}(|k^0|) \tag{II.20}$$

and $\chi_{\ge 2m}(|k^0|)$ denotes the characteristic function of $\{k^0 : |k^0| \ge 2m\}$.

It follows that $\mu_0(k^0)$ is holomorphic for $|k^0| > 2m$, and that $|\mathrm{Im}\,\mu_0(k^0)| < \frac{3}{4}(m-\varepsilon)$ for $0 < \mathrm{Im}\,k^0 < \frac{7}{2}m$ (with $k^0 = u+iv$, we have

$$|\mathrm{Im}\,\mu_0(k^0)| = \left|\frac{v}{2(u^2+v^2)}\left(m_B^2 - m^2 - \frac{u^2+v^2}{3}\right)\chi_{\ge 2m}(|k^0|)\right| \le \tfrac{3}{4}(m-\varepsilon)$$

for $|v| < \frac{7}{2}m$ and $\sqrt{u^2+v^2} \ge 2m$).

The function $\chi_{\geq 2m}$ spoils the holomorphy on a disc of radius $2m$ about the origin in the $k^0$-plane. But since the region of interest to us is a neighbourhood of $k^0 = 3im$, we shall not mind about this point. Note also that

$$\begin{aligned}
|\langle \phi, \varepsilon_1(\lambda, k^0)\rangle| &= \left| \int dp_2 \phi\left(\mu(k^0), \frac{\mu(k^0)}{2} + p_2\right)(1+\hat{T}_2^*(\lambda, \zeta_1))^{-1}(0, p_2) \right| \\
&= |((1+\hat{T}_2^*(\lambda, \zeta_1))^{-1}\phi_{\mu(k^0)})(0)| \\
&\leq \mathcal{O}(1)\, \|(1+\hat{T}_2^*(\lambda, \zeta_1))^{-1}\phi_{\mu(k^0)}\|_{A_2} \\
&\leq \mathcal{O}(1)\, \|\phi_{\mu(k^0)}\|_{A_2} \\
&\leq \mathcal{O}(1)\, \|\phi\|_{A_3}
\end{aligned}$$

so that $\varepsilon_1(\lambda, k^0) \in A_3'^*$ for $|\mathrm{Im}\, k^0| < \frac{7}{2}m$, $|\lambda|$ small, as asserted above.

We are now able to define the rank-one operator. Let $\gamma_{11}(\lambda, k^0) \in \mathcal{L}(A_3', A_3'^*)$ be defined by:

$$\gamma_{11}(\lambda, k^0)\psi = 27Z(\lambda)^2 \cdot 2\zeta_1(\lambda) \cdot \hat{r}(\lambda, \zeta_1(\lambda))t(\lambda, k^0)\varepsilon_1(\lambda, k^0)\widehat{\varepsilon_1\psi}(\mu(k^0)) \tag{II.21}$$

for $\psi \in A_3'$ (or $A_3$) and

$$\begin{aligned}
\widehat{\varepsilon_1\psi}(p_1) &= \int dq_2 \psi\left(p_1, \frac{p_1}{2} - q_2\right)\hat{H}(\lambda, \zeta_1(\lambda), q_2), \\
t(\lambda, k^0) &= \int_{|p_1^{(1)}| \leq 4m} dp_1 \left[\left(\frac{k}{3} + p_1\right)^2 + m^2\right]^{-1} \left[\left(\frac{2k}{3} - p_1\right)^2 + m_B^2\right]^{-1}.
\end{aligned} \tag{II.22}$$

We analyse $t(\lambda, k^0)$ in Lemma II.7 below. By now we consider, for $\phi, \psi \in A_3'$, $\phi \in A_{3,\infty}'$:

$$J^*(\lambda, k^0) = J_{222}(\lambda, k^0) - \langle \phi, \gamma_{11}(\lambda, k^0)\psi\rangle. \tag{II.23}$$

We are going to show that $J^*(\lambda, k^0)$ is bounded as $k \to i(m + m_B)$ (and can even be continued across the cut $[i(m + m_B), \infty)$, see below). This result justifies our previous claim that the singular part of $J_{222}(\lambda, k^0)$ (and so that of $b_{11}(\lambda, k^0)$) is contained in a rank-one operator, namely $\gamma_{11}(\lambda, k^0)$ (and Lemma II.7 will show that $\gamma_{11}$ is indeed singular at $k^0 = i(m + m_B)$). On the other hand, it is possible to show that $J^*(\lambda, k^0)$ has a square root branch point at $k^0 = i(m + m_B)$ but, as remarked above, it is bounded on a neighbourhood of it. This suggests that changing to $\omega' = \sqrt{k^{02} + (m + m_B)^2}$ would remove this singularity and this is indeed the case. Nevertheless there remains the branch point at $k^0 = 3im$ or, in the $\omega'$-variable, at $\omega' = \pm i\eta(\lambda)$, with $\eta(\lambda) = \sqrt{9m^2 - (m + m_B)^2}$. Since $\eta(\lambda) \to 0$ as $\lambda \to 0$, we must be careful. The way we have chosen to deal with this problem is the following. We define a new variable

$$\omega = \frac{1}{\eta(\lambda)}\sqrt{k^{02} + (m + m_B)^2},$$

with

$$\eta(\lambda) = \sqrt{9m^2 - (m + m_B)^2} \tag{II.24}$$

(We choose the square root with positive real part). This is defined for any $\lambda > 0$ and since the transformation is also a scaling, the branch point at $k^0 = 3im$

remains fixed at $\omega = \pm i$. Of course, we loose differentiability at $\lambda = 0$. Let also

$$\mathscr{B} = \{k^0 : 0 < \operatorname{Im} k^0 < \tfrac{7}{2}m, \operatorname{Im} k^0 \notin [(m+m_B), \tfrac{7}{2}m), |k^0| > 2m\}$$
$$\hat{\mathscr{B}} = \left\{\omega : \omega = \frac{1}{\eta(\lambda)}(k^{02} + (m+m_B)^2)^{1/2}, k^0 \in \mathscr{B}\right\}. \tag{II.25}$$

The transformation $k^0 \to \omega$ is a conformal map of $\mathscr{B}$ onto $\hat{\mathscr{B}}$. Let also $\hat{\mathscr{B}}' = \hat{\mathscr{B}} \cup -\hat{\mathscr{B}} \cup$ (connecting line).

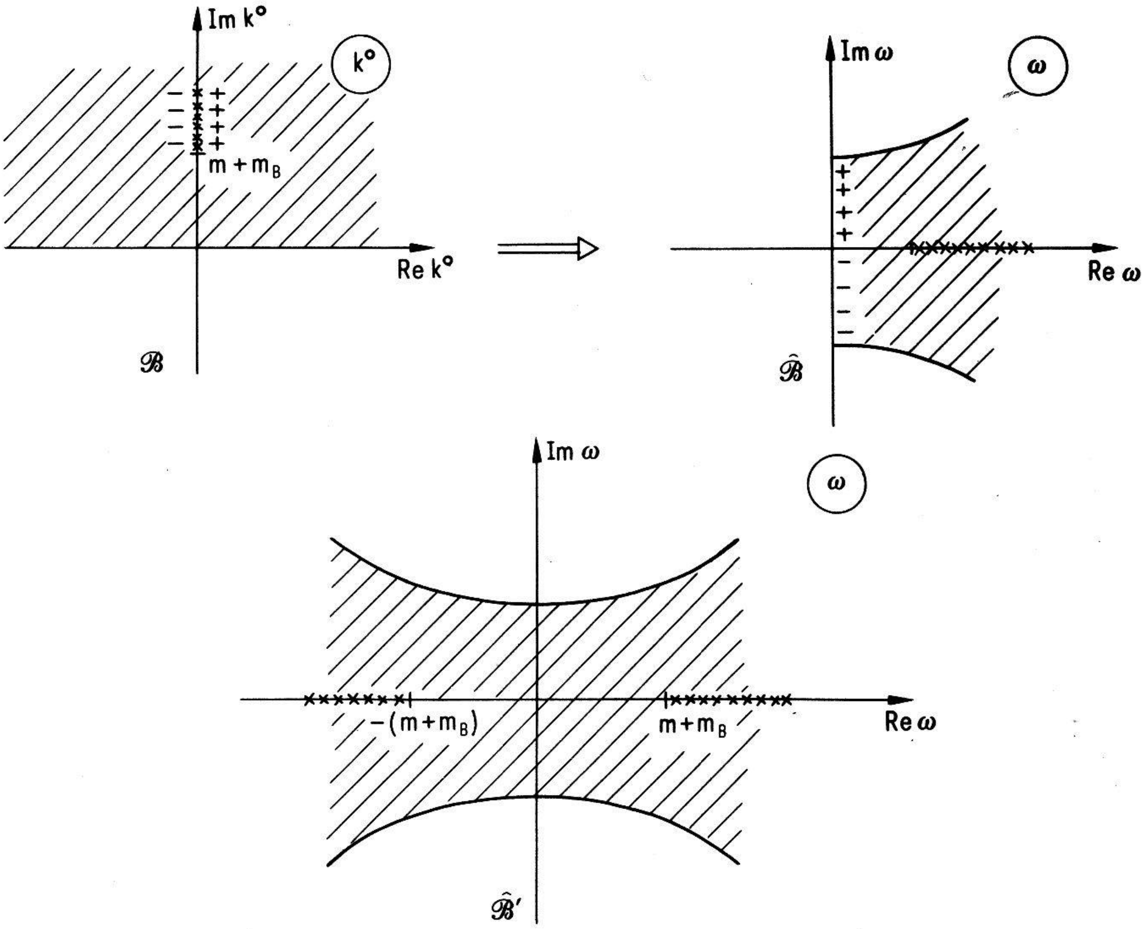



Figure 4

Note that if a function $f(k^0)$ can be analytically continued across the imaginary axis at $m + m_B < \operatorname{Im} k^0 < \frac{7}{2}m$ then $\hat{f}(\omega) \equiv f(k^0)$ for $\omega = [1/\eta(\lambda)](k^{02} + (m+m_B)^2)^{1/2}$ is holomorphic in $\hat{\mathscr{B}}'$ (this is the case of $J_1(\lambda, k^0)$ and $J_{21}(\lambda, k^0)$). On the other hand, if a function $f(k^0)$ is holomorphic in $\mathscr{C}$ then $\hat{f}(\omega) \equiv f(k^0)$ is holomorphic in $\hat{\mathscr{B}}'_C$, where

$$\hat{\mathscr{B}}'_C = \hat{\mathscr{B}}' \setminus \{\omega : \omega = i\alpha, \alpha \text{ real}, |\alpha| > 1\}. \tag{II.26}$$

This is the case for $J_{221}(\lambda, k^0)$.

We have one last definition before stating our result on $J^*(\lambda, k^0)$. Let

$$\hat{\mathscr{B}}(\beta) = \{\omega \in \hat{\mathscr{B}}' : |\operatorname{Re} \omega| + |\operatorname{Im} \omega| < \beta \quad \text{if} \quad \operatorname{Re} \omega < 0\}$$
$$\hat{\mathscr{B}}_C(\beta) = \{\omega \in \hat{\mathscr{B}}'_C : |\operatorname{Re} \omega| + |\operatorname{Im} \omega| < \beta \quad \text{if} \quad \operatorname{Re} \omega < 0\}. \tag{II.27}$$

We write $J^*(\lambda, k^0)$ in the following way:

$$J^*(\lambda, k^0) = J_{222}(\lambda, k^0) - \langle \phi, \gamma_{11}(\lambda, k^0)\psi \rangle = I_1(\lambda, k^0) + I_2(\lambda, k^0) \tag{II.28}$$

$$\begin{aligned} I_1(\lambda, k^0) &= 27Z(\lambda)^2 \cdot 2\zeta_1(\lambda) \cdot \hat{r}(\lambda; \zeta_1(\lambda)) \\ &\times \int_{|p_1^{(1)}| \le 4m} dp_1 \left[\left(\frac{k}{3} + p_1\right)^2 + m^2\right]^{-1} \\ &\times \left[\left(\frac{2k}{3} - p_1\right)^2 + m_B^2\right]^{-1} (h(p_1) - h(\mu(k^0))) \end{aligned} \tag{II.29}$$

$$\begin{aligned} h(p_1) &= \int dp_2\, dq_2 \phi_{p_1}(p_2)\psi_{p_1}(-q_2)\hat{H}(\lambda, \zeta_1(\lambda), p_2)\hat{H}(\lambda, \zeta_1(\lambda), q_2) \\ &= \widehat{\phi\varepsilon_1}(p_1)\widehat{\varepsilon_1\psi}(p_1) \end{aligned} \tag{II.30}$$

$$\begin{aligned} I_2(\lambda, k^0) &= 27Z(\lambda)^2 \hat{r}(\lambda, \zeta_1(\lambda)) \\ &\times \int_{|p_1^{(1)}| \le 4m} dp_1 \left[\left(\frac{k}{3} + p_1\right)^2 + m^2\right]^{-1} \\ &\times \left[\left(\left(\frac{2k}{3} - p_1\right)^2 + 4m^2\right)^{1/2} + \zeta_1(\lambda)\right]^{-1} h(p_1) \end{aligned} \tag{II.31}$$

**Lemma II.5.** *Let $\phi, \psi \in A_3'$, $\phi \in A_{3,\infty}'$. Then $\hat{I}_1(\lambda, \omega)$ defined in (II.29) is holomorphic in $\hat{\mathscr{B}}([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$ and in this domain it satisfies, uniformly in $\omega$ and $\lambda > 0$ small:*

$$|\hat{I}_1(\lambda, \omega)| \le \mathcal{O}(1)\, \|\phi\|_{A_{3,\infty}} \|\psi\|_{A_3}.$$

**Lemma II.6.** *Let $\phi, \psi \in A_3'$, $\psi \in A_{3,\infty}'$. Then $\hat{I}_2(\lambda, \omega)$ defined in (II.31) is holomorphic in $\hat{\mathscr{B}}_C'$ and in this domain it satisfies, uniformly in $\omega$ and $\lambda > 0$ small:*

$$|\hat{I}_2(\lambda, \omega)| \le \mathcal{O}(1)\, \|\phi\|_{A_{3,\infty}} \|\psi\|_{A_3}.$$

*Proof of* II.5: Let $c(\lambda) = 27Z(\lambda)^2 \cdot 2\zeta_1(\lambda) \cdot \hat{r}(\lambda, \zeta_1(\lambda))$.

$$I_1(\lambda, k^0) = I_{11}(\lambda, k^0) + I_{12}(\lambda, k^0),$$

$$\begin{aligned} I_{11}(\lambda, k^0) &= c(\lambda) \int_{|p_1^{(1)}| \le 4m} dp_1 \left[\left(\frac{k}{3} + p_1\right)^2 + m^2\right]^{-1} \\ &\times \left[\left(\frac{2k}{3} - p_1\right)^2 + m_B^2\right]^{-1} (h(p_1^{(0)}, p_1^{(1)}) - h(\mu_0(k^0), p_1^{(1)})), \end{aligned}$$

$$\begin{aligned} I_{12}(\lambda, k^0) &= c(\lambda) \int_{|p_1^{(1)}| \le 4m} dp_1 \left[\left(\frac{k}{3} + p_1\right)^2 + m^2\right]^{-1} \\ &\times \left[\left(\frac{2k}{3} - p_1\right)^2 + m_B^2\right]^{-1} (h(\mu_0(k^0), p_1^{(1)}) - h(\mu_0(k^0), 0)). \end{aligned}$$

In the first term we put:

$$\frac{1}{\left(\frac{k}{3}+p_1\right)^2+m^2}\cdot\frac{1}{\left(\frac{2k}{3}-p_1\right)^2+m_B^2}$$
$$=\frac{1}{m_B^2-m^2+\frac{k^{02}}{3}-2k^0p_1^{(0)}}\left[\frac{1}{\left(\frac{k}{3}+p_1\right)^2+m^2}-\frac{1}{\left(\frac{2k}{3}-p_1\right)^2+m_B^2}\right]$$

so that we can write

$$I_{11}(\lambda, k^0)=c(\lambda)\int_{|p_1^{(1)}|\le 4m} dp_1\,\frac{h(p_1^{(0)}, p_1^{(1)})-h(\mu_0(k^0), p_1^{(1)})}{2k^0(\mu_0(k^0)-p_1^{(0)})}$$
$$\times\left[\frac{1}{\left(\frac{k}{3}+p_1\right)^2+m^2}-\frac{1}{\left(\frac{2k}{3}-p_1\right)^2+m_B^2}\right]$$

and each of the two resulting terms can be analytically continued to $0<\operatorname{Im} k^0<\frac{7}{2}m$ by suitable shifting the contour of the $p_1^{(0)}$-integration. Note that $\hat{\mu}_0(\omega)$ is holomorphic in $\mathscr{B}'$ and this implies that $\hat{I}_{11}(\lambda, \omega)$ is holomorphic in this region.

The uniform bound follows by noting that

$$\sup_{|\operatorname{Im} p_1^{(i)}|<\delta_i^{(3)}} |w(p_1)(h(p_1^{(0)}, p_1^{(1)})-h(\mu_0(k^0), p_1^{(1)})|\le \mathcal{O}(1)\,\|\phi\|_{A_{3,\infty}}\,\|\psi\|_{A_3}$$

and that the integrals

$$\int_{|p_1^{(1)}|\le 4m} dp_1\,\frac{(p_1^{(0)2}+p_1^{(1)2}+1)^{2/3}}{[\mu_0(k^0)-p_1^{(0)}]\left[\left(p_1^{(0)}+\operatorname{Re}\frac{k^0}{3}\right)^2+p_1^{(1)2}+1\right]}$$

and

$$\int_{|p^{(1)}|\le 4m} dp_1\,\frac{(p_1^{(0)2}+p_1^{(1)2}+1)^{2/3}}{[\mu_0(k^0)-p_1^{(0)}][(p_1^{(0)}-\frac{2}{3}\operatorname{Re} k^0)^2+p_1^{(1)2}+1]}$$

are uniformly bounded, since we can suppose $|p_1^{(0)}-\mu_0(k^0)|>1$ and $|\operatorname{Re} k^0|<4m$ (see Lemma II.2).

Consider now the term $I_{12}(\lambda, k^0)$. We can do the $p_1^{(0)}$-integration by residues, obtaining, with $g(a)=\sqrt{p_1^{(1)2}+a^2}$:

$$I_{12}(\lambda, k^0)=c(\lambda)\pi\int_{|p_1^{(1)}|\le 4m} dp_1^{(1)}(h(\mu_0(k^0), p_1^{(1)})-h(\mu_0(k^0), 0))$$
$$\times\left[\frac{1}{g(m)}\cdot\frac{1}{[-k^0+i(g(m_B)+g(m))][-k^0-i(g(m_B)-g(m))]}\right.$$
$$\left.+\frac{1}{g(m_B)}\cdot\frac{1}{[k^0+i(g(m_B)+g(m))][k^0+i(g(m_B)-g(m))]}\right.$$
$$=\pi\cdot c(\lambda)\int_{|p_1^{(1)}|\le 4m} dp_1^{(1)}(h(\mu_0(k^0), p_1^{(1)})-h(\mu_0(k^0), 0))$$
$$+\left[\frac{1}{g(m)}\cdot\frac{k^{02}+m_B^2-m^2+2ik^0g(m)}{(k^{02}+m_B^2-m^2)^2+4k^{02}(p_1^{(1)2}+m^2)}\right.$$
$$\left.+\frac{1}{g(m_B)}\cdot\frac{k^{02}+m^2-m_B^2+2ik^0g(m_B)}{(k^{02}+m^2-m_B^2)^2+4k^{02}(p_1^{(1)2}+m_B^2)}\right].$$

Note that

$$(k^{02}+m_B^2-m^2)^2+4k^{02}m_B^2=(k^{02}+(m+m_B)^2)(k^{02}+(m_B-m)^2)$$
$$=(k^{02}+m^2-m_B^2)^2+4k^{02}m^2$$

so that the imaginary terms cancel and we are left with

$$I_{12}(\lambda, k^0)=\pi\cdot c(\lambda)\int_{|p_1^{(1)}|\le 4m} dp_1^{(1)}(h(\mu_0(k^0), p_1^{(1)})-h(\mu_0(k^0),0))$$
$$\times\left[\frac{1}{g(m)}\frac{k^{02}+m_B^2-m^2}{(k^{02}+(m_B+m)^2)(k^{02}+(m_B-m)^2)+4k^{02}p_1^{(1)2}}\right.$$
$$\left.+\frac{1}{g(m_B)}\cdot\frac{k^{02}+m^2-m_B^2}{(k^{02}+(m_B+m)^2)(k^{02}+(m_B-m)^2)+4k^{02}(p_1^{(1)2}}\right]$$

We introduce the $\omega$-variable:

$$k^{02}+(m_B+m)^2=\eta(\lambda)^2\omega^2,$$
$$\frac{k^{02}+(m_B-m)^2}{4k^{02}}=\frac{4m'm_B'-\omega^2}{4(m'+m_B')^2-\omega^2}\equiv s(\omega),\qquad m'=\frac{m}{\eta(\lambda)},\qquad m_B'=\frac{m_B}{\eta(\lambda)},$$
$$\frac{k^{02}+m_B^2-m^2}{4k^{02}}\equiv s_1(\omega),$$
$$\frac{k^{02}+m^2-m_B^2}{4k^{02}}\equiv s_2(\omega).$$

We have, thus:

$$\hat{I}_{12}(\lambda,\omega)=\pi\cdot c(\lambda)\int_{|p_1^{(1)}|\le 4m} dp_1^{(1)}(h(\hat{\mu}_0(\omega), p_1^{(1)})-h(\hat{\mu}_0(\omega),0))$$
$$\times\left[\frac{s_1(\omega)}{g(m)[\eta(\lambda)^2\omega^2 s(\omega)+p_1^{(1)2}]}+\frac{s_2(\omega)}{g(m_B)[\eta(\lambda)^2\omega^2 s(\omega)+p_1^{(1)2}]}\right].$$

Both terms are analysed in exactly the same way. Note that $s_1(\omega)$ and $s_2(\omega)$ are holomorphic and bounded on a large disc centered at $\omega=0$ (in fact they are holomorphic on a disc of radius $(m+m_B)\eta(\lambda)^{-1}$). Write the denominators in the form:

$$\frac{1}{\eta(\lambda)^2\omega^2 s(\omega)p_1^{(1)2}}=\frac{1}{2p_1^{(1)}}\left[\frac{1}{p_1^{(1)}+i\eta(\lambda)\omega s^{1/2}(\omega)}+\frac{1}{p_1^{(1)}-i\eta(\lambda)\omega s^{1/2}(\omega)}\right].$$

Choose the determination of $s^{1/2}(\omega)=u+iv$ for which $u>0$. We claim that (see the proof in Appendix B), with $\omega=x+iy$:

i) if $x>0$, $vy<0$.
ii) if $x<0$, $vy>0$.
iii) $s_0<|s^{1/2}(\omega)|<1$, some $s_0>0$.

We analyse the term $(p_1^{(1)}+i\eta(\lambda)\omega s^{1/2}(\omega))$:

$$(p_1^{(1)}+i\eta(\lambda)\omega s^{1/2}(\omega))^{-1}=(p_1^{(1)}+\eta(\lambda)i(xu-vy)-\eta(\lambda)(xv+yu))^{-1}$$
$$=(p_1^{(1)}+il_1+l_2)^{-1}.$$

If $x>0$, the denominator never vanishes since $xu-yv>0$. If $x=0$, then $v=0$ and $l_1=0$, and so there is a real zero $p_1^{(1)}=\eta(\lambda)yu$ on the integration path. So the integral defining $\hat{I}_{12}(\lambda, \omega)$ is a holomorphic function of $\omega$ for $x = \text{Re}\,\omega > 0$. But we can take advantage of the holomorphy of $h(\hat{\mu}_0(\omega), p_1^{(1)})$ to deform the contour of the $p_1^{(1)}$-integration and analytically continue $\hat{I}_{12}(\lambda, \omega)$ to the left half-plane. This is done according to the picture below.

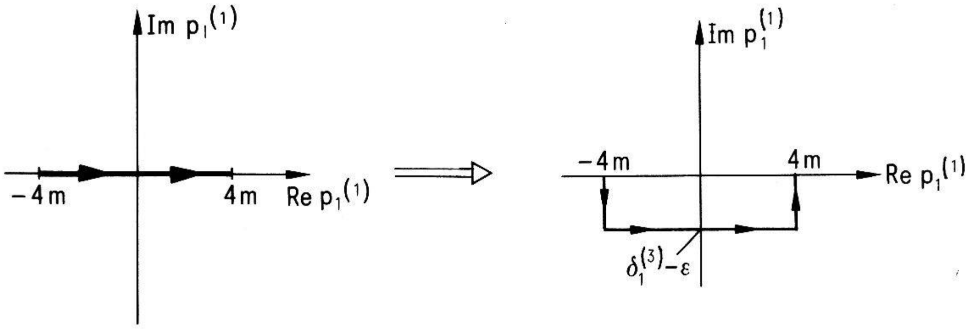



Figure 5

We can now go with $\omega$ to the left half-plane as long as $|l_1| < \delta_1^{(3)} - 2\varepsilon$. A very crude bound is $|l_1| \leq (|x| + |y|)\eta(\lambda)$ and so we can analytically continue $\hat{I}_{12}(\lambda, \omega)$ to the left half-plane in the domain

$$\left\{\omega : \text{Re}\,\omega < 0, |\text{Re}\,\omega| + |\text{Im}\,\omega| < \frac{\delta_1^{(3)} - 2\varepsilon}{\eta(\lambda)}\right\}$$

(in the case of $(p_1^{(1)} - i\eta(\lambda)\omega s^{1/2}(\omega))$ the contour is deformed in the opposite sense). This defines an analytic continuation of $\hat{I}_{12}(\lambda, \omega)$ to $\mathcal{B}([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$. The uniform bound follows as in the case of $\hat{I}_{11}(\lambda, \omega)$. This completes the proof.

□

*Proof of Lemma* II.6. We have to show that $I_2(\lambda, k^0)$ is holomorphic and satisfies the bound for $k^0 \in \mathscr{C}$. This implies holomorphy in $\mathcal{B}'_C$. As in the proof of Lemma II.4, we consider separately the regions $|p_1^{(0)}| \lesssim 6m$. In the region $|p_1^{(0)}| > 6m$, both functions in the integrand are holomorphic in $\mathscr{C}$. The bound is also easy to show. In the region $|p_1^{(0)}| < 6m$, we note that the integrand is analytic in $k^0 \in \mathscr{C}$ and that

$$\int_{\substack{|p_1^{(1)}| \leq 4m \\ |p_1^{(0)}| \leq 6m}} dp_1 \left[\left(\frac{k}{3} + p_1\right)^2 + m^2\right]^{-1} \left[\left(\frac{2k}{3} - p_1\right)^2 + 4m^2\right]^{-1/2} < \infty$$

uniformly in $k^0 \in \mathscr{C}$. Since we always choose the determination of the square root which has positive real part, this also proves our result for $((2k/3 - p_1)^2 + 4m^2)^{1/2} + \zeta_1(\lambda)$ once we note that $\zeta_1(\lambda) > 0$. □

Our next task is the analysis of the singular part, $\gamma_{11}(\lambda, k^0)$. From its definition (see (II.21)) we can see that its analytical properties are those of $t(\lambda, k^0)$ defined in (II.22). We state our result on $t(\lambda, k^0)$.

**Lemma II.7.** *Let* $\hat{t}(\lambda, \omega)$ *be defined by* (II.22) *and set*

$$\hat{t}(\lambda, \omega) = \frac{1}{\eta(\lambda)\omega}\,\hat{t}_0(\lambda, \omega).$$

*Then $\hat{t}_0(\lambda, \omega)$ is holomorphic in $\mathcal{B}([\delta_1^{(3)}-2\varepsilon]/\eta(\lambda))$ and in this domain it is uniformly bounded in $\omega$ and $\lambda>0$ small. We have also:*

$$\hat{t}_0(\lambda, 0)=\pi^2\cdot\frac{1}{\sqrt{m(\lambda)m_B(\lambda)}}$$

*Proof.* We use the same notation as in Lemma II.5. Performing the $p_1^{(0)}$-integration by residues we get, as before:

$$(*)\quad t(\lambda, k^0)=\pi\int_{|p_1^{(1)}|\le 4m} dp_1^{(1)} \times\left[\frac{k^{02}+m_B^2-m^2}{g(m)[(k^{02}+(m_B+m)^2)(k^{02}+(m_B-m)^2)+4k^{02}p_1^{(1)2}}+m\leftrightarrow m_B\right].$$

Again we introduce the $\omega$-variable, obtaining:

$$\hat{t}(\lambda, \omega)=\pi\int_{|p_1^{(1)}|\le 4m} dp_1^{(1)}\left[\frac{s_1(\omega)}{g(m)[\eta(\lambda)^2\omega^2 s(\omega)+p_1^{(1)2}]}+\frac{s_1(\omega)}{g(m_B)[\eta(\lambda)^2\omega^2 s(\omega)+p_1^{(1)2}]}\right].$$

Now write the denominators as:

$$\frac{1}{\eta(\lambda)^2\omega^2 s(\omega)+p_1^{(1)2}}=\frac{1}{2i\eta(\lambda)\omega s^{1/2}(\omega)}\times\left[\frac{1}{p_1^{(1)}-i\eta(\lambda)\omega s^{1/2}(\omega)}-\frac{1}{p_1^{(1)}+i\eta(\lambda)\omega s^{1/2}(\omega)}\right].$$

From this representation, it is clear that we can repeat the proof of Lemma II.5 to show the holomorphy of $\hat{t}_0(\lambda, \omega)$ in $\mathcal{B}([\delta_1^{(3)}-2\varepsilon]/\eta(\lambda))$.

On the other hand, for $k^0=i\kappa$, $(m_B-m)<\kappa<(m_B+m)$, we can explicitly do the integral marked (*) above. We use

$$\int dx\,\frac{1}{(x^2+c^2)\sqrt{x^2+a^2}}=\frac{1}{c\sqrt{a^2-c^2}}\operatorname{arctg}\frac{\sqrt{a^2-c^2}\,x}{c\sqrt{x^2+a^2}},\qquad a^2>c^2,$$

and obtain:

$$t(\lambda, i\kappa)=\pi(-\kappa^2+(m_B+m)^2)^{-1/2}(\kappa^2-(m_B-m)^2)^{-1/2}\times\left[\operatorname{arctg}\frac{(\kappa^2-m_B^2+m^2)p_1^{(1)}}{(\kappa^2-(m_B-m)^2)^{1/2}(-\kappa^2+(m_B+m)^2)^{1/2}}\cdot\frac{1}{\sqrt{p_1^{(1)2}+m^2}}\Bigg|_{-4m}^{4m}+m\leftrightarrow m_B\right].$$

When $\kappa\to m+m_B(\omega\to 0)$, the last factor goes to $2\pi$, so

$$\hat{t}_0(\lambda, 0)=2\pi^2\cdot\frac{1}{\sqrt{4mm_B}}=\frac{\pi^2}{\sqrt{mm_B}}$$

since $(\kappa^2-(m_B-m)^2)^{-1/2} \to (4mm_B)^{-1/2}$ as $\kappa \to (m+m_B)$. This completes the proof. $\square$

This lemma completes also the analysis of $b_{11}(\lambda, k^0)$. There remain the other $b_{ij}(\lambda, k^0)$, defined by the rule (II.10). It is clear that by a change of variables, all results on $b_{11}(\lambda, k^0)$ carry over to $b_{ij}(\lambda, k^0)$. Let us see briefly what things look like. Define

$$\begin{aligned} \phi^{(1)}_{p_1}(p_2) &= \phi\left(p_1, \frac{p_1}{2}+p_1\right), \\ \phi^{(2)}_{p_1}(p_2) &= \phi\left(-\frac{p_1}{2}+p_2, \frac{p_1}{2}+p_2\right), \\ \phi^{(3)}_{p_1}(p_2) &= \phi\left(-\frac{p_1}{2}-p_2, -p_1\right), \end{aligned} \tag{II.32}$$

with analogous definitions for $\psi^{(1)}_{p_1}(q_2)$. The index $i$ in $\phi^{(i)}_{p_1}$, $\psi^{(i)}_{p_1}$ refers to the channel. Our previous $\phi_{p_1}$, $\psi_{p_1}$ coincide with $\phi^{(1)}_{p_1}$, $\psi^{(1)}_{p_1}$.

One can verify that

$$\begin{aligned} &\langle \phi, b_{ij}(\lambda, k^0)\psi\rangle \\ &= 54\pi \int dp_1 R_1\left(\lambda, \frac{k}{3}+p_1\right) \int dp_2\, dq_2 \phi^{(i)}_{p_1}(p_2) R\left(\lambda, \frac{2k}{3}-p_1, p_2, q_2\right)\psi^{(j)}_{p_1}(-q_2). \end{aligned} \tag{II.33}$$

For $\theta(p_1, p_2) \in A'_3$, we define $\varepsilon_i(\lambda, k^0) \in \mathscr{L}(A'_3, A'^*_3)$, $i = 1, 2, 3$:

$$\begin{aligned} &\langle \theta, \varepsilon_i(\lambda, k^0)\rangle = \widehat{\theta\varepsilon_i}(\mu(k^0)), \\ &\widehat{\theta\varepsilon_i}(p_1) = \int dp_2 \theta^{(i)}_{p_1}(p_2)\hat{H}(\lambda, \zeta_1(\lambda), p_2) \end{aligned} \tag{II.34}$$

and, as in (II.21), $\widehat{\varepsilon_i\theta}(p_1) = \int dp_2 \theta^{(i)}_{p_1}(-p_2)\hat{H}(\lambda, \zeta_1(\lambda), p_2)$.

Let also

$$\gamma_{ij}(\lambda, k^0)\psi \equiv 27 Z(\lambda)^2 \cdot 2\zeta_1(\lambda) \cdot \hat{r}(\lambda, \zeta_1(\lambda)) t(\lambda, k^0)\varepsilon_i(\lambda, k^0)\widehat{\varepsilon_j\psi}(\mu(k^0)) \tag{II.35}$$

and

$$\alpha_{ij}(\lambda, k^0) = b_{ij}(\lambda, k^0) - \gamma_{ij}(\lambda, k^0). \tag{II.36}$$

Then our previous lemmas and the above definitions imply the following theorem:

**Theorem II.1.** *Let $\eta(\lambda) = \sqrt{9m^2-(m+m_B(\lambda))^2}$ and let also $\phi, \psi \in A'_3$, $\phi \in A'_{3,\infty}$. Then $\langle \phi, \hat{\alpha}_{ij}(\lambda, \omega)\psi\rangle$ and $\eta(\lambda)\omega\langle \phi, \hat{\gamma}_{ij}(\lambda, \omega)\psi\rangle$ are holomorphic functions in $\omega \in \mathscr{B}_C([\delta^{(3)}_1 - 2\varepsilon]/\eta(\lambda))$. Furthermore, in this region we have, uniformly in $\omega$ and $\lambda > 0$ small:*

$$|\langle \phi, \hat{\alpha}_{ij}(\lambda, \omega)\psi\rangle| \le \mathcal{O}(1)\, \|\phi\|_{A_{3,\infty}} \|\psi\|_{A_3}.$$

*Proof.* Follows from the preceding discussion. $\square$

We turn to the discussion of $R_{2B}(\lambda, k^0)$. Recall (II.11)

$$R_{2B}(\lambda, k^0) = \left(\sum_{i,j=1}^{3} b_{ij}^{-1}(\lambda, k^0)\right)^{-1}.$$

Write this in the form (we omit sometimes the $\lambda$, $k^0$-dependence)

$$R_{2B} = \tfrac{1}{9}(b_{11}+b_{12}+b_{13})\left[\left(\sum_{i,j=1}^{3} b_{ij}^{-1}\right)\cdot \tfrac{1}{9}(b_{11}+b_{12}+b_{13})\right]^{-1}.$$

We have as a first result:

**Lemma II.8.** *On the space $A_3$, the following identities hold:*
i) $(b_{11}^{-1}+b_{12}^{-1}+b_{13}^{-1})\cdot\frac{1}{9}\cdot(b_{11}+b_{12}+b_{13}) = 1$
ii) $(b_{i1}^{-1}+b_{i2}^{-1}+b_{i3}^{-1})\cdot\frac{1}{9}\cdot(b_{11}+b_{12}+b_{13}) = 1+\mathcal{O}(\lambda\,|\omega|)$, $i=2,3$.

*Proof*
i) it suffices to remark that $b_{11}=b_{12}=b_{13}$ acting on $A_3$.
ii) we write

$$\begin{aligned}(b_{i1}^{-1}&+b_{i2}^{-1}+b_{i3}^{-1})\cdot\tfrac{1}{9}\cdot(b_{11}+b_{12}+b_{13})\\ &=(b_{i1}^{-1}+b_{i2}^{-1}+b_{i3}^{-1})\cdot\tfrac{1}{9}\cdot(b_{i1}+b_{i2}+b_{i3})\\ &\quad+(b_{i1}^{-1}+b_{i2}^{-1}+b_{i3}^{-1})\cdot\tfrac{1}{9}\cdot(b_{11}+b_{12}+b_{13}-b_{i1}-b_{i2}-b_{i3}).\end{aligned}$$

As in i) above, the first term equals 1 when acting on $A_3$. In the second, we note that, on $A_3$,

$$\begin{aligned}(b_{i1}^{-1}&+b_{i2}^{-1}+b_{i3}^{-1})\cdot\tfrac{1}{9}\cdot(b_{11}-b_{i1}+b_{12}-b_{i2}+b_{13}-b_{i3})\\ &=(b_{i1}^{-1}+b_{i2}^{-1}+b_{i3}^{-1})\cdot\tfrac{1}{9}\cdot(\alpha_{11}-\alpha_{i1}+\alpha_{12}-\alpha_{i2}+\alpha_{13}-\alpha_{i3}).\end{aligned}$$

Now we remark that each $b_{ij}^{-1}$ is $O(|\omega|)$ as $|\omega|\to 0$. Then it suffices to show, to complete the proof, that $\alpha_{1j}-\alpha_{ij}=O(\lambda)$. But this follows from $b_{1j}-b_{ij}=O(\lambda/|\omega|)$ since each $b_{ij}(\lambda, k^0)$ is $C^\infty$ in $\lambda$ for $\lambda\geq 0$ small and they coincide at $\lambda=0$. □

From Lemma II.8, we can conclude that, acting on $A_3$, $[(\sum_{i,j=1}^{3} b_{ij}^{-1})\cdot\frac{1}{9}(b_{11}+b_{12}+b_{13})]^{-1}=\frac{1}{3}+O(\lambda\,|\omega|)$ and we have the following result on $R_{2B}(\lambda, k^0)$.
Write

$$\hat{R}_{2B}(\lambda,\omega)=\hat{\beta}_1(\lambda,\omega)+\hat{\beta}_2(\lambda,\omega),$$

$$\hat{\beta}_1(\lambda,\omega)\psi = 2Z(\lambda)^2\zeta_1(\lambda)\hat{r}(\lambda,\zeta_1(\lambda))\hat{t}(\lambda,\omega)\hat{\varepsilon}_1(\lambda,\omega)\times[\widehat{\varepsilon_1\psi}(\hat{\mu}(\omega))+\widehat{\varepsilon_2\psi}(\hat{\mu}(\omega))+\widehat{\varepsilon_3\psi}(\hat{\mu}(\omega))] \tag{II.37}$$

For $\psi\in A_3$, we have:

$$\hat{\beta}_1(\lambda,\omega)\psi=\hat{c}^*(\lambda,\omega)(\eta(\lambda)\omega)^{-1}\hat{\varepsilon}_1(\lambda,\omega)\widehat{\varepsilon_1\psi}(\hat{\mu}(\omega)) \tag{II.38}$$

with

$$\hat{c}^*(\lambda,\omega)=6Z(\lambda)^2\zeta_1(\lambda)\hat{r}(\lambda,\zeta_1(\lambda))\hat{t}_0(\lambda,\omega). \tag{II.39}$$

**Theorem II.2.** *Let* $\phi,\psi\in A_3$, $\phi\in A_{3,\infty}$. *Then* $\langle\phi,\hat{\beta}_2(\lambda,\omega)\psi\rangle$ *and*

*$\eta(\lambda)\omega\langle\phi, \hat\beta_1(\lambda, \omega)\psi\rangle$ are holomorphic in $\omega \in \hat{\mathcal{B}}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$. In this region we have, uniformly in $\omega$ and $\lambda > 0$ small.*

$$|\langle\phi, \hat\beta_2(\lambda, \omega)\psi\rangle| \le \mathcal{O}(1)\, \|\phi\|_{A_{3,\infty}} \|\psi\|_{A_3}.$$

*Proof.* Follows from the preceding considerations. □

## III. The analysis of $R_3(\lambda, k^0)$ and $S_2(\lambda, k^0)$

The analysis of $R_3(\lambda, k^0)$ has two main ingredients: the knowledge of the analytic structure of $R_{2B}(\lambda, k^0)$ and $K_3(\lambda, k^0)$. These two things are not really independent since in defining $K_3$ we have extracted from $R_3^{-1}$ all of its non-connected part (namely $R_{2B}^{-1}$). So in $K_3 = R_3^{-1} - R_{2B}^{-1}$ we expect that in addition to three-particle irreducibility (in the $\mathbf{x} \to \mathbf{y}$ channel) we have also connectedness in all channels. This is indeed what can be proven in models to which Spencer's method of $t$-derivatives applies, as it is the case for weakly coupled $\lambda P(\phi)_2$ models. In [CD], Corollary 2.1, it is shown that the suitable $t$-derivatives of $K_3(\lambda, t)$ (in fact it is proven for $K_n$, $n \ge 1$) vanish at $t = 0$ so as to give, applying Spencer's method, the decay:

$$\begin{aligned} |K_3(\lambda, x_1, x_2, x_3, y_1, y_2, y_3)| \le \mathcal{O}(1) \exp[&-(4m - \varepsilon) \cdot \tfrac{1}{3}|x_1 + x_2 + x_3 - y_1 - y_2 - y_3| \\ &-(m - \varepsilon)(|x_1 - x_2| + |x_2 - x_3| + |y_1 - y_2| + |y_2 - y_3|)]. \end{aligned} \tag{III.1}$$

This result can be translated in $p$-space by saying that $K_3(\lambda, k^0, p_1, p_2, q_1, q_2)$ is holomorphic in

$$\begin{aligned} &|\mathrm{Im}\, k^0| < 4m - \varepsilon, \\ &|\mathrm{Im}\, p_1^{(0)}|, |\mathrm{Im}\, p_2^{(0)}|, |\mathrm{Im}\, q_1^{(0)}|, |\mathrm{Im}\, q_2^{(0)}| < \tfrac{3}{4}m - \varepsilon = \delta_0^{(3)}, \\ &|\mathrm{Im}\, p_1^{(1)}|, |\mathrm{Im}\, p_2^{(1)}|, |\mathrm{Im}\, q_1^{(1)}|, |\mathrm{Im}\, q_2^{(1)}| < \tfrac{1}{4}m - \varepsilon = \delta_1^{(3)}. \end{aligned} \tag{III.2}$$

Note that holomorphy in the $p$, $q$ variables expresses the connectedness of $K_3$ and is in some sense equivalent to it (see [B]).

For our purposes we need some results on $K_3$ going a little further, namely:

i) $K_3(\lambda, k^0, p_i, q_i)$ is bounded in the region (III.2).

ii) $K_3(\lambda, k^0, p_i, q_i)$ is a $C^\infty$ function of $\lambda \ge 0$ small.

These two properties can be derived from the representation of $K_3$ as a convergent Neumann series:

$$K_3 = (1 + R_{2B}^{-1}G_3)^{-1} R_{2B}^{-1} G_3 R_{2B}^{-1}, \qquad G_3 = R_3 - R_{2B}.$$

If $C(x - y)$ denotes the free covariance and $C_3^{-1} \equiv C^{-1} \otimes C^{-1} \otimes C^{-1}$, the singularities of $C_3^{-1}G_3$ and of $C_3^{-1}G_3C_3^{-1}$ are isolated using integration by parts, as in [S] and [K]. We sketch a proof of this in Appendix D.

Note that $K_3(\lambda, k, p_i, \cdot) \in A_3$ since it is invariant under the transformations (I.11) and it is bounded (in fact $|K_3(\lambda, k^0, p_i, q_i| \le O(\lambda)$, since $K_3(0, k^0, p_i, q_i) = 0$).

Given these properties, we are able to study the analytic structure of $R_3(\lambda, k^0)$. Recall equation (I.10):

$$R_3(\lambda, k^0) = R_{2B}(\lambda, k^0) - R_{2B}(\lambda, k^0) K_3(\lambda, k^0) R_3(\lambda, k^0).$$

Let

$$V(\lambda, k^0) = K_3(\lambda, k^0)R_{2B}(\lambda, k^0). \tag{III.3}$$

According to the decomposition of $R_{2B}$ given in (II.37), we decompose also:

$$\begin{aligned} V(\lambda, k^0) &= V_1(\lambda, k^0) + V_2(\lambda, k^0), \\ V_i(\lambda, k^0) &= K_3(\lambda, k^0)\beta_i(\lambda, k^0), \qquad i = 1, 2. \end{aligned} \tag{III.4}$$

Since $K_3$ is holomorphic and bounded, we can apply Theorem II.2 to conclude that $V_1(\lambda, k^0)$ is a rank-one operator in $\mathscr{L}(A_3)$ given explicitly by, for any $\psi \in A_3$:

$$(\hat{V}_1(\lambda, \omega)\psi)(p_1, p_2) = \hat{c}^*(\lambda, \omega)(\eta(\lambda)\omega)^{-1}\widehat{\varepsilon_1\psi}(\hat{\mu}(\omega))\hat{K}^*(\lambda, \omega, p_1, p_2) \tag{III.5}$$

with $\hat{K}^*(\lambda, \omega, p_1, p_2) \in A_3$ given by:

$$\hat{K}^*(\lambda, \omega, p_1, p_2) = \int dq_2 \hat{K}_3\left(\lambda, \omega, p_1, p_2, \hat{\mu}(\omega), \frac{\hat{\mu}(\omega)}{2} + q_2\right)\hat{H}(\lambda, \zeta_1(\lambda), q_2) \tag{III.6}$$

We quote a first result.

**Proposition III.1.** *The operators $\eta(\lambda)\omega\hat{V}_1(\lambda, \omega)$ and $\hat{V}_2(\lambda, \omega)$ are holomorphic in $\omega \in \hat{\mathscr{B}}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$. Furthermore, we have in this domain*

$$\|\hat{V}_2(\lambda, \omega)\| \leq \mathcal{O}(\lambda).$$

*Proof.* Given the properties of $\hat{\beta}_1(\lambda, \omega)$ and $\hat{\beta}_2(\lambda, \omega)$ in Theorem II.2, and the fact that $|\hat{K}_3(\lambda, \omega, p_i, q_i)| \leq O(\lambda)$, the result follows as in Theorem II.3 of [DE]. □

As a preparation for the main theorem, we prove here the following:

**Lemma III.1.** *Let $\hat{V}_1(\lambda, \omega)$ be defined in* (III.5). *Then we have $\|\omega\hat{V}_1(\lambda, \omega)\| \leq O(\lambda)$ for $\omega \in \hat{\mathscr{B}}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$.*

*Proof.* Given that $|\hat{K}_3(\lambda, \omega)| \leq O(\lambda)$, the result follows once we prove that $\hat{c}^*(\lambda, \omega)\eta(\lambda)^{-1}| \leq O(1)$. But this is a consequence of $\zeta_1(\lambda)\eta(\lambda)^{-1} \leq O(1)$. □

Recall now that

$$\begin{aligned} R_3(\lambda, k^0) &= R_{2B}(\lambda, k^0)(1 + K_3(\lambda, k^0)R_{2B}(\lambda, k^0))^{-1} \\ &= R_{2B}(\lambda, k^0)(1 - V_1(\lambda, k^0) + V_2(\lambda, k^0))^{-1}. \end{aligned}$$

Turn to the $\omega$-variable:

$$\hat{R}_3(\lambda, \omega) = \hat{R}_{2B}(\lambda, \omega)(1 + (1 + \hat{V}_2(\lambda, \omega))^{-1}\hat{V}_1(\lambda, \omega))^{-1}(1 + \hat{V}_2(\lambda, \omega))^{-1},$$

where the existence of $(1 + \hat{V}_2(\lambda, \omega))^{-1}$ follows from $\|\hat{V}_2(\lambda, \omega)\| \leq O(\lambda)$ for $\omega \in \hat{\mathscr{B}}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$, see Proposition III.1. It follows that $\hat{R}_3(\lambda, \omega)$ is meromorphic in $\hat{\mathscr{B}}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$, with poles at those values of $\omega$ where the trace of the rank-one operator $(1 + \hat{V}_2(\lambda, \omega))^{-1}\hat{V}_1(\lambda, \omega)$ is equal to $-1$. Let

$$\begin{aligned} \hat{F}_1(\lambda, \omega) &= \mathrm{Tr}\,[(1 + \hat{V}_2(\lambda, \omega))^{-1}\hat{V}_1(\lambda, \omega)] \\ &= \frac{\hat{c}^*(\lambda, \omega)}{\eta(\lambda)\omega}\widehat{\varepsilon_1\psi_0}(\hat{\mu}(\omega)) \end{aligned} \tag{III.7}$$

where we have defined $\psi_0 \in A_3$:

$$\psi_0 = (1 + \hat{V}_2(\lambda, \omega))^{-1} \hat{K}^*(\lambda, \omega). \tag{III.8}$$

We write, explicitly:

$$\begin{aligned}\widehat{\varepsilon_1 \psi_0}(\hat{\mu}(\omega)) = \int dp_2\, dp_1'\, dp_2'\, dq_2 \quad & \hat{H}(\lambda, \zeta_1(\lambda), p_2) \\ & \times (1 + \hat{V}_2)^{-1}\left(\lambda, \omega, \hat{\mu}(\omega), \frac{\hat{\mu}(\omega)}{2} - p_2, -p_1', -p_2'\right) \\ & \times \hat{K}_3\left(\lambda, \omega, p_1', p_2', \hat{\mu}(\omega), \frac{\hat{\mu}(\omega)}{2} + q_2\right) \hat{H}(\lambda, \zeta_1(\lambda), q_2).\end{aligned} \tag{III.9}$$

**Lemma III.2.**

i) $\hat{H}(\lambda, \zeta_1(\lambda), \cdot) = \delta(\cdot) + \mathcal{O}_{A_2}(\lambda)$.

ii) $(1 + \hat{V}_2)^{-1}(\lambda, \omega, p_1, p_2, q_1, q_2) = \delta(p_1 - q_1)\, \delta(p_2 - q_2) + \mathcal{O}(\lambda)$.

*Proof*

i) follows from

$$\hat{H}(\lambda, \zeta_1(\lambda), \cdot) = (1 + \hat{T}_2(\lambda, \zeta_1(\lambda)))^{-1}(0, \cdot)$$

and $\|\hat{T}_2(\lambda, \zeta_1(\lambda))\|_{\mathscr{L}(A_2)} \leq O_{A_2}(\lambda)$, see (I.15) and [DE].

ii) follows from Proposition III.1. □

We can state now our first theorem:

**Theorem III.1.** *$\hat{F}_1(\lambda, \omega) = -1$ has one real solution $\omega_1(\lambda)$ in $\mathscr{B}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$. In addition, we have that $\hat{K}_3(\lambda, \omega = 0, \hat{\mu}(0), \hat{\mu}(0)/2, \hat{\mu}(0), \hat{\mu}(0)/2) \gtrless 0$ implies $\omega_1(\lambda) \lessgtr 0$ respectively.*

*Proof.* Since $|\hat{F}_1(\lambda, \omega) \leq O(\lambda\, |\omega|^{-1})$ (this is a consequence of Lemma III.1), the solutions of $\hat{F}_1(\lambda, \omega) = -1$ can only occur for small $|\omega|$. We look thus for solutions inside the curve $\gamma = \{\omega : |\omega| = r\}$ for a fixed and small $r$. Let $\hat{G}_1(\lambda, \omega) = \omega \hat{F}_1(\lambda, \omega)$ and note that $\hat{F}_1(\lambda, \omega) = -1$ is equivalent to $\hat{G}_1(\lambda, \omega) + \omega = 0$. Consider, then:

$$\begin{aligned}\hat{H}(\lambda, \omega) = \hat{G}_1(\lambda, \omega) + \omega &= (\hat{G}_1(\lambda, 0) + \omega) + (\hat{G}_1(\lambda, \omega) - \hat{G}_1(\lambda, 0)) \\ &\equiv \hat{H}_1(\lambda, \omega) + \hat{H}_2(\lambda, \omega).\end{aligned}$$

We see that $\hat{H}_1(\lambda, \omega)$ has a real zero at $\omega_{01}(\lambda) = -\hat{G}_1(\lambda, 0)$. Remark also that, by Lemma III.2, and for $\lambda$ small, $\hat{K}_3(\lambda, 0, \hat{\mu}(0), \hat{\mu}(0)/2, \hat{\mu}(0), \hat{\mu}(0)/2) \gtrless 0$ implies $\hat{G}_1(\lambda, 0) \gtrless 0$ and so $\omega_{01}(\lambda) \lessgtr 0$, respectively (this follows because $\hat{c}^*(\lambda, 0)\eta(\lambda)^{-1}$ is positive). Let $\omega_{01}(\lambda) > 0$ and consider the semi-circle $\gamma_1 \cup \gamma_2$, with

$$\gamma_1 = \{\omega : |\omega| = r, \operatorname{Re} \omega > 0\},$$

$$\gamma_2 = \{\omega : \omega = iy, -r \leq y \leq r\}.$$

On $\gamma_1$, using that $\hat{G}_1(\lambda, \omega)$ is holomorphic inside a circle of unit radius, contained in $\mathscr{B}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$, we have $|\hat{H}_2(\lambda, \omega)| \leq r/2$ for sufficiently small $\lambda$. Also $|\hat{H}_1(\lambda, \omega)| > r/2$ for $\lambda$ sufficiently small, so that on $\gamma_1$, $|\hat{H}_2(\lambda, \omega)| \leq r/2 < |\hat{H}_1(\lambda, \omega)|$.

On the other hand, we have the bound

$$|\hat{H}_2(\lambda, \omega)| \le \mathcal{O}(\lambda\ |\omega|)$$

uniformly for $|\omega| \le r$, so that on $\gamma_2$

$$|\hat{H}_2(\lambda, \omega)| < |y| < |\hat{H}_1(\lambda, \omega)|$$

for $\lambda$ sufficiently small.

So $|\hat{H}_2(\lambda, \omega)| < |\hat{H}_1(\lambda, \omega)|$ on $\gamma_1 \cup \gamma_2$ and by Rouché's theorem, $\hat{H}(\lambda, \omega)$ has a unique zero $\omega_1(\lambda) > 0$ inside $\gamma_1 \cup \gamma_2$. Since the zero is unique, it is real. If $\omega_{01} < 0$ we use the same argument to show that $\omega_1(\lambda) < 0$. □

One last result we shall need about $\hat{R}_3(\lambda, \omega)$ is the following:

**Theorem III.2.** *$\hat{R}_3(\lambda, \omega) = \hat{\sigma}_1(\lambda, \omega) + \hat{\sigma}_2(\lambda, \omega)$ with $(\omega - \omega_1(\lambda))\ \hat{\sigma}_1(\lambda, \omega)$ and $\hat{\sigma}_2(\lambda, \omega) \in \mathcal{L}(A_3, A_3^*)$ holomorphic in $\omega \in \mathcal{B}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$. Furthermore,*

$$\hat{\sigma}_1(\lambda, \omega) = \hat{d}(\lambda, \omega)(\omega - \omega_1(\lambda))^{-1}(1 + \hat{V}_2^*(\lambda, \omega))^{-1}\hat{\varepsilon}_1(\lambda, \omega)$$
$$\times \langle \cdot, (1 + \hat{V}_2^*(\lambda, \omega))^{-1}\hat{\varepsilon}_1(\lambda, \omega)\rangle,$$

$$\hat{\sigma}_2(\lambda, \omega) = \hat{\beta}_2(\lambda, \omega)(1 + \hat{V}_2(\lambda, \omega))^{-1},$$

*and*

$$\hat{d}(\lambda, \omega) = \frac{\omega - \omega_1(\lambda)}{\omega + \hat{G}_1(\lambda, \omega)} \frac{\hat{c}^*(\lambda, \omega)}{\eta(\lambda)}.$$

*Proof.* Since $(1 + \hat{V}_2(\lambda, \omega))^{-1}\hat{V}_1(\lambda, \omega)$ is rank-one, we have, with $\mathrm{Tr}\,(1 + \hat{V}_2(\lambda, \omega))^{-1}\hat{V}_1(\lambda, \omega) = \hat{F}_1(\lambda, \omega) \neq -1$.

$$[1 + (1 + \hat{V}_2(\lambda, \omega))^{-1}\hat{V}_1(\lambda, \omega)]^{-1} = 1 - (1 + \hat{V}_2(\lambda, \omega))^{-1}\hat{V}_1(\lambda, \omega)[1 + \hat{F}_1(\lambda, \omega)]^{-1}.$$

Hence we can write $\hat{R}_3(\lambda, \omega)$ as follows:

$$\begin{aligned}
\hat{R}_3 &= \hat{R}_{2B}(1 + (1 + \hat{V}_2)^{-1}\hat{V}_1)^{-1}(1 + \hat{V}_2)^{-1} \\
&= \hat{R}_{2B}(1 - (1 + \hat{V}_2)^{-1}\hat{V}_1(1 + \hat{F}_1)^{-1})(1 + \hat{V}_2)^{-1} \\
&= \hat{\beta}_2(1 + \hat{V}_2)^{-1} - \hat{\beta}_2 \frac{(1 + \hat{V}_2)^{-1}\hat{V}_1(1 + \hat{V}_2)^{-1}}{1 + \hat{F}_1} \\
&\quad + \hat{\beta}_1\left(1 - \frac{(1 + \hat{V}_2)^{-1}\hat{V}_1(1 + \hat{V}_2)^{-1}}{1 + \hat{F}_1}\right) + \hat{\beta}_1\left(1 - \frac{(1 + \hat{V}_2)^{-1}\hat{V}_1}{1 + \hat{F}_1}\right)(1 + \hat{V}_2)^{-1} \\
&= \hat{\sigma}_2 - \hat{\beta}_2 \frac{(1 + \hat{V}_2)^{-1}\hat{V}_1(1 + \hat{V}_2)^{-1}}{1 + \hat{F}_1} \\
&\quad + \frac{\hat{c}^*}{\eta(\lambda)(\omega + \omega\hat{F}_1)}\,\hat{\varepsilon}_1(\lambda, \omega)\langle(1 + \hat{V}_2)^{-1}\,\cdot, \hat{\varepsilon}_1(\lambda, \omega)\rangle
\end{aligned}$$

where we have used that:

$$\hat{\beta}_1(1 + \hat{F}_1 - (1 + \hat{V}_2)^{-1}\hat{V}_1) = \hat{\beta}_1$$

since, for a general $\psi$:

$$\hat{\beta}_1(1+\hat{V}_2)^{-1}\hat{V}_1\psi = \hat{\beta}_1(1+\hat{V}_2)^{-1}\cdot\frac{\hat{c}^*}{\eta(\lambda)\omega}\widehat{\varepsilon_1\psi}(\hat{\mu}(\omega))\hat{K}^*(\lambda,\omega), \quad \text{see (III.5)},$$

$$=\frac{\hat{c}^*}{\eta(\lambda)\omega}\cdot\widehat{\varepsilon_1\psi}(\hat{\mu}(\omega))\cdot\hat{\beta}_1\psi_0 \qquad (\psi_0=(1+\hat{V}_2)^{-1}\hat{K}^*, \quad \text{see (III.8)}),$$

$$=\left(\frac{\hat{c}^*}{\eta(\lambda)\omega}\right)^2\widehat{\varepsilon_1\psi}(\hat{\mu}(\omega))\widehat{\varepsilon_1\psi_0}(\hat{\mu}(\omega))\hat{\varepsilon}_1(\lambda,\omega).$$

On the other hand,

$$\hat{\beta}_1(1+\hat{F}_1)\psi=\hat{\beta}_1\psi+\hat{F}_1\frac{\hat{c}^*}{\eta(\lambda)\omega}\widehat{\varepsilon_1\psi}(\hat{\mu}(\omega))\hat{\varepsilon}_1(\lambda,\omega)$$

$$=\hat{\beta}_1\psi+\left(\frac{\hat{c}^*}{\eta(\lambda)\omega}\right)^2\widehat{\varepsilon_1\psi_0}(\hat{\mu}(\omega))\widehat{\varepsilon_1\psi}(\hat{\mu}(\omega))\hat{\varepsilon}_1(\lambda,\omega).$$

Returning to $\hat{R}_3$:

$$\hat{R}_3=\hat{\sigma}_2+\frac{\hat{c}^*}{\eta(\lambda)(\omega+\omega\hat{F}_1)}(\hat{\varepsilon}_1(\lambda,\omega)-\hat{\beta}_2\psi_0)\langle(1+\hat{V}_2)^{-1}\cdot,\hat{\varepsilon}_1(\lambda,\omega)\rangle$$

$$=\hat{\sigma}_2+\frac{\hat{c}^*}{\eta(\lambda)(\omega+\omega\hat{F}_1)}(1-\hat{\beta}_2(1+\hat{V}_2)^{-1}\hat{K}_3)\hat{\varepsilon}_1(\lambda,\omega)\langle(1+\hat{V}_2)^{-1}\cdot,\hat{\varepsilon}_1(\lambda,\omega)\rangle$$

since

$$\hat{\beta}_2\psi_0=\hat{\beta}_2(1+\hat{V}_2)^{-1}\hat{K}^*=\hat{\beta}_2(1+\hat{V}_2)^{-1}\hat{K}_3\hat{\varepsilon}_1(\lambda,\omega)$$

$$\hat{R}_3=\hat{\sigma}_2+\frac{\hat{c}^*}{\eta(\lambda)(\omega+\omega\hat{F}_1)}(1+\hat{V}_2^*)^{-1}\hat{\varepsilon}_1(\lambda,\omega)\langle\cdot,(1+\hat{V}_2^*)^{-1}\hat{\varepsilon}_1(\lambda,\omega)\rangle$$

where we have used that

$$1-\hat{\beta}_2(1+\hat{V}_2)^{-1}\hat{K}_3=(1+\hat{V}_2^*)^{-1}, \quad \text{with} \quad \hat{V}_2^*=\hat{\beta}_2\hat{K}_3.$$

Note finally that $\omega+\omega\hat{F}_1(\lambda,\omega)=\hat{H}(\lambda,\omega)$ is holomorphic in $\mathscr{B}_C([\delta_1^{(3)}-2\varepsilon]/\eta(\lambda))$ and has a unique zero $\omega_1(\lambda)$, so that we can write

$$\frac{\hat{c}^*(\lambda,\omega)}{\eta(\lambda)(\omega+\omega\hat{F}_1(\lambda,\omega))}=\frac{1}{\omega-\omega_1(\lambda)}\hat{d}(\lambda,\omega)$$

with

$$\hat{d}(\lambda,\omega)=\frac{\omega-\omega_1(\lambda)}{\omega+\omega\hat{F}_1(\lambda,\omega)}\cdot\frac{\hat{c}^*(\lambda,\omega)}{\eta(\lambda)}$$

holomorphic in $\mathscr{B}_C([\delta_1^{(3)}-2\varepsilon]/\eta(\lambda))$. □

The spectrum of the energy-momentum operator is directly connected to the Schwinger functions via the Osterwalder–Schrader theorem [OS]. Up to now, we have studied the 2-particle irreducible six-point function and we must now carry out our results to the Schwinger functions. We shall see that one way of doing so is to look at the two-point Schwinger function, $S_2(\lambda, x-y)$. Let $\tilde{S}_2(\lambda, k^0)$ be its Fourier transform, and let $C(k^0)=(k^{02}+m_0^2)^{-1}$ (as before, we put $k^1=0$). Define also the one particle irreducible two point function $K_1(\lambda, k^0)$ by:

$$K_1(\lambda, k^0) = \tilde{S}_2(\lambda, k^0)^{-1} - C(k^0)^{-1},$$

or

$$\tilde{S}_2(\lambda, k^0) = C(k^0) - C(k^0)K_1(\lambda, k^0)\tilde{S}_2(\lambda, k^0). \tag{III.10}$$

In an even theory, we have Spencer's result [S] that $K_1(\lambda, k^0)$ is holomorphic and bounded by $O(\lambda)$ for $|\mathrm{Im}\, k^0| < 3m - \varepsilon$. From this result we can conclude the existence of a pole of $\tilde{S}_2(\lambda, k^0) = (1 + C(k^0)K_1(\lambda, k^0))^{-1}C(k^0)$ because $1 + C(k^0)K_1(\lambda, k^0)$ has a zero at some value $k^0 = im(\lambda)$ near the pole of $C(k^0)$ at $k^{02} = -m_0^2$. This pole corresponds to the mass of the lightest particle described by the theory. We shall now see that $1 + C(k^0)K_1(\lambda, k^0) = 0$ can have a solution near $k^{02} = -(m + m_B)^2$ because of a pole of $K_1(\lambda, k^0)$. We begin our study of $K_1(\lambda, k^0)$ in $x$-space, considering:

$$\int dx' C^{-1}(x - x')\langle\phi(x')\phi(y)\rangle = \delta(x - y) - \lambda\langle P'(x)\phi(y)\rangle$$

$$= \delta(x - y) - \lambda C(x - y)\langle P''(x)\rangle + \lambda^2 \int dx' \langle P'(x)P'(x')\rangle C(x' - y).$$

This follows by integration by parts, and we use $P(x)$ as a shorthand notation for $P(\phi(x))$, and $P$ is the Wick-ordered interaction polynomial. The primes on $P$ stand for derivatives. We have also:

$$\begin{aligned} K_1 C &= S_2^{-1}C - 1 \\ &= (1 - C^{-1}S_2)S_2^{-1}C(1 - C^{-1}S_2) + (1 - C^{-1}S_2) \\ &= \lambda^2 \int dz_1\, dz_2\, dz_3 \langle P'(x)\phi(z_1)\rangle S_2^{-1}(z_1, z_2)C(z_2 - z_3)\langle P'(z_3)\phi(y)\rangle \\ &\quad - \lambda^2 \int dx' \langle P'(x)P'(x')\rangle C(x' - y) + \lambda C(x - y)\langle P''(x)\rangle \\ &= \lambda^2 \int dz_1\, dz_2\, dz_3 \langle P'(x)\phi(z_1)\rangle S_2^{-1}(z_1, z_2)\langle\phi(z_2)P'(z_3)\rangle C(z_3 - y) \\ &\quad - \lambda^2 \int dx' \langle P'(x)P'(x')\rangle C(x' - y) + \lambda C(x - y)\langle P''(x)\rangle, \end{aligned}$$

where we have used

$$\begin{aligned} \lambda \int dz_3 C(z_2 - z_3)\langle P'(z_3)\phi(y)\rangle &= \lambda \int dz_3 \langle\phi(z_2)P'(z_3)\rangle C(z_3 - y) \\ &= -\langle\phi(z_2)\phi(y)\rangle + C(z_2 - y). \end{aligned}$$

We can express $K_1$ in the following way (recall the definition of $\mathbf{P}_n$ in Chapter I):

$$K_1(\lambda, x - y) = -\lambda^2\langle P'(x)(1 - \mathbf{P}_1)P'(y)\rangle + \lambda\, \delta(x - y)\langle P''(x)\rangle. \tag{III.11}$$

At this point we use the fact that $P$ is even to rewrite $K_1$ as:

$$\begin{aligned} K_1(\lambda, x - y) = &-\lambda^2\langle P'(x)\mathbf{P}_3 P'(y)\rangle - \lambda^2\langle P'(x)(1 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_3)P'(y)\rangle \\ &+ \lambda\, \delta(x - y)\langle P''(x)\rangle, \end{aligned} \tag{III.12}$$

since $\mathbf{P}_0 P' = \mathbf{P}_2 P' = 0$ for even $P$.

We define:

$$B_1(\lambda, x-y) = -\lambda^2 \langle P'(x)\mathbf{P}_3 P'(y)\rangle.$$

$$D_2(\lambda, x-y) = -\lambda^2 \langle P'(x)(1-\mathbf{P}_0-\mathbf{P}_1-\mathbf{P}_2-\mathbf{P}_3)P'(y)\rangle, \tag{III.13}$$

$$D_1(\lambda, x-y) = \lambda\, \delta(x-y)\langle P''(x)\rangle.$$

We write explicitly $B_1(\lambda, x-y)$:

$$B_1(\lambda, x-y) = -\lambda^2 \int dz_1\, dz_2\, dz_3\, dz_1'\, dz_2'\, dz_3' \langle P'(x)(1-\mathbf{P}_1)\phi(z_1)\phi(z_2)\phi(z_3)\rangle$$
$$\times R_3^{-1}(z_1, z_2, z_3, z_1', z_2', z_3')\langle \phi(z_1')\phi(z_2')\phi(z_3')(1-\mathbf{P}_1)P'(y)\rangle$$

and we let

$$L_3(\lambda, x, y_1, y_2, y_3) = \lambda \int dz_1\, dz_2\, dz_3 \langle P'(x)(1-\mathbf{P}_1)\phi(z_1)\phi(z_2)\phi(z_3)\rangle$$
$$\times R_3^{-1}(z_1, z_2, z_3, y_1, y_2, y_3). \tag{III.14}$$

With $\mathbf{z} = (z_1, z_2, z_3)$, we write:

$$B_1(\lambda, x-y) = -\int d\mathbf{z}\, d\mathbf{z}' L_3(\lambda, x; \mathbf{z}) R_3(\lambda, \mathbf{z}, \mathbf{z}') L_3^*(\lambda, \mathbf{z}'; y). \tag{III.15}$$

Introduce

$$\tau = x - \tfrac{1}{3}(y_1+y_2+y_3), \qquad \xi_1 = y_1 - y_2, \qquad \xi_2 = y_2 - y_3$$

and consider the Fourier transform of $L_3(\lambda, x; y_1, y_2, y_3)$ $(=L_3(\lambda, \tau, \xi_1, \xi_2)$ by translation invariance):

$$L_3(\lambda, k, p_1, p_2) = (2\pi)^{-2} \int d\tau\, d\xi_1\, d\xi_2 e^{i(k\tau + p_1\xi_1 + p_2\xi_2)} L_3(\lambda, \tau, \xi_1, \xi_2)$$

Similarly, with

$$\tau' = \tfrac{1}{3}(y_1+y_2+y_3) - x, \qquad \xi_1 = y_1 - y_2, \qquad \xi_2 = y_2 - y_3$$

we Fourier transform $L_3^*(\lambda, y_1, y_2, y_3; x)$ $(=L_3^*(\lambda, \tau', \xi_1, \xi_2)$ by translation invariance):

$$L_3^*(\lambda, k, p_1, p_2) = (2\pi)^{-2} \int d\tau\, d\xi_1\, d\xi_2 e^{i(k\tau' + p_1\xi_1 + p_2\xi_2)} L_3^*(\lambda, \tau', \xi_1, \xi_2).$$

*Remark.* Using that $R_3^{-1}(\lambda, \mathbf{x}, \mathbf{y}) = R_3^{-1}(\lambda, \mathbf{y}, \mathbf{x})$ and that

$$\langle P'(x)(1-\mathbf{P}_1)\phi(y_1)\phi(y_2)\phi(y_3)\rangle = \langle \phi(y_1)\phi(y_2)\phi(y_3)(1-\mathbf{P}_1)P'(x)\rangle$$

we can conclude that $L_3(\lambda, x; y_1, y_2, y_3) = L_3^*(\lambda, y_1, y_2, y_3; x)$, or that $L_3^*(\lambda, \tau', \xi_1, \xi_2) = L_3(\lambda, -\tau', \xi_1, \xi_2)$. This implies that $L_3^*(\lambda, k, p_1, p_2) = L_3(\lambda, -k, p_1, p_2)$.

Equation (III.15) reads as follows, in momentum space:

$$B_1(\lambda, k^0) = -\int dp_1\, dp_2\, dq_1\, dq_2 L_3(\lambda, k^0, p_1, p_2) R_3(\lambda, k^0, -p_1, -p_2, q_1, q_2)$$
$$\times L_3^*(\lambda, k^0, -q_1, -q_2), \tag{III.16}$$

where we have put as usual $k^1 = 0$.

Consider also:

$$D_2(\lambda, k^0) = -\frac{\lambda^2}{2\pi} \int d\tau e^{ik^0\tau} \langle P'(\tau)(1 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_3) P'(0)\rangle,$$
$$D_1(\lambda, k^0) = \frac{\lambda\alpha_0}{2\pi}, \qquad \alpha_0 = \langle P''(x)\rangle = \langle P''(0)\rangle. \tag{III.17}$$

We should know, to go ahead, what are the analyticity properties of $D_1(\lambda, k^0)$, $D_2(\lambda, k^0)$, $L_3(\lambda, k^0, p_1, p_2)$. The result for $D_1(\lambda, k^0)$ is trivial, since it is a constant. Kernels like $D_2(\lambda, k^0)$, defined with the Euclidean projectors $\mathbf{P}_n$, are expected to be 3-particle irreducible. In weakly coupled $\lambda P(\phi)_2$, where the modified cluster expansion of Spencer [S] converges, one can show that (see the proof of Theorem I.1 in [CD]) the suitable $t$-derivatives of $D_2(\lambda, t, x, y)$ vanish at $t = 0$ so as to give the expected decay:

$$|D_2(\lambda, x - y)| \leq \text{const.} \exp[-(4m - \varepsilon)\,|x - y|].$$

This property can be translated in momentum space by saying that $D_2(\lambda, k^0)$ is holomorphic in $|\text{Im } k^0| < 4m - \varepsilon$.

A similar result can be proven for $L_3(\lambda, k^0, p_1, p_2)$. This is essentially done in [CD] (see their proof of Theorem II.1) and the result is again that the suitable $t$-derivatives of $L_3(\lambda, t, x; y_1, y_2, y_3)$ vanish at $t = 0$ so as to give the decay (assuming that Spencer's method works, as is the case in weakly coupled $\lambda P(\phi)_2$):

$$|L_3(\lambda, x; y_1, y_2, y_3)| \leq \text{const.} \exp[-(4m - \varepsilon)\,|x - \tfrac{1}{3}(y_1 + y_2 + y_3)|$$
$$-(m - \varepsilon)(|y_1 - y_2| + |y_2 - y_3|)].$$

This means that, in momentum space, $L_3(\lambda, k^0, p_1, p_2)$ is holomorphic in $|\text{Im } k^0| < 4m - \varepsilon$, $|\text{Im } p_1^{(0)}|$, $|\text{Im } p_2^{(0)}| < \frac{3}{4}(m - \varepsilon)$, $|\text{Im } p_1^{(1)}|$, $|\text{Im } p_2^{(1)}| < \frac{1}{4}(m - \varepsilon)$. Regarding $L_3(\lambda, k^0, p_1, p_2)$, one can prove that it satisfies the following two properties (the proof is sketched in Appendix D).

i) $L_3(\lambda, k^0, p_1, p_2)$ is bounded in the above region.

ii) it is a $C^\infty$ function of $\lambda$ for $\lambda \geq 0$ small. (III.18)

We have already used similar properties when dealing with $K_3$. As in the case of $K_3$, these properties in the present case can also be derived by isolating the singularities of $C_3^{-1} \cdot \langle \phi(x_1)\phi(x_2)\phi(x_3)(1 - \mathbf{P}_1)P'(y)\rangle$ using integration by parts.

The same technique can also be applied to show that $D_2(\lambda, k^0)$ is bounded by $O(\lambda^2)$. Introducing the $\omega$-variable and taking into account the above discussion, we conclude that $\hat{D}_1(\lambda, \omega)$, $\hat{D}_2(\lambda, \omega)$, $\hat{L}_3(\lambda, \omega, p_i)$ and $\hat{L}_3^*(\lambda, \omega, p_i)$ are holomorphic and bounded by $O(\lambda)$ in $\omega \in \hat{\mathscr{B}}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$.

We further analyze $\hat{B}_1(\lambda, \omega)$. Recall Theorem III.2 and consider:

$$\begin{aligned} \hat{B}_1(\lambda, \omega) &= \hat{B}(\lambda, \omega) + \hat{D}_3(\lambda, \omega), \\ \hat{B}(\lambda, \omega) &= -\hat{L}_3(\lambda, \omega)\hat{\sigma}_1(\lambda, \omega)\hat{L}_3^*(\lambda, \omega), \\ D_3(\lambda, \omega) &= -\hat{L}_3(\lambda, \omega)\hat{\sigma}_2(\lambda, \omega)\hat{L}_3^*(\lambda, \omega). \end{aligned} \tag{III.19}$$

Define also

$$\hat{D}(\lambda, \omega) = \hat{D}_1(\lambda, \omega) + \hat{D}_2(\lambda, \omega) + \hat{D}_3(\lambda, \omega) \tag{III.20}$$

so that

$$\hat{K}_1(\lambda, \omega) = \hat{B}(\lambda, \omega) + \hat{D}(\lambda, \omega). \tag{III.21}$$

Note also that $C(k^0) = \hat{C}(\omega)$ is holomorphic and uniformly bounded for $\omega \in \hat{\mathscr{B}}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$ since we stay away from $k^0 = im_0$, the pole of $C(k^0)$. We can summarise the preceding analysis in the following.

**Proposition III.2**

i) *$\hat{D}(\lambda, \omega)$ and $(\omega - \omega_1(\lambda))\hat{B}(\lambda, \omega)$ are holomorphic and uniformly bounded in $\omega \in \hat{\mathscr{B}}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$. In addition,*

$$|\hat{D}(\lambda, \omega)| \leq \mathcal{O}(\lambda).$$

ii) *The same for $\hat{C}(\omega)\hat{D}(\lambda, \omega)$ and $(\omega - \omega_1(\lambda))\hat{C}(\omega)\hat{B}(\lambda, \omega)$, with $|\hat{C}(\omega)\hat{D}(\lambda, \omega)| \leq O(\lambda)$.*

*These statements are true uniformly in $\lambda > 0$ small.* □

Consider the two point function:

$$\begin{aligned} \tilde{S}_2(\lambda, \omega) &= (1 + \hat{C}(\omega)\hat{K}_1(\lambda, \omega))^{-1}\hat{C}(\omega) \\ &= (1 + \hat{C}(\omega)\hat{B}(\lambda, \omega) + \hat{C}(\omega)\hat{D}(\lambda, \omega))^{-1}\hat{C}(\omega) \\ &= (1 + (1 + \hat{C}(\omega)\hat{D}(\lambda, \omega))^{-1}\hat{C}(\omega)\hat{B}(\lambda, \omega))^{-1}(1 + \hat{C}(\omega)\hat{D}(\lambda, \omega))^{-1}\hat{C}(\omega) \end{aligned}$$

where we have used that $(1 + \hat{C}(\omega)\hat{D}(\omega))$ is bounded away from zero, for small $\lambda > 0$, so that $(1 + \hat{C}(\omega)\hat{D}(\lambda, \omega))^{-1}$ is holomorphic and uniformly bounded in $\omega \in \hat{\mathscr{B}}_C([\delta_1^{(3)} - 2\varepsilon]/\eta(\lambda))$ and $\lambda > 0$ small.

Define

$$\hat{F}_2(\lambda, \omega) = (1 + \hat{C}(\omega)\hat{D}(\lambda, \omega))^{-1}\hat{C}(\omega)\hat{B}(\lambda, \omega). \tag{III.22}$$

Then it is clear that poles of $\hat{S}_2(\lambda, \omega)$ correspond to the solutions of $\hat{F}_2(\lambda, \omega) = -1$.

**Theorem III.3.** *Assume $\partial_\lambda \hat{K}_3(\lambda = 0, \omega = 0, \hat{\mu}(0), \hat{\mu}(0)/2, \hat{\mu}(0), \hat{\mu}(0)/2) \neq 0$. Then $\hat{F}_2(\lambda, \omega) + 1 = 0$ has one real solution $\omega_2(\lambda)$ which has the same sign as $\omega_1(\lambda)$.*

*Proof.* Consider $\hat{L}_3(\lambda, \omega, p_1, p_2)$ and $\hat{L}_3^*(\lambda, \omega, p_1, p_2) \in A_3$ (this follows from i) and ii) in (III.18).

Define

$$\begin{aligned} \hat{l}(\lambda, \omega) &= \langle (1 + \hat{V}_2(\lambda, \omega))^{-1}\hat{L}_3(\lambda, \omega), \hat{\varepsilon}_1(\lambda, \omega)\rangle, \\ \hat{l}^*(\lambda, \omega) &= \langle (1 + \hat{V}_2(\lambda, \omega)^{-1}\hat{L}_3^*(\lambda, \omega), \hat{\varepsilon}_1(\lambda, \omega)\rangle. \end{aligned}$$

We can write $\hat{B}(\lambda, \omega)$ as follows: (see (III.19) and Theorem III.2)

$$\hat{B}(\lambda, \omega) = -(\omega - \omega_1(\lambda))^{-1}\, \hat{d}(\lambda, \omega)\hat{l}(\lambda, \omega)\hat{l}^*(\lambda, \omega).$$

Let

$$\hat{Q}(\lambda, \omega) = (1+\hat{C}(\omega)\hat{D}(\lambda, \omega))^{-1}\hat{C}(\omega)\, \hat{d}(\lambda, \omega)\hat{l}(\lambda, \omega)\hat{l}^*(\lambda, \omega).$$

Then $\hat{F}_2(\lambda, \omega)+1=0$ is equivalent to

$$\hat{J}(\lambda, \omega) = \omega_1(\lambda) - \omega + \hat{Q}(\lambda, \omega) = 0$$

or, with $\hat{J}_1(\lambda, \omega) = \omega_1(\lambda) - \omega + \hat{Q}(\lambda, 0)$

$$\hat{J}_2(\lambda, \omega) = \hat{Q}(\lambda, \omega) - \hat{Q}(\lambda, 0),$$

$$\hat{J}_1(\lambda, \omega) + \hat{J}_2(\lambda, \omega) = 0.$$

The function $\hat{J}_1(\lambda, \omega)$ has a real zero at

$$\omega_{02}(\lambda) = \omega_1(\lambda) + \hat{Q}(\lambda, 0).$$

One can also verify that:

i) $\hat{l}(\lambda, 0)\hat{l}^*(\lambda, 0) = |\hat{l}(\lambda, 0)|^2 > 0$
ii) $\hat{d}(\lambda, 0) = [-\omega_1(\lambda)/\hat{G}_1(\lambda, 0)][\hat{c}^*(\lambda, 0)/\eta(\lambda)] > 0$ because $\omega_1(\lambda) > 0 \leftrightarrow \hat{G}_1(\lambda, 0) < 0$ and $\hat{c}^*(\lambda, 0) > 0$
iii) $\hat{C}(0) < 0$, $|\hat{C}(0)| \le O(1)$
iv) $(1+\hat{C}(0)\hat{D}(\lambda, 0))^{-1} > 0$, $|(1+\hat{C}(0)\hat{D}(\lambda, 0))^{-1}| \le O(1)$.

The result is that $\hat{Q}(\lambda, 0) < 0$. We shall now verify that $|Q(\lambda, 0)| < |\omega_1(\lambda)|$ so that $\omega_{02}(\lambda)$ has the same sign as $\omega_1(\lambda)$. To see this, note that, by our assumption on $\partial_\lambda K_3$, we have $\hat{G}_1(\lambda, 0) = \lambda\nu(\lambda)$, with $|\nu(\lambda)| \ge \nu_0 > 0$ for small $\lambda$. On the other hand, $|\hat{l}(\lambda, 0)| \le O(\lambda)$, so that $|\hat{Q}(\lambda, 0)| \le O(1)\, |\omega_1(\lambda)| \cdot O(\lambda^2)/\lambda = O(1)\lambda\, |\omega_1(\lambda)|$. For small $\lambda$, we have then $|Q(\lambda, 0)| < |\omega_1(\lambda)|$. Assume now that $\omega_1(\lambda) > 0$, so that $\omega_{02}(\lambda) > 0$. Consider again a semi-circle $\gamma_1 \cup \gamma_2$,

$$\gamma_1 = \{\omega : |\omega| = r, \operatorname{Re}\omega > 0\},$$

$$\gamma_2 = \{\omega : \omega = iy, -r \le y \le r\}$$

for $r$ fixed and small.

Since $\hat{Q}(\lambda, \omega)$ is $O(\lambda^2)$, on $\gamma_1$ we have:

$$|\hat{J}_2(\lambda, \omega)| < \frac{r}{2} < |\hat{J}_1(\lambda, \omega)|.$$

In addition, we have the bound for $|\omega| < r$:

$$|\hat{Q}(\lambda, \omega) - \hat{Q}(\lambda, 0)| \le O(\lambda^2\, |\omega|) \quad \text{(this follows from the holomorphy)}$$

so that on $\gamma_2$ we have:

$$|\hat{J}_2(\lambda, \omega)| < |y| < |\hat{J}_1(\lambda, \omega)|.$$

Since $|\hat{J}_2(\lambda, \omega)| < |\hat{J}_1(\lambda, \omega)|$ on $\gamma_1 \cup \gamma_2$, we conclude by Rouchè's theorem that $\hat{J}(\lambda, \omega)$ has a unique (and real) zero $\omega_2(\lambda) > 0$ inside $\gamma_1 \cup \gamma_2$. If $\omega_1(\lambda) < 0$, a similar argument shows that $\omega_2(\lambda) < 0$. □

We close this chapter by observing that the physical region of the $\omega$-plane is the half plane $\operatorname{Re}\omega > 0$. So $\omega_2(\lambda) > 0$ lies on the physical region. In order to decide whether or not this pole of $\hat{S}_2(\lambda, \omega)$ at $\omega_2(\lambda) > 0$ corresponds to a point in the mass-spectrum, we have to analyse the residue of $\hat{S}_2(\lambda, \omega)$ at $\omega = \omega_2(\lambda)$. This is done in the next chapter, together with a brief analysis of the connection of the mass-spectrum to the poles of the Schwinger functions.

## IV. The spectrum of $P_0$

We begin this chapter by showing that the residue of $\hat{S}_2(\lambda, \omega)$ at $\omega = \omega_2(\lambda)$ is not zero. Then we make the correspondence between the functions already studied and the four and six point Schwinger functions. Finally we establish the relation between Schwinger functions and points of the mass spectrum.

In this chapter we shall use the notation $k^0 = i\kappa$, and we assume that we are dealing with a model such that $\omega_2(\lambda) > 0$ (and so $\omega_1(\lambda) > 0$).

Let $m_3(\lambda)$ be defined by:

$$m_3(\lambda)^2 = (m(\lambda) + m_B(\lambda))^2 - \eta(\lambda)^2 \omega_2(\lambda)^2. \tag{IV.1}$$

Let also

$$Z_3(\lambda)^2 = \lim_{\kappa \to m_3(\lambda)} (-\kappa^2 + m_3(\lambda)^2) \tilde{S}_2(\lambda, \kappa). \tag{IV.2}$$

We have the following result:

**Proposition IV.1.** *Assume* $\hat{L}_3(\lambda, \omega = 0, \hat{\mu}(0), \hat{\mu}(0)/2) \neq 0$. *Then* $Z_3(\lambda)^2 > 0$.

*Proof.* According to the definition of $Z_3(\lambda)^2$, this is equivalent to prove that

$$2\omega_2(\lambda) \lim_{\omega \to \omega_2(\lambda)} (\omega - \omega_2(\lambda)) \hat{S}_2(\lambda, \omega) > 0.$$

Recall the notation used in the proof of Theorem III.3. We have:

$$\begin{aligned} \hat{S}_2(\lambda, \omega) &= (1 + \hat{C}(\omega)\hat{D}(\lambda, \omega))^{-1} \hat{C}(\omega) \\ &\quad \cdot \left(1 - (1 + \hat{C}(\omega)\hat{D}(\lambda, \omega))^{-1} \hat{C}(\omega) \frac{\hat{d}(\lambda, \omega)\hat{l}(\lambda, \omega)\hat{l}^*(\lambda, \omega)}{\omega - \omega_1(\lambda)}\right)^{-1} \\ &= (1 + \hat{C}(\omega)\hat{D}(\lambda, \omega))^{-1} \hat{C}(\omega) \frac{\omega_1(\lambda) - \omega}{\hat{J}(\lambda, \omega)}. \end{aligned}$$

But since $\hat{J}(\lambda, \omega)$ is holomorphic and has a zero at $\omega_2(\lambda)$, we can write:

$$\hat{J}(\lambda, \omega) = (\omega - \omega_2(\lambda))[\partial_\omega \hat{J}(\lambda, \omega_2(\lambda)) + \mathcal{O}(\omega - \omega_2(\lambda))].$$

Note that

$$\partial_\omega \hat{J}(\lambda, \omega) = -1 + \partial_\omega \hat{Q}(\lambda, \omega)$$

and that

$$|\partial_\omega \hat{Q}(\lambda, \omega_2(\lambda))| \leq \mathcal{O}(\lambda^2).$$

So, we have

$$Z_3(\lambda)^2 = 2\omega_2(\lambda)(1+\hat{C}(\omega_2(\lambda))\hat{D}(\lambda, \omega_2(\lambda)))^{-1}\hat{C}(\omega_2(\lambda))\frac{\omega_1(\lambda)-\omega_2(\lambda)}{-1+\partial_\omega Q(\lambda, \omega_2(\lambda))}.$$

Our condition on $L_3$ guarantees that $\omega_1(\lambda)-\omega_2(\lambda)>0$. On the other hand, $\hat{C}(\omega_2(\lambda))<0$, so that $Z_3(\lambda)^2>0$ as asserted. □

This result shows that $m_3(\lambda)$ is the mass of a particle, the three-particle bound-state. Even though we have not proven any $C^\infty$ property in the $\lambda$-variable, we calculate in Appendix C the first correction of $m_3(\lambda)$ with respect to $(m(\lambda)+m_B(\lambda))$. In the remainder of this chapter, we show that $\kappa_1(\lambda)^2 = (m(\lambda)+m_B(\lambda))^2-\eta(\lambda)^2\omega_1(\lambda)^2$ is not a point in the mass-spectrum. The situation resembles very much the two-particle problem in the case of a non-even model, where the pole of the 1-particle irreducible four point function is not a point in the mass-spectrum but rather induces a pole of the two point function, see [K] and [GJ3].

We shall use the following notation: points of $\mathbf{R}^2$ are denoted by $x=(x^0, x^1)$ and when the (imaginary) time component is zero, we write $\vec{x}=(0, x^1)$.

Considering products of (Euclidean) fields, we define the usual Wick-ordering and a modified one, where instead of using $C(x-y)=\int e^{ik(x-y)}(k^2+m_0^2)^{-1}\,d^2k$ in the contractions, we use $S_2(\lambda, x-y)$. We denote, for example:

$$\begin{aligned} &:\phi(x)\phi(y): = \phi(x)\phi(y)-C(x-y)\\ &\vdots\,\phi(x)\phi(y)\,\vdots = \phi(x)\phi(y)-S_2(\lambda, x-y)\end{aligned} \tag{IV.3}$$

Let

$$S_6^T(\lambda, \mathbf{x}, \mathbf{y}) = \langle \vdots\,\phi(x_1)\phi(x_2)\phi(x_3)\,\vdots\ \vdots\,\phi(y_1)\phi(y_2)\phi(y_3)\,\vdots\rangle \tag{IV.4}$$

with the usual notation $\langle\cdot\rangle$ for Schwinger functions. Let also

$$S_4^C(\lambda; x_1; x_2; x_3; z) = \langle\phi(x_1); \phi(x_2); \phi(x_3); \phi(z)\rangle_C \tag{IV.5}$$

where we use the semicolons to indicate truncation as, for instance, in [S]. In an even model,

$$\begin{aligned} &\langle\phi(x_1); \phi(x_2); \phi(x_3); \phi(z)\rangle_C\\ &\quad = \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(z)\rangle - \langle\phi(x_1)\phi(x_2)\rangle\langle\phi(x_3)\phi(z)\rangle\\ &\qquad - \langle\phi(x_1)\phi(x_3)\rangle\langle\phi(z)\phi(x_2)\rangle - \langle\phi(x_1)\phi(z)\rangle\langle\phi(x_2)\phi(x_3)\rangle.\end{aligned} \tag{IV.6}$$

Recall the definition of $R_3(\lambda, \mathbf{x}, \mathbf{y})$. We claim that

$$\begin{aligned} R_3(\lambda, \mathbf{x}, \mathbf{y}) = S_6^T(\lambda, \mathbf{x}, \mathbf{y}) - \int dz_1\, dz_2 S_4^C(\lambda, x_1; x_2; x_3; z_1)\\ \times S_2^{-1}(\lambda, z_1, z_2) S_4^C(\lambda; z_2; y_1; y_2; y_3).\end{aligned} \tag{IV.7}$$

This can be explicitly verified using (IV.6) and the definition of ⋮Wick-order⋮.

Consider also

$$\begin{aligned} &\langle P'(x)(1-\mathbf{P}_1)\phi(y_1)\phi(y_2)\phi(y_3)\rangle = \langle P'(x)\phi(y_1)\phi(y_2)\phi(y_3)\rangle\\ &\quad - \int dz_1\, dz_2 \langle P'(x)\phi(z_1)\rangle S_2^{-1}(\lambda, z_1, z_2)\langle\phi(z_2)\phi(y_1)\phi(y_2)\phi(y_3)\rangle\end{aligned} \tag{IV.8}$$

One can also verify that:

$$\langle P'(x)(1-\mathbf{P}_1)\phi(y_1)\phi(y_2)\phi(y_3)\rangle = \langle P'(x); \phi(y_1); \phi(y_2); \phi(y_3)\rangle_C$$
$$- \int dz_1\, dz_2 \langle P'(x)\phi(z_1)\rangle S_2^{-1}(\lambda, z_1, z_2)\langle\phi(z_2); \phi(y_1); \phi(y_2); \phi(y_3)\rangle_C. \qquad \text{(IV.9)}$$

We want to express the Schwinger functions $S_6^T(\mathbf{x}, \mathbf{y})$ and $S_4^C(x_1, x_2, x_3, z)$ in terms of the functions already studied. This is done in the next proposition:

**Proposition IV.2**

$$S_4^C(\lambda; x; y_1; y_2; y_3) = -(S_2 L_3 R_3)(\lambda; x, y_1, y_2, y_3)$$
$$S_6^T(\lambda, \mathbf{x}, \mathbf{y}) = R_3(\lambda, \mathbf{x}, \mathbf{y}) + (R_3 L_3 S_2 L_3^* R_3)(\lambda, \mathbf{x}, \mathbf{y})$$

*or, in momentum space, with* $k^0 = i\kappa$,

$$\tilde{S}_4^C(\lambda, \kappa, q_1, q_2) = -\tilde{S}_2(\lambda, \kappa)\int dp_1\, dp_2 L_3(\lambda, \kappa, p_1, p_2) R_3(\lambda, \kappa, -p_1, -p_2, q_1, q_2)$$

$$\tilde{S}_6^T(\lambda, \kappa, p_i, q_i) = R_3(\lambda, \kappa, p_i, q_i) + \int dp_i'\, dq_i' R_3(\lambda, \kappa, p_i, p_i')$$
$$\times L_3(\lambda, \kappa, -p_i')\tilde{S}_2(\lambda, \kappa) L_3^*(\lambda, \kappa, q_i') R_3(\lambda, \kappa, -q_i', q_i).$$

*Proof.* Recall the definition of $L_3(\lambda; x; \mathbf{y})$ to write:

$$L_3 R_3(\lambda; x; \mathbf{y}) = \lambda\langle P'(x)(1-\mathbf{P}_1)\phi(y_1)\phi(y_2)\phi(y_3)\rangle$$
$$= \lambda\langle P'(x); \phi(y_1); \phi(y_2); \phi(y_3)\rangle_C - \lambda\int dz_1\, dz_2$$
$$\times\langle P'(x)\phi(z_1)\rangle S_2^{-1}(\lambda, z_1, z_2)\langle\phi(z_2); \phi(y_1); \phi(y_2); \phi(y_3)\rangle_C$$

by (IV.9). Integrating by parts, we have:

$$\lambda(P'(x); \phi(y_1); \phi(y_2); \phi(y_3)\rangle_C$$
$$= -\int dz_2 C^{-1}(x-z_2)\langle\phi(z_2); \phi(y_1); \phi(y_2); \phi(y_3)\rangle_C$$

and

$$\lambda\int dz_1\langle P'(x)\phi(z_1) S_2^{-1}(\lambda, z_1, z_2) = K_1(\lambda, x-z_2)$$

so that

$$-L_3 R_3(\lambda; x; \mathbf{y})$$
$$= \int dz_1 (C^{-1}(x-z_2) + K_1(\lambda, x-z_2))\langle\phi(z_2); \phi(y_1); \phi(y_2); \phi(y_3)\rangle_C.$$

We conclude, then:

$$S_4^C(\lambda; x; y_1; y_2; y_3) = -S_2 L_3 R_3(\lambda, x, y_1, y_2, y_3).$$

By taking Fourier transform, we have the equation in momentum space.

Consider now

$$S_6^T(\lambda, \mathbf{x}, \mathbf{y}) = R_3(\lambda, \mathbf{x}, \mathbf{y})$$
$$+ \int dz_1\, dz_2 S_4^C(\lambda; x_1; x_2; x_3; z_1) S_2^{-1}(\lambda, z_1, z_2) S_4^C(\lambda; z_2; y_1; y_2; y_3).$$

We use the preceding result to write

$$S_6^T(\lambda, \mathbf{x}, \mathbf{y}) = R_3(\lambda, \mathbf{x}, \mathbf{y}) + (R_3 L_3 S_2 L_3^* R_3)(\lambda, \mathbf{x}, \mathbf{y}).$$

Again by taking the Fourier transform, we have the equation in momentum space. □

Consider again $\kappa_1(\lambda) = \sqrt{(m(\lambda)+m_B(\lambda))^2 - \eta(\lambda)^2\omega_1(\lambda)^2}$. Since it is a pole of $K_1(\lambda, \kappa)$, it is a zero of $\tilde{S}_2(\lambda, \kappa)$. We show in the next proposition that both $S_4^C(\lambda, \kappa)$ and $S_6^T(\lambda, \kappa)$ are bounded as $\kappa \to \kappa_1(\lambda)$. This is done in terms of the $\omega$-variable:

**Proposition IV.3.** *$\hat{S}_4^C(\lambda, \omega)$ and $\tilde{S}_6^T(\lambda, \omega)$ are bounded on a neighbourhood of $\omega_1(\lambda)$.*

*Proof.* According to the representation of $\tilde{S}_4^C$ given in Proposition IV.2, it is clear that the pole of $\hat{R}_3(\lambda, \omega)$ at $\omega = \omega_1(\lambda)$ is compensated by the zero of $\hat{S}_2(\lambda, \omega)$ at $\omega = \omega_1(\lambda)$, so that $\hat{S}_4^C(\lambda, \omega_1(\lambda))$ is finite. We analyse $\tilde{S}_6^T$ in more detail.

Let $\bar{\varepsilon}(\lambda, \omega) \in A_3^*$ be defined by

$$\bar{\varepsilon}(\lambda, \omega) = (1 + \hat{V}_2^*(\lambda, \omega))^{-1} \hat{\varepsilon}_1(\lambda, \omega)$$

so that we can write

$$\hat{R}_3(\lambda, \omega) = (\omega - \omega_1(\lambda))^{-1}\, \hat{d}(\lambda, \omega) \bar{\varepsilon}(\lambda, \omega) \langle \cdot, \bar{\varepsilon}(\lambda, \omega)\rangle + (\text{Reg}),$$

where (Reg) stands for terms which are regular at $\omega_1(\lambda)$. Let

$$\hat{l}^{(*)}(\lambda, \omega) = \langle (1 + \hat{V}_2(\lambda, \omega))^{-1} \hat{L}_3^{(*)}(\lambda, \omega), \hat{\varepsilon}_1(\lambda, \omega)\rangle = \langle \hat{L}_3^{(*)}(\lambda, \omega), \bar{\varepsilon}(\lambda, \omega)\rangle$$

with $L_3^{(*)} = L_3$ or $L_3^*$. We express the two point function as follows:

$$\begin{aligned} \hat{S}_2(\lambda, \omega) &= (1 + \hat{C}(\omega)\hat{K}_1(\lambda, \omega))^{-1} \hat{C}(\omega) \\ &= (1 - (\omega - \omega_1(\lambda))^{-1} \hat{C}(\omega)\, \hat{d}(\lambda, \omega) \hat{l}(\lambda, \omega) \hat{l}^*(\lambda, \omega) + (\text{Reg}))^{-1} \hat{C}(\omega) \end{aligned}$$

Note that $\hat{C}(\omega)\, \hat{d}(\lambda, \omega)\hat{l}(\lambda, \omega)\hat{l}^*(\lambda, \omega)$ is bounded away from zero as $\omega \to \omega_1(\lambda)$. So

$$\begin{aligned} \hat{S}_2(\lambda, \omega) &= (\omega - \omega_1(\lambda))(\hat{C}(\omega)\, \hat{d}(\lambda, \omega)\hat{l}(\lambda, \omega)\hat{l}^*(\lambda, \omega))^{-1} \\ &\quad \times [1 + (\omega - \omega_1(\lambda))(\text{Reg})]^{-1} \hat{C}(\omega) \\ &= (\omega - \omega_1(\lambda))(\hat{d}(\lambda, \omega)\hat{l}(\lambda, \omega)\hat{l}^*(\lambda, \omega))^{-1} + (\omega - \omega_1(\lambda))^2 (\text{Reg}). \end{aligned}$$

We can thus write, using Proposition IV.2:

$$\begin{aligned} \hat{S}_6^T(\lambda, \omega) &= (\omega - \omega_1(\lambda))^{-1}\, \hat{d}(\lambda, \omega)\bar{\varepsilon}(\lambda, \omega)\langle \cdot, \bar{\varepsilon}(\lambda, \omega)\rangle \\ &\quad - (\omega - \omega_1(\lambda))^{-1}\, \hat{d}(\lambda, \omega)\hat{l}(\lambda, \omega)\bar{\varepsilon}(\lambda, \omega) \cdot (\omega - \omega_1(\lambda)) \\ &\quad \times (\hat{d}(\lambda, \omega)\hat{l}(\lambda, \omega)\hat{l}^*(\lambda, \omega))^{-1} \\ &\quad \cdot (\omega - \omega_1(\lambda))^{-1}\, \hat{d}(\lambda, \omega)\hat{l}^*(\lambda, \omega)\langle \cdot, \bar{\varepsilon}(\lambda, \omega)\rangle + (\text{Reg}) \\ &= (\text{Reg}) \end{aligned}$$

and this proves the proposition. □

We next want to briefly sketch the connection between the spectrum of the energy operator and the singularities of Schwinger functions. We will see that the relevant functions to analyse in order to know the spectrum in the odd subspace up to energies of order $4m-\varepsilon$ are exactly $\tilde{S}_2(\lambda, \kappa)$, $\tilde{S}_4^C(\lambda, \kappa)$, $\tilde{S}_6^T(\lambda, \kappa)$. This result together with our preceding analysis demonstrates that $\kappa = m_3(\lambda)$ and $\kappa = m(\lambda)$ are the only points, below $m(\lambda)+m_B(\lambda)$, in the spectrum of the energy operator restricted to the odd subspace of the physical Hilbert space.

This connection is a standard result and relies on the proof by Glimm, Jaffe and Spencer [GJS1] that the subspace of energy less than $n(m(\lambda)-\varepsilon)$ is spanned, in the physical Hilbert space, by vectors of the form

$$\Omega,\ e^{tP_0}E_n\phi_j(h_j)\Omega, \qquad j \le n-1,$$

where $\Omega$ is the physical vacuum, $E_n$ is the orthogonal projection onto the subspace of energy less than $n(m(\lambda)-\varepsilon)$, $P_0$ is the energy operator, and

$$\phi_j(h_j) = \int d\vec{x}_1 \cdots d\vec{x}_j h_j(\vec{x}_1, \ldots, \vec{x}_j) :\phi(\vec{x}_1)\cdots \phi(\vec{x}_j): \tag{IV.10}$$

with $h_j \in L^2(\mathbf{R}^j)$.

In an even $P(\phi)_2$ model, this means that the odd subspace of energy less than $4(m-\varepsilon)$ is spanned by vectors of the form:

$$e^{tP_0}E_4(\phi_1(h_1)\Omega, \qquad e^{tP_0}E_4\phi_3(h_3)\Omega.$$

Since the difference

$$\int d\vec{x}_1\, d\vec{x}_2\, d\vec{x}_3 h_3(\vec{x}_1, \vec{x}_2, \vec{x}_3)(:\phi(\vec{x}_1)\phi(\vec{x}_2)\phi(\vec{x}_3): - :\phi(\vec{x}_1)\phi(\vec{x}_2)\phi(\vec{x}_3):)$$

is an element of the form $\phi_1(h_1)$, we can take the :Wick dots: in (IV.10). By taking into account the invariance of $\Omega$ and the covariance of $\phi(x)$ under the action of the Poincaré group, we can restrict ourselves to the span of

$$e^{tP_0+ixP_1}E_4\phi_1(f_1)\Omega, \qquad e^{tP_0+ixP_1}E_4\phi_3(f_3)\Omega,$$

where

$$\phi_1(f_1) = f_1\phi(0), \qquad f_1 \text{ a constant},$$

$$\phi_3(f_3) = \int d\vec{\xi}_1\, d\vec{\xi}_2 f_3(\vec{\xi}_1, \vec{\xi}_2) :\phi(0)\phi(-\vec{\xi}_1)\phi(-\vec{\xi}_1-\vec{\xi}_2):$$

and $f_3 \in L^2(\mathbf{R}^2)$, with $P_1$ the momentum operator.

We consider matrix elements of $\delta(P_1)(P_0-\kappa)^{-1}$ and a term which is analytic in $\kappa$ on a neighbourhood of the real axis for $0<\kappa<m(\lambda)+m_B(\lambda)$ will be simply denoted by (Reg). With $\theta(t, x, f) = e^{tP_0+ixP_1}E_4(\phi_1(f_1)+\phi_3(f_3))\Omega$, we have (the scalar product in the physical Hilbert space is denoted by $(\cdot, \cdot)$):

$$\begin{aligned} N(\kappa) &= (\theta(t, x, f), \delta(P_1)(P_0-\kappa)^{-1}\theta(s, y, g)) \\ &= \int_{-\infty}^{+\infty} d\tau_1' \int_0^{\infty} d\tau_0' e^{\tau_0'\kappa}(E_4(\phi_1(f_1)+\phi_3(f_3))\Omega, \\ &\quad \times e^{-(\tau_0'-t-s)P_0}e^{i(\tau_1'-x+y)P_1}E_4(\phi_1(g_1)+\phi_3(g_3))\Omega) \end{aligned}$$

$$= \int_{-\infty}^{+\infty} d\tau_1 \int_{-s-t}^{\infty} d\tau_0 e^{\tau_0\kappa}(E_4(\phi_1(f_1)+\phi_3(f_3))\Omega,$$
$$\times e^{-\tau_0 P_0 + i\tau_1 P_1} E_4(\phi_1(g_1)+\phi_3(g_3))\Omega)$$
$$= \int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\tau_0 e^{(t+s+\tau_0)\kappa}(E_4(\phi_1(f_1)+\phi_3(f_3))\Omega,$$
$$\times e^{-|\tau_0| P_0 + i\tau_1 P_1} E_4(\phi_1(g_1)+\phi_3(g_3))\Omega + (\mathrm{Reg})$$

where we have used that

$$\int_{-s-t}^{0} d\tau_0 e^{\tau_0\kappa}(E_4(\phi_1(f_1)+\phi_3(f_3))\Omega,\ e^{-\tau_0 P_0 + i\tau_1 P_1} E_4(\phi_1(g_1)+\phi_3(g_3))\Omega)$$

and

$$\int_{-\infty}^{0} d\tau_0 e^{\tau_0\kappa}(E_4(\phi_1(f_1)+\phi_3(f_3))\Omega,\ e^{-|\tau_0| P_0 + i\tau_1 P_1} E_4(\phi_1(g_1)+\phi_3(g_3))\Omega)$$

are analytic in $\kappa$ for $\mathrm{Re}\,\kappa > 0$.

Hence

$$N(\kappa) = \int_{-\infty}^{+\infty} d\tau_1 \int_{-\infty}^{+\infty} d\tau_0 e^{(t+s+\tau_0)\kappa}((\phi_1(f_1)+\phi_3(f_3))\Omega,\ e^{-|\tau_0| P_0 + i\tau_1 P_1}$$
$$\times (\phi_1(g_1)+\phi_3(g_3))\Omega) + (\mathrm{Reg})$$

where we used, with $E_4^{\perp} = 1 - E_4$, that

$$\int_{-\infty}^{+\infty} d\tau_0 e^{\tau_0\kappa}((\phi_1(f_1)+\phi_3(f_3))\Omega,\ e^{-|\tau_0| P_0 + i\tau_1 P_1} E_4^{\perp}(\phi_1(g_1)+\phi_3(g_3))\Omega)$$

is analytic in $0 < \mathrm{Re}\,\kappa < m(\lambda) + m_B(\lambda)$, since

$$|(\phi(f)\Omega,\ e^{-|\tau_0| P_0 + i\tau_1 P_1} E_4^{\perp}\phi(g)\Omega)| \le e^{-4m|\tau_0|} \|\phi(f)\Omega\|\, \|\phi(g)\Omega\|.$$

According to the Osterwalder–Schrader reconstruction theorem [OS], we get, with

$$(1 \otimes \tilde{f}_3)(p_1, p_2) = \tilde{f}_3(\vec{p}_1, \vec{p}_2) \in A_3$$

for $f_3 \in C_0^{\infty}(\mathbf{R}^2)$ and $k = (i\kappa, 0)$ the momentum conjugate to $\tau$:

$$N(\kappa) = f_1 \cdot g_1 \tilde{S}_2(\lambda, \kappa) + f_1\langle 1 \otimes \tilde{g}_3, \tilde{S}_4^C(\lambda, \kappa)\rangle$$
$$+ g_1\langle 1 \otimes \tilde{f}_3, \tilde{S}_4^C(\lambda, \kappa)\rangle + \langle 1 \otimes \tilde{f}_3, \tilde{S}_6^T(\lambda, \kappa) 1 \otimes \tilde{g}_3\rangle + (\mathrm{Reg}).$$

This proves our previous claim that the spectrum of $P_0$ can be studied by means of $\tilde{S}_2$, $\tilde{S}_4^C$ and $\tilde{S}_6^T$.

Consider now the value $\kappa = m_3(\lambda)$. By the above, it is a point on the spectrum of $P_0$ restricted to the odd subspace of $\mathscr{H}$. A standard analysis (see [K], cf. also [SZ]) shows that the number of particles with mass $m_3(\lambda)$ is bounded by the rank of the bilinear form

$$\langle f, g\rangle_0 \equiv \frac{1}{2\pi i} \oint_{\gamma} d\kappa [f_1 g_1 \tilde{S}_2(\lambda, \kappa) + f_1\langle g_3, \tilde{S}_4^C(\lambda, \kappa)\rangle_{A_3}$$
$$+ g_1\langle f_3, \tilde{S}_4^C(\lambda, \kappa)\rangle_{A_3} + \langle f_3, \tilde{S}_6^T(\lambda, \kappa) g_3\rangle_{A_3}]$$

with $f=(f_1, f_3)$, $g=(g_1, g_3)\in \mathbf{R}\times A_3$ and $\gamma$ a simple curve around $m_3(\lambda)$ in the complex $\kappa$-plane. Using Proposition IV.2, the fact that $R_3(\lambda, \kappa)$ is holomorphic on a neighbourhood of $\kappa = m_3(\lambda)$ and that

$$Z_3(\lambda)^2 = \frac{1}{2\pi i}\oint_\gamma d\kappa \tilde{S}_2(\lambda, \kappa) \neq 0,$$

we get:

$$\begin{aligned}\langle f, g\rangle_0 &= Z_3(\lambda)^2[f_1 g_1 - f_1\langle g_3, \varphi\rangle_{A_3} - g_1\langle f_3, \varphi\rangle_{A_3} + \langle g_3, \varphi\rangle_{A_3}\langle f_3, \varphi\rangle_{A_3}]\\ &= Z_3(\lambda)^2(f_1 - \langle f_3, \varphi\rangle_{A_3})(g_1 - \langle g_3, \varphi\rangle_{A_3}),\end{aligned}$$

with $\varphi(\lambda, p_1, p_2) = \int dp_i' R_3(\lambda, m_3, p_i, p_i')L_3(\lambda, m_3, -p_i') \in A_3^*$. We conclude that $\langle f, g\rangle_0 \neq 0$ unless $\langle f, f\rangle_0$ or $\langle g, g\rangle_0 = 0$, that is, $\langle \cdot, \cdot\rangle_0$ is rank one. As a consequence, we see that there is only one particle with mass $m_3(\lambda)$ associated to the corresponding point on the energy spectrum.

## V. Examples

In this chapter, we consider a class of models which can be discussed by our method. This class of even theories is the following:

$$P(\phi) = -\phi^4 + a_6\phi^6 + \sum_{n=4}^{N} a_{2n}\phi^{2n}, \qquad a_{2n} > 0. \tag{V.1}$$

The existence of a two-particle bound state for these models has been established in [DE], so that our discussion applies. We next verify that the assumption in Proposition IV.1 about $\hat{L}_3$ is always satisfied, namely:

**Proposition V.1.** *Let $P(\phi)$ be as in* (V.1). *Then*

$$\frac{1}{\lambda}L_3(\lambda, \kappa, p, q) = -\frac{4}{(2\pi)^2} + \mathcal{O}(\lambda)$$

*Proof*

$$\frac{1}{\lambda}L_3(\lambda; x; \mathbf{y}) = \int d\mathbf{y}'\langle P'(x)(1-\mathbf{P}_1)\phi(y_1')\phi(y_2')\phi(y_3')\rangle R_3^{-1}(\lambda, \mathbf{y}', \mathbf{y})$$

We calculate the lowest order in $\lambda$ of the integrand. Clearly,

$$\begin{aligned}&\langle P'(x)\mathbf{P}_1\phi(y_1')\phi(y_2')\phi(y_3')\rangle\\ &\quad = \int dz_1\, dz_2 \langle P'(x)\phi(z_1)\rangle S_2^{-1}(z_1, z_2)\langle \phi(z_2)\phi(y_1')\phi(y_2')\phi(y_3')\rangle\end{aligned}$$

is zero in lowest order, since $\langle P'(x)\phi(z_1)\rangle$ is zero in lowest order. In the other term, only $:\phi^4:$ contributes in this order. We denote a term of order zero in $\lambda$ by $\langle\cdot\rangle_0$ or $X^{(0)}$.

$$-4\int d\mathbf{y}'\langle:\phi^3(x):\ \phi(y_1')\phi(y_2')\phi(y_3')\rangle_0(R_3^{(0)})^{-1}(\mathbf{y}',\mathbf{y})$$

$$=-4\cdot\frac{3!}{6}\int d\mathbf{y}'C(x-y_1')C(x-y_1')C(x-y_3')C^{-1}(y_1'-y_1)C^{-1}(y_2'-y_2)C^{-1}(y_3'-y_3)$$

$$=-4\delta(x-y_1)\,\delta(y_1-y_2)\,\delta(y_2-y_3).$$

By taking the Fourier transform, we obtain our result. □

Having a $:\phi^6:$ term in the polynomial simplifies the analysis because in this case we have a first order contribution to $K_3$. Our result is the following:

**Theorem V.1.** *Let $P(\phi)$ be as in* (V.1). *If $a_6<0$ there is a three-particle bound state near (and below) the threshold $m(\lambda)+m_B(\lambda)$. If $a_6>0$ no such bound state occurs.*

*Proof.* Our discussion in Chapters III and IV shows that the pole of $\tilde{S}_2(\lambda,\kappa)$ is in the physical sheet of the energy-plane if $\omega_2(\lambda)>0$. On the other hand, $\omega_2(\lambda)>0$ is a consequence of $\hat{K}_3(\lambda, 0, \hat{\mu}(0), \hat{\mu}(0)/2, \hat{\mu}(0), \hat{\mu}(0)/2)<0$. This is in turn equivalent, for small $\lambda$, to the condition $\partial_\lambda\hat{K}_3(0, 0, \hat{\mu}(0), \hat{\mu}(0)/2, \hat{\mu}(0), \hat{\mu}(0)/2)<0$. Clearly, $\partial_\lambda K_3(\lambda=0)$ is the first order contribution to $K_3(\lambda)$. The $:\phi^6:$ term in $P(\phi)$ guarantees that $\partial_\lambda K_3(\lambda=0)\neq 0$. We shall see that $\partial_\lambda K_3(\lambda=0)$ is in the form of a positive constant times $a_6$, so that the sign of $a_6$ decides on the presence or absence of three-particle bound states as asserted in the theorem.

Let $R_0(\mathbf{x},\mathbf{y})=6C(x_1-y_1)C(x_2-y_2)C(x_3-y_3)$ and let $X^{(1)}$ denote the first order contribution to $X(\lambda)$. We have:

$$K_3^{(1)}=R_0^{-1}(R_{2B}^{(1)}-R_3^{(1)})R_0^{-1}$$

$$R_{2B}^{(1)}=-R_0M^{(1)}R_0$$

$$M^{(1)}(\mathbf{x},\mathbf{y})=\tfrac{1}{3}(C^{-1}(x_1-y_1)K_2^{(1)}(x_2,x_3,y_2,y_3)$$
$$+C^{-1}(x_2-y_2)K_2^{(1)}(x_1,x_3,y_1,y_3)+C^{-1}(x_3-y_3)K_2^{(1)}(x_1,x_2,y_1,y_2)).$$

Because of the $-:\phi^4:$ term, we can write:

$$K_2^{(1)}(x_i,x_j,y_i,y_j)=-6\delta(x_i-x_j)\,\delta(x_j-y_i)\,\delta(y_i-t_j).$$

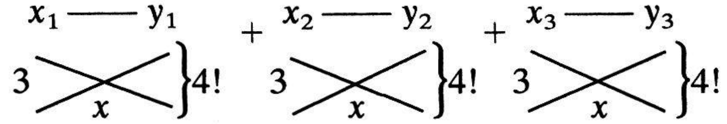
Note that the $:\phi^6:$ term does not contribute in first order. We conclude:

$$R_0^{-1}R_{2B}^{(1)}R_0^{-1}(\mathbf{x},\mathbf{y})=2(C^{-1}(x_1-y_1)\,\delta(x_2-x_3)\,\delta(x_3-y_2)\,\delta(y_2-y_3)$$
$$+C^{-1}(x_2-y_2)\,\delta(x_1-x_3)\,\delta(x_3-y_1)\,d(y_1-y_3)$$
$$+C^{-1}(x_3-y_3)\,\delta(x_1-x_2)\,\delta(x_2-y_1)\,\delta(y_1-y_2)) \qquad \text{(V.2)}$$

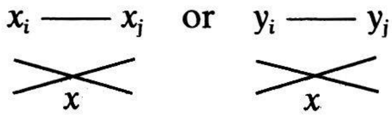
We next calculate $R_3^{(1)}$:

$$R_3^{(1)}(\mathbf{x}, \mathbf{y}) = \int dx \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(y_1)\phi(y_2)\phi(y_3)(:\phi^4(x): - a_6 :\phi^6(x):)\rangle_0$$
$$- \int dz_1\, dz_2\, dx \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(z_1) :\phi^4(x):\rangle_0$$
$$\times C^{-1}(z_1, z_2)\langle \phi(z_2)\phi(y_1)\phi(y_2)\phi(y_3)\rangle_0$$
$$- \int dz_1\, dz_2\, dx \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(z_1)\rangle_0$$
$$\times C^{-1}(z_1, z_2)\langle \phi(z_2)\phi(y_1)\phi(y_2)\phi(y_3) :\phi^4(x):\rangle_0$$

Note that there is no contribution in first order to $S_2^{-1}(x, y)$. The contribution of the terms in $:\phi^4:$ can be written, in graphical language:

$\bar{R}_3^{(1)}(\mathbf{x}, \mathbf{y}) =$  (V.3)


since we can see that all contributions of the form



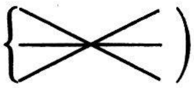

are cancelled.

If we now amputate each term in (V.3), we have:

$$R_0^{-1}\bar{R}_3^{(1)}R_0^{-1}(\mathbf{x}, \mathbf{y}) = \frac{3 \cdot 4!}{6 \cdot 6}(C^{-1}(x_1 - y_1)\,\delta(x_2 - x_3)\,\delta(x_3 - y_2)\,\delta(y_2 - y_3)$$
$$+ C^{-1}(x_2 - y_2)\,\delta(x_1 - x_3)\,\delta(x_3 - y_1)\,\delta(y_1 - y_3)$$
$$+ C^{-1}(x_3 - y_3)\,\delta(x_1 - x_2)\,\delta(x_2 - y_1)\,\delta(y_1 - y_2))$$

which exactly cancels (V.2).

As expected, the $:\phi^4:$ term does not contribute to $K_3^{(1)}$. So the only contribution comes from

$\bar{\bar{R}}_3^{(1)}(\mathbf{x}, \mathbf{y}) = -a_6 \cdot \Big(6!\,\Big\{$  $\Big)$

and

$$K_3^{(1)} = -R_0^{-1}\bar{\bar{R}}_3^{(1)}R_0^{-1}$$
$$= a_6 \cdot \frac{6!}{6 \cdot 6}\,\delta(x_1 - x_2)\,\delta(x_2 - x_3)\,\delta(x_3 - y_1)\,\delta(y_1 - y_2)\,\delta(y_2 - y_3).$$

The Fourier transform reads:

$$\tilde{K}_3^{(1)}(\kappa, p_i, q_i) = \frac{20}{(2\pi)^3} \cdot a_6$$

and the theorem follows. □

## Appendix A

To obtain $R_{03}(\lambda, k, p_i, q_i)$:

$$R_{03}(\lambda, k, p_i, q_i) = \frac{6}{(2\pi)^5} \int d\tau\, d\xi_i\, d\eta_i$$
$$\times \exp\left[i(k\tau + p_1\xi_1 + p_2\xi_2 + q_1\eta_1 + q_2\eta_2)\right]$$
$$\times R_1(\lambda, x_1 - y_1) R_1(\lambda, x_2 - y_2) R_1(\lambda, x_3 - y_3).$$

One changes to the variables

$$\begin{array}{lcl} x_1 - y_1 = \omega_1 & & \tau = \frac{1}{3}(\omega_1 + \omega_2 + \omega_3) \\ x_2 - y_2 = \omega_2 & \text{with inverse} & \xi_1 = \frac{1}{2}(\omega_1 - \omega_2 + \omega_3) \\ x_3 - y_3 = \omega_3 & \text{relations} & \xi_2 = \frac{1}{2}(\omega_2 - \omega_3 + \omega_5) \\ \quad \omega_4 = \xi_1 + \eta_1 & & \eta_1 = \frac{1}{2}(-\omega_1 + \omega_2 + \omega_4) \\ \quad \omega_5 = \xi_2 + \eta_2 & & \eta_2 = \frac{1}{2}(-\omega_2 + \omega_3 + \omega_5) \end{array}$$

the Jacobian is $\frac{1}{4}$, so that:

$$R_{03}(\lambda, k, p_i, q_i) = \frac{6}{(2\pi)^5} \cdot \frac{1}{4} \int d\omega_1 \cdots d\omega_5 R_1(\lambda, \omega_1) R_1(\lambda, \omega_2) R_1(\lambda, \omega_3)$$
$$\times \exp\left[i\left(\omega_1\left(\frac{k}{3} + \frac{p_1}{2} - \frac{p_1}{2}\right) + \omega_2\left(\frac{k}{3} - \frac{p_1}{2} + \frac{q_1}{2} + \frac{p_2}{2} - \frac{q_2}{2}\right)\right.\right.$$
$$\left.\left. + \omega_3\left(\frac{k}{3} - \frac{p_2}{2} + \frac{q_2}{2}\right) + \omega_4\left(\frac{p_1}{2} + \frac{q_1}{2}\right) + \omega_5\left(\frac{p_2}{2} + \frac{q_2}{2}\right)\right)\right]$$

Now, $(1/2\pi) \int d\omega_1 e^{i\omega_1 Q} R_1(\omega_1) = \tilde{R}_1(Q)$. So,

$$R_{03}(\lambda, k, p_i, q_i) = 6 \cdot \frac{1}{4} R_1\left(\lambda, \frac{k}{3} - \frac{p_1}{2} - \frac{q_1}{2}\right)$$
$$\times R_1\left(\lambda, \frac{k}{3} - \frac{p_1}{2} + \frac{q_1}{2} + \frac{p_2}{2} - \frac{q_2}{2}\right)$$
$$\times R_1\left(\lambda, \frac{k}{3} - \frac{p_2}{2} + \frac{q_2}{2}\right) \cdot \frac{1}{(2\pi)^2}$$
$$\times \int d\omega_4\, d\omega_5 \exp\left[i\left(\frac{\omega_4}{2}(p_1 + q_1) + \frac{\omega_5}{2}(p_2 + q_2)\right)\right]$$

The last expression is equal to $4 \cdot 4\pi^2\, \delta(p_1 + q_1)\, \delta(p_2 + q_2)$ and the final expression for $R_{03}$ is:

$$R_{03}(\lambda, k, p_i, q_i) = 6 \cdot 4\pi^2\, \delta(p_1 + q_1) \delta(p_2 + q_2) R_1\left(\lambda, \frac{k}{3} + p_1\right)$$
$$\times R_1\left(\lambda, \frac{k}{3} - p_1 + p_2\right) R_1\left(\lambda, \frac{k}{3} - p_2\right).$$

To obtain $M_{11}(\lambda, k, p_i, q_i)$ (it is the same for $b_{11}(\lambda, k, p_i, q_i)$):

$$M_{11}(\lambda, k, p_i, q_i)=\frac{1}{9}\cdot\frac{1}{(2\pi)^3}\int d\tau\, d\xi_i\, d\eta_i e^{i(k\tau+p_1\xi_1+\cdots)}$$
$$\times R_1^{-1}(\lambda, x_1-y_1)K_2(\lambda, x_2, x_3, y_2, y_3).$$

Change to the variables

$$\begin{aligned}
x_1-y_1&=\omega_1 & & & \tau&=\tfrac{1}{2}(2\omega_4+\omega_1)\\
x_2-x_3&=\omega_2 & &\text{with inverse} & \xi_1&=\tfrac{1}{2}(\omega_1-\omega_2-2\omega_4+\omega_5)\\
y_2-y_3&=\omega_3 & &\text{relations} & \xi_2&=\omega_2\\
\tfrac{1}{2}(x_2+x_3-y_2-y_3)&=\omega_4 & & & \eta_1&=\tfrac{1}{2}(-\omega_1-\omega_3+\omega_5)\\
x_1+y_1-y_2-y_3&=\omega_5 & & & \eta_2&=\omega_3
\end{aligned}$$

The Jacobian is $\frac{1}{2}$.

Using that

$$\tilde{R}_1^{-1}(\lambda, Q)=\frac{1}{2\pi}\int e^{i\omega_1 Q}R_1^{-1}(\lambda, \omega_1),$$

$$\tilde{K}_2(\lambda, k', p, q)=\frac{1}{2\pi}\int d\omega_2\, d\omega_3\, d\omega_4 e^{i(k'\omega_4+p\omega_3+q\omega_2)}K_2(\lambda, \omega_4, \omega_3, \omega_2)$$

we have:

$$M_{11}(\lambda, k, p_i, q_i)=\frac{1}{9}\cdot\frac{1}{2}\cdot\frac{1}{2\pi}R_1^{-1}\left(\lambda, \frac{k}{3}+\frac{p_1}{2}-\frac{q_1}{2}\right)$$
$$\times K_2\left(\lambda, \frac{2k}{3}-p_1, \frac{-p_1}{2}+p_2, \frac{-q_1}{2}+q_2\right)\int d\omega_5 e^{i(\omega_5/2)(p_1+q_1)}$$
$$=\frac{2\pi}{9}\delta(p_1+q_1)R_1^{-1}\left(\lambda, \frac{k}{3}+p_1\right)K_2\left(\lambda, \frac{2k}{3}-p_1, \frac{-p_1}{2}+p_2, \frac{p_1}{2}+q_2\right).$$

## Appendix B

As explained in the text, $\omega=x+iy$ and we will prove our result for $|x|<\frac{5}{2}m'(\lambda)$, no restriction on $y$. This region contains $\hat{\mathscr{B}}_C([\delta_1^{(3)}-2\varepsilon]/\eta(\lambda))$.

Write $s(\omega)$ in the form:

$$s(\omega)=\frac{4m'm_B'-\omega^2}{4(m'+m_B')^2-\omega^2}=\frac{4m'm_B'-x^2+y^2-2ixy}{4(m'+m_B')^2-x^2+y^2-2ixy}$$

We show, first of all, that $|s(\omega)|<1$. To this end, we will compare the modulus of both terms:

$$|4m'm_B'-x^2+y^2-2ixy|^2=(4m'm_B'-x^2+y^2)^2+4x^2y^2$$
$$|4(m'+m_B')^2-x^2+y^2-2ixy|^2=(4(m'+m_B')^2-x^2+y^2)^2+4x^2y^2.$$

We have therefore:

$$|4m'm_B'-x^2+y^2-2ixy|^2-|4(m'+m_B')^2-x^2+y^2-2ixy|^2$$
$$=(4m'm_B'-x^2+y^2)^2-(4(m'+m_B')^2-x^2+y^2)^2.$$

But, for $|x|<2\sqrt{m'm_B'}$,

$$4m'm_B'-x^2+y^2<4(m'+m_B')^2-x^2+y^2,$$

so that

$$|4m'm_B'-x^2+y^2-2ixy|^2-|4(m'+m_B')^2-x^2+y^2-2ixy|^2<0$$

and the result follows.

We continue our analysis writing:

$$s(\omega)=\frac{(4m'm_B'-x^2y^2)(4(m'+m_B')^2-x^2+y^2)+4x^2y^2-8ixy(m'^2+m_B'^2+m'm_B')}{(4(m'+m_B')^2-x^2+y^2)^2+4x^2y^2} \tag{B.1}$$

From this formula, we can also see that $|s(\omega)|>s_0$, since the real part of $s(\omega)$ never vanishes (and is positive) if $|x|<2\sqrt{m'm_B'}$. Considering the region $|x|<\frac{5}{2}m'$, we have our result, since $2\sqrt{m'm_B'}\sim 2\sqrt{2}m'>\frac{5}{2}m'$.

We note again that the real part of $s(\omega)$ is always positive, and that we choose the determination of $s^{1/2}(\omega)=u+iv$ which has $u>0$. This implies that $v$ has the same sign as the imaginary part of $s(\omega)$.

Looking at (B.1), we see that $\operatorname{Im} s(\omega)=-xyf(x,y)$, where $f(x,y)$ is always positive. It then follows that:

i) if $x>0$, $\operatorname{Im} s(\omega)$ has the opposite sign of $y$, and $vy<0$.
ii) if $x<0$, $\operatorname{Im} s(\omega)$ and $y$ have the same sign, and $vy>0$.

This completes our proof.

## Appendix C

In this appendix we calculate the lowest order in $\lambda$ of $\eta(\lambda)^2\omega_2(\lambda)^2$. Given that $m_3(\lambda)=\sqrt{(m(\lambda)+m_B(\lambda))^2-\eta(\lambda)^2\omega_2(\lambda)^2}$, we can write

$$m_3(\lambda)=m(\lambda)+m_B(\lambda)-\frac{\eta(\lambda)^2\omega_2(\lambda)^2}{2(m+m_B)}+O((\eta(\lambda)\omega_2(\lambda))^4).$$

We will see that $\eta(\lambda)^2\omega_2(\lambda)^2\sim O(\lambda^4)$, but this is not the only term contributing to $\lambda^4$: there is a $\lambda^3$ and a $\lambda^4$-term coming from $m_B(\lambda)$. Since these terms come entirely from the two-body problem, we will not calculate them here (in fact, calculate the $\lambda^4$-term in $m_B(\lambda)$ requires the knowledge of $\zeta_1'(0)$, $\zeta_1''(0)$ and $\zeta_1'''(0)$ with $\zeta_1(\lambda)$ the pole of $\hat{R}(\lambda,\zeta)$; this is quite a long calculation).

We come thus to calculate $\omega_2(\lambda)$. The first order in $\lambda$ is given by $\partial_\lambda\omega_2(0)$, that is, $\omega_2(\lambda)=\partial_\lambda\omega_2(0)\cdot\lambda+O(\lambda^2)$ (it is clear that $\omega_2(0)=0$). Recall now the notation in the proof of Theorem III.3. We have that $\hat{J}(\lambda,\omega_2(\lambda))=0$, so that

$$\partial_\lambda\omega_2(0)=-\frac{\partial_\lambda\hat{J}(0,0)}{\partial_\omega\hat{J}(0,0)}$$

But $\hat{J}(\lambda, \omega) = \omega_1(\lambda) - \omega + \hat{Q}(\lambda, \omega)$, and

$$\partial_\omega \hat{J}(0,0) = -1 + \partial_\omega \hat{Q}(0,0) = -1.$$

So $\partial_\lambda \omega_2(0) = \partial_\lambda \hat{J}(0,0) = \partial_\lambda \omega_1(0) + \partial_\lambda \hat{Q}(0,0)$. But $\hat{Q}(\lambda,\omega) = O(\lambda^2)$, so that $\partial_\lambda \hat{Q}(0,0) = 0$.

The problem is then to calculate $\partial_\lambda \omega_1(0)$. Recall now the notation in the proof of Theorem III.1. We have that $\omega_1(\lambda)$ is the solution of $\hat{H}(\lambda, \omega_1(\lambda)) = 0$ where $\hat{H}(\lambda,\omega) = \omega + \hat{G}_1(\lambda,\omega)$

$$\hat{G}_1(\lambda,\omega) = \frac{\hat{c}^*(\lambda,\omega)}{\eta(\lambda)} \widehat{\varepsilon_1\psi_0}(\hat{\mu}(\omega)$$

$$\hat{c}^*(\lambda,\omega) = 6Z(\lambda)^2 \zeta_1(\lambda) \hat{r}(\lambda, \zeta_1(\lambda)) \hat{t}_0(\lambda,\omega)$$

$$\begin{aligned}\widehat{\varepsilon_1\psi_0}(\hat{\mu}(\omega)) = \int & dp_2\, dq_2\, dp_1'\, dp_2' \hat{H}(\lambda, \zeta_1(\lambda), p_2) \\ & \times (1+\hat{V}_2)^{-1}\left(\lambda, \omega, \hat{\mu}(\omega), \frac{\hat{\mu}(\omega)}{2} - p_2, p_1', -p_2'\right) \\ & \times \hat{K}_3\left(\lambda, \omega, p_1', p_2', \hat{\mu}(\omega), \frac{\hat{\mu}(\omega)}{2} + q_2\right) \hat{H}(\lambda, \zeta_1(\lambda), q_2).\end{aligned}$$

Again we have:

$$\partial_\lambda \omega_1(0) = \frac{\partial_\lambda \hat{H}(0,0)}{\partial_\omega \hat{H}(0,0)} = -\partial_\lambda \hat{H}(0,0) = -\partial_\lambda \hat{G}_1(0,0).$$

But

$$\partial_\lambda \hat{G}_1(0,0) = \lim_{\lambda\to 0} \frac{\hat{c}^*(\lambda,0)}{\eta(\lambda)} \partial_\lambda K_3\left(\lambda, \kappa(\lambda), \mu(\kappa(\lambda)), \frac{\mu(\kappa(\lambda))}{2}, \mu(\kappa(\lambda)), \frac{\mu(\kappa(\lambda))}{2}\right),$$

$$\kappa(\lambda) = m(\lambda) + m_B(\lambda)$$

One can verify that

i) $\lim_{\lambda\to 0} \zeta_1(\lambda)/\eta(\lambda) = \sqrt{\tfrac{2}{3}}$
ii) $\hat{r}(0,0) = \pi/m_0$, $Z(0)^2 = 1$.
iii) $\hat{t}_0(0,0) = \pi^2/\sqrt{2}m_0$

where $m_0 = m(0)$. Note that $\kappa(\lambda) \to 3m_0$ as $\lambda \to 0$, and that $\mu(3m_0) = 0$.

Let $\alpha_2 = \partial_\lambda K_3(0, 3m_0, 0, 0, 0, 0)$. Putting everything together:

$$\partial_\lambda \omega_1(0) = 6 \cdot \sqrt{\frac{2}{3}} \frac{\pi}{m_0} \frac{\pi^2}{\sqrt{2}m_0} \alpha_2 = \frac{6}{\sqrt{3}} \frac{\pi^3}{m_0^2} \alpha_2$$

Note now that

$$\eta(\lambda) = \sqrt{\tfrac{3}{2}} \zeta_1(\lambda) + \mathcal{O}(\zeta_1(\lambda)^2) = -\sqrt{\frac{3}{2}} \frac{\pi}{m_0} \alpha_1 \cdot \lambda + \mathcal{O}(\lambda^2)$$

where $\alpha_1 = 3\, \partial_\lambda K_2(0, 2m_0, 0, 0)$. The net result is that

$$\eta(\lambda)\omega_2(\lambda) = -\sqrt{\frac{3}{2}} \frac{\pi}{m_0} \alpha_1 \cdot \frac{6}{\sqrt{3}} \frac{\pi^3}{m_0^2} \alpha_2 \lambda^2 + \mathcal{O}(\lambda^3)$$

and

$$m_3(\lambda)=(m(\lambda)+m_B(\lambda))\left(1-9\frac{\pi^8}{m_0^6}\alpha_1^2\alpha_2^2\frac{\lambda^4}{(m(\lambda)+m_B(\lambda))^2}+\mathcal{O}(\lambda^8)\right)$$
$$=(m(\lambda)+m_B(\lambda))\left(1-\frac{\pi^8}{m_0^8}\alpha_1^2\alpha_2^2\lambda^4+\mathcal{O}(\lambda^6)\right).$$

## Appendix D

In this appendix, we show how to isolate the local singularities of $K_3=R_3^{-1}-R_{2B}^{-1}$. It turns out that $K_3(\lambda,\mathbf{x},\mathbf{y})$ can be defined by a convergent (for $\lambda$ small) Neumann series and is locally regular in the sense that it has at most $\delta$-functions singularities. It then follows that its Fourier transform, in the $k$, $p_i$, $q_i$ variables, is a bounded function (integrability at infinity in position space follows from the exponential decay in the difference variables $\tau$, $\xi_i$, $\eta_i$).[7] The same local regularity property holds for $L_3(\lambda; x, \mathbf{y})$ and $D_2(\lambda; x, y)$ and they are therefore bounded in momentum space.

We begin by considering again (see (IV.7)):

$$R_3(\lambda,\mathbf{x},\mathbf{y})=\langle\phi(x_1)\phi(x_2)\phi(x_3)(1-\mathbf{P}_1)\phi(y_1)\phi(y_2)\phi(y_3)\rangle$$
$$=\langle:\phi(x_1)\phi(x_2)\phi(x_3):\ :\phi(y_1)\phi(y_2)\phi(y_3):\rangle$$
$$-\int dz_1\,dz_2\langle\phi(x_1);\phi(x_2);\phi(x_3);\phi(z_1)\rangle_C$$
$$\times\Gamma(z_1,z_2)\langle\phi(z_2);\phi(y_1);\phi(y_2);\phi(y_3)\rangle_C$$

where we have used the notation:

$$\Gamma(z_1,z_2)=S_2^{-1}(\lambda,z_1,z_2).$$

The technique is to isolate the singularities of $C_3^{-1}R_3$ using integration by parts, where $C_3^{-1}=C^{-1}\otimes C^{-1}\otimes C^{-1}$, $C(x,y)$ the free covariance.

We consider each term separately. We use the notation $C_{x_1}^{-1}\langle\phi(x_1)\cdots\rangle$ to denote $\int dx_1' C^{-1}(x_1,x_1')\langle\phi(x_1')\cdots\rangle$. We also use

$$\langle x_1x_2\cdots x_n\rangle\equiv\langle\phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle\quad\text{and}\quad P(x)\equiv P(\phi(x)).$$

Our first result is:

$$C_{x_3}^{-1}C_{x_2}^{-1}C_{x_1}^{-1}\langle x_1;x_2;x_3;z\rangle_C=-\lambda^3(P'(x_1);P'(x_2);P'(x_3);z\rangle_C$$
$$+\lambda^2\sum_{i=1}^{3}\delta(x_{\alpha_1}-x_{\alpha_2})\langle P'(x_i);P''(x_{\alpha_1});z\rangle_C \tag{D.1}$$
$$-\lambda\,\delta(x_2-x_1)\,\delta(x_3-x_2)\langle P'''(x_1)z\rangle$$

where $\alpha=\{1,2,3\}\backslash\{i\}$.

---

[7] To be precise, we should prove the local regularity properties for the kernel $K_3(\lambda,t,\mathbf{x},\mathbf{y})$, defined with the covariance $C(t,x,y)$ used by Spencer [S] in its modified cluster expansion. We omit the index $t$, since it does not play a role in the proof of local regularity.

This formula, as well as those derived below, is obtained by repeated use of integration by parts. We give the corresponding result on $C_{x_3}^{-1}C_{x_2}^{-1}C_{x_1}^{-1}\langle\vdots x_1x_2x_3\vdots\ \vdots y_1y_2y_3\vdots\rangle$ in three steps.

i) $C_{x_1}^{-1}\langle\vdots x_1x_2x_3\vdots\ \vdots y_1y_2y_3\vdots\rangle$

$$= C_{x_1}^{-1}R_{03}(\lambda,\mathbf{x},\mathbf{y}) + \sum_{i=1}^{3} C_{x_1}^{-1}\langle x_1y_i\rangle\langle x_2;\, x_3;\, y_{\alpha_1};\, y_{\alpha_2}\rangle_C$$

$$+\lambda\sum_{i=1}^{3}\langle P'(x_1)y_i\rangle(\langle x_2;\, x_3;\, y_{\alpha_1};\, y_{\alpha_2}\rangle_C+\langle x_2y_{\alpha_1}\rangle\langle x_3y_{\alpha_2}\rangle+\langle x_2y_{\alpha_2}\rangle\langle x_3y_{\alpha_1}\rangle)$$

$$+\sum_{i=1}^{3} C_{x_1}^{-1}\langle x_1;\, x_2;\, x_3;\, y_i\rangle_C\langle y_{\alpha_1}y_{\alpha_2}\rangle$$

$$-\lambda\langle P'(x_1)x_2x_3y_1y_2y_3\rangle+\lambda\langle P'(x_1)x_2\rangle\langle x_3y_1y_2y_3\rangle$$

$$+\lambda\langle P'(x_1)x_3\rangle\langle x_2y_1y_2y_3\rangle+\lambda\langle x_2x_3\rangle\langle P'(x_1)y_1y_2y_3\rangle.$$

ii) $C_{x_2}^{-1}C_{x_1}^{-1}\langle\vdots x_1x_2x_3\vdots\ \vdots y_1y_2y_3\vdots\rangle$

$$= C_{x_2}^{-1}C_{x_1}^{-1}\left(R_{03}(\lambda,\mathbf{x},\mathbf{y})+\sum_{i=1}^{3}\langle x_1y_i\rangle\langle x_2;\, x_3;\, y_{\alpha_1};\, y_{\alpha_2}\rangle_C\right)$$

$$+\sum_{i=1}^{3} C_{x_2}^{-1}C_{x_1}^{-1}\langle x_2y_i\rangle\langle x_1;\, x_3;\, y_{\alpha_1};\, y_{\alpha_2}\rangle_C$$

$$+\sum_{i=1}^{3} C_{x_2}^{-1}C_{x_1}^{-1}\langle x_1;\, x_2;\, x_3;\, y_i\rangle_C\langle y_{\alpha_1}y_{\alpha_2}\rangle$$

$$+\lambda\sum_{i=1}^{3}(\langle P'(x_1)y_i\rangle C_{x_2}^{-1}\langle x_2;\, x_3;\, y_{\alpha_1};\, y_{\alpha_2}\rangle_C+\langle P'(x_2)y_i\rangle\langle x_1;\, x_3;\, y_{\alpha_1};\, y_{\alpha_2}\rangle_C)$$

$$-\lambda\,\delta(x_1-x_2)(\langle P''(x_1)x_3y_1y_2y_3\rangle-\langle P''(x_1)\rangle\langle x_3y_1y_2y_3\rangle)$$

$$+\lambda^2(\langle P'(x_1)P'(x_2)x_3y_1y_2y_3\rangle-\langle P'(x_1)P'(x_2)\rangle\langle x_3y_1y_2y_3\rangle$$

$$-\langle P'(x_2)x_3\rangle\langle P'(x_1)y_1y_2y_3\rangle-\langle P'(x_1)x_3\rangle\langle P'(x_2)y_1y_2y_3\rangle)$$

$$-\lambda^2\sum_{i=1}^{3}\langle P'(x_1)y_i\rangle(\langle P'(x_2)y_{\alpha_1}\rangle\langle x_3y_{\alpha_2}\rangle+\langle P'(x_2)y_{\alpha_2}\rangle\langle x_3y_{\alpha_1}\rangle)$$

iii) $C_{x_3}^{-1}C_{x_2}^{-1}C_{x_1}^{-1}\langle\vdots x_1x_2x_3\vdots\ \vdots y_1y_2y_3\vdots\rangle$

$$= C_{x_3}^{-1}C_{x_2}^{-1}C_{x_1}^{-1}R_{03}(\lambda,\mathbf{x},\mathbf{y})$$

$$+\sum_{i,j=1}^{3} C_{x_3}^{-1}C_{x_2}^{-1}C_{x_1}^{-1}\langle x_iy_j\rangle\langle x_{\alpha_1};\, x_{\alpha_2};\, y_{\beta_1};\, y_{\beta_2}\rangle_C$$

$$+\sum_{i=1}^{3} C_{x_3}^{-1}C_{x_2}^{-1}C_{x_1}^{-1}\langle x_1;\, x_2;\, x_3;\, y_i\rangle_C\langle y_{\alpha_1}y_{\alpha_2}\rangle$$

$$+\lambda\sum_{i=1}^{3}(\langle P'(x_1)y_i\rangle C_{x_2}^{-1}C_{x_3}^{-1}\langle x_2;\, x_3;\, y_{\alpha_1};\, y_{\alpha_2}\rangle_C$$

$$+\langle P'(x_2)y_i\rangle C_{x_3}^{-1}C_{x_1}^{-1}\langle x_1;\, x_3;\, y_{\alpha_1};\, y_{\alpha_2}\rangle_C$$

$$+\langle P'(x_3)y_i\rangle C_{x_2}^{-1}C_{x_1}^{-1}\langle x_1;\, x_2;\, y_{\alpha_1};\, y_{\alpha_2}\rangle_C)$$

$$-\lambda\,\delta(x_1-x_2)\,\delta(x_2-x_3)\langle P'''(x_1)y_1y_2y_3\rangle$$

$$+\lambda^2 \sum_{i=1}^{3} \delta(x_{\alpha_1}-x_{\alpha_2})(\langle P'(x_i)P''(x_{\alpha_1})y_1y_2y_3\rangle - \langle P''(x_{\alpha_1})\rangle\langle P'(x_i)y_1y_2y_3\rangle)$$

$$-\lambda^3(\langle P'(x_1)P'(x_2)P'(x_3)y_1y_2y_3\rangle - \sum_{i=1}^{3} \langle P'(x_{\alpha_1})P'(x_{\alpha_2})\rangle\langle P'(x_i)y_1y_2y_3\rangle)$$

$$+\lambda^3 \sum_{\pi} \langle P'(x_1)y_{\pi_1}\rangle\langle P'(x_2)y_{\pi_2}\rangle\langle P'(x_3)y_{\pi_3}\rangle$$

where $\pi$ ranges over the six permutations $(\pi_1, \pi_2, \pi_3)$ of $(1, 2, 3)$. We combine this last result with (D.1), to get:

$$\begin{aligned}
&C_{x_3}^{-1}C_{x_2}^{-1}C_{x_1}^{-1}R_3(\lambda, \mathbf{x}, \mathbf{y}) \\
&\quad = C_{x_3}^{-1}C_{x_2}^{-1}C_{x_1}^{-1}(R_{03}(\lambda, \mathbf{x}, \mathbf{y}) + \sum_{i,j=1}^{3} \langle x_iy_j\rangle\langle x_{\alpha_1}; x_{\alpha_2}; y_{\beta_1}; y_{\beta_2}\rangle_C) \\
&\quad + \lambda \sum_{i,j=1}^{3} \langle P'(x_i)y_j\rangle C_{x_{\alpha_2}}^{-1}C_{x_{\alpha_1}}^{-1}\langle x_{\alpha_1}; x_{\alpha_2}; y_{\beta_1}; y_{\beta_2}\rangle_c \\
&\quad - \lambda\, \delta(x_1-x_2)\, \delta(x_2-x_3)\langle P'''(x_1)(1-\mathbf{P}_1)y_1y_2y_3\rangle \\
&\quad + \lambda^2 \sum_{i=1}^{3} \delta(x_{\alpha_1}-x_{\alpha_2})(\langle P'(x_i)P''(x_{\alpha_1})(1-\mathbf{P}_1)y_1y_2y_3\rangle \\
&\quad - \langle P''(x_{\alpha_1})\rangle\langle P'(x_i)(1-\mathbf{P}_1)y_1y_2y_3\rangle) - \lambda^3(\langle P'(x_1)P'(x_2)P'(x_3)(1-\mathbf{P}_1)y_1y_2y_3\rangle \\
&\quad - \sum_{i=1}^{3} \langle P'(x_{\alpha_1})P'(x_{\alpha_2})\rangle\langle P'(x_i)(1-\mathbf{P}_1)y_1y_2y_3\rangle).
\end{aligned}$$

Define $A_3$ by:

$$\begin{aligned}
A_3(\lambda, \mathbf{x}, \mathbf{y}) = C_{x_3}^{-1}C_{x_2}^{-1}C_{x_1}^{-1}(&R_3(\lambda, \mathbf{x}, \mathbf{y}) - R_{03}(\lambda, \mathbf{x}, \mathbf{y}) \\
&- \sum_{i,j=1}^{3} \langle x_iy_j\rangle\langle x_{\alpha_1}; x_{\alpha_2}; y_{\beta_1}; y_{\beta_2}\rangle_C),
\end{aligned}$$

so that our formula can be written

$$\begin{aligned}
R_3(\lambda, \mathbf{x}, \mathbf{y}) &= R_{03}(\lambda, \mathbf{x}, \mathbf{y}) \\
&+ \sum_{i,j=1}^{3} S_2(\lambda, x_i, y_j)S_4^C(\lambda, x_{\alpha_1}; x_{\alpha}2; y_{\beta_1}; y_{\beta_2}) + C_{x_1}C_{x_2}C_{x_3}A_3. \qquad \text{(D.2)}
\end{aligned}$$

We come back for a while to $R_{2B}$. We have seen (see Chapter I) that on a space of symmetric functions the operator $M(\mathbf{x}, \mathbf{y})$ has the form:

$$M(\lambda, \mathbf{x}, \mathbf{y}) = \frac{1}{3} \sum_{i=1}^{3} S_2^{-1}(\lambda, x_i, y_i)K_2(\lambda, x_{\alpha_1}, x_{\alpha_2}, y_{\alpha_1}, y_{\alpha_2}).$$

On the other hand, the Neumann series for $R_{2B}$ can be written:

$$\begin{aligned}
R_{2B} &= \left(\sum_{n=0}^{\infty} (-1)^n (R_{03}M)^n\right)R_{03} \\
&= \left(\sum_{n=0}^{\infty} (-1)^n \left(\sum_{i=1}^{3} 1 \underset{i}{\otimes} R_{02}K_2\right)^n\right)R_{03}
\end{aligned}$$

where we use the notation $1\otimes_i R_{02}K_2=\delta(x_i-y_i)R_{02}K_2(x_{\alpha_1}, x_{\alpha_2}, y_{\alpha_1}, y_{\alpha_2})$. This can be put in the form:

$$\begin{aligned} R_{2B} &= -2R_0+3\sum_{i=1}^{3} S_2\underset{i}{\otimes} R_2+b \\ &= 3\sum_{i=1}^{3} S_2\underset{i}{\otimes} S_4^C+R_{03}+b, \end{aligned}$$

where we have used that $R_2-R_{02}=S_4^C$ and $b$ stands for the sum of all terms on which the lines 'cross', for instance $R_{03}M_1R_{03}M_2R_{03}$, etc.

Turn again to $R_3$ and consider (D.2):

$$R_3=R_{03}+3\sum_{i=1}^{3}\langle x_iy_i\rangle\langle x_{\alpha_1}; x_{\alpha_2}; y_{\alpha_1}; y_{\alpha_2}\rangle_C+C_{x_1}C_{x_2}C_{x_3}A_3$$

where we have used that, on symmetric function,

$$\sum_{i,j}\langle x_iy_j\rangle\langle x_{\alpha_1}; x_{\alpha_2}; y_{\beta_1}; y_{\beta_2}\rangle_C=3\sum_i\langle x_iy_i\rangle\langle x_{\alpha_1}; x_{\alpha_2}; y_{\alpha_1}; y_{\alpha_2}\rangle_C.$$

Then,

$$R_3-R_{2B}=C_{x_1}C_{x_2}C_{x_3}A_3-b.$$

Consider the above expansion for $R_{2B}$ and, for each term $X$ in this expansion, consider $R_{03}^{-1}XR_{03}^{-1}$. This is a product of the form $M_{i_1}R_{03}M_{i_2}R_{03}\cdots R_{03}M_{i_n}$. A factor $M_i$ has the form $\Gamma\otimes_i K_2$ and is potentially singular because of the $\Gamma$-factor, which contains a $C^{-1}$ part. Note that if $i_1=i_2=\cdots=i_n$, then this $\Gamma$ factor remains at the end. But this cannot happen if the lines 'cross' at least once. In this case, one can see that all factors $\Gamma$ are cancelled by some factor $S_2$ contained in $R_{03}$, and the net result contains only $S_2$ and $K_2$ factors. These are locally regular (in the above sense).

We can therefore conclude that $R_{03}^{-1}bR_{03}^{-1}$ is locally regular. In addition it is exponentially decreasing in $x_i-x_j$, $y_i-y_j$, $x_i-y_j$ (this follows from the fact that $S_2(x-y)$ and $K_2(x_1, x_2, y_1, y_2)$ are exponentially decreasing).

Similarly, consider $R_{03}^{-1}C_{x_1}C_{x_2}C_{x_3}A_3$. Using that $R_{03}^{-1}=\frac{1}{6}\Gamma\otimes\Gamma\otimes\Gamma$ and that $\Gamma=K_1+C^{-1}$, we get: $R_{03}^{-1}C_{x_1}C_{x_2}C_{x_3}A_3=A_3+A_3'$, where $A_3'$ is a sum of terms of the form $\int dx'\prod_{i=1}^3\sigma_i(x_i-x_i')A_3(\mathbf{x}', \mathbf{y})$, and $\sigma_i(x_i-x_i')=\delta(x_i-x_i')$ or $K_1C(x_i-x_i')$. From the form of $A_3$, we see that both $A_3$ and $A_3'$ are locally regular (in the above sense) and exponentially decreasing. Furthermore, the $y$-variables of $A_3$ and $A_3'$ can be put equal, since they appear truncated or in the combination $(1-\mathbf{P}_1)y_1y_2y_3$, which cancels the singularities at coinciding points. The same is true of $R_{03}^{-1}b$. We can thus define, for $\lambda$ small, the inverse $(1+R_{2B}^{-1}G_3)^{-1}$, with $G_3=R_3-R_{2b}=C_{x_1}C_{x_2}C_{x_3}A_3-b$, using that $R_{2B}^{-1}=R_{03}^{-1}+\frac{1}{3}\sum_i\Gamma\otimes_i K_2$. The worst term, $R_{03}^{-1}G_3=R_{03}^{-1}C_{x_1}C_{x_2}C_{x_3}A_3-R_{03}^{-1}b$, is analysed in the way sketched above. The other term is even better, since there is only one $\Gamma$-factor.

Finally, we note that $R_{2B}^{-1}G_3R_{2B}^{-1}$ is also locally regular. We consider again the most singular (potentially) part $R_{03}^{-1}G_3R_{03}^{-1}$: the term $R_{03}^{-1}bR_{03}^{-1}$ was already discussed above. In the other term, $A_3R_{03}^{-1}+A_3'R_{03}^{-1}$, one should again integrate by parts to show that only $\delta$-function singularities remain. But this follows again from the fact that the $y$-variables in $A_3$ appear truncated or in the combination $(1-\mathbf{P}_1)y_1y_2y_3$.

We conclude, then, that

$$K_3 = R_3^{-1} - R_{2B}^{-1} = (1 + R_{2B}^{-1} G_3)^{-1} R_{2B}^{-1} G_3 R_{2B}^{-1}$$

is defined, as a distribution, by a convergent Neumann series (for $\lambda$ small) and that it is locally regular and exponentially decreasing. It then follows that its Fourier transform, in the $k$, $p_i$, $q_i$-variables, is a bounded function, in fact bounded by $O(\lambda)$ since $G_3$ is $O(\lambda)$.

The discussion for $L_3$ is essentially done, since:

$$L_3^*(\lambda, \mathbf{x}, y) = \lambda \int R_3^{-1}(\lambda, \mathbf{x}, \mathbf{y}') \langle y_1' y_2' y_3' (1 - \mathbf{P}_1) P'(x) \rangle \, d\mathbf{y}'$$

$$= \lambda \int d\mathbf{y}' (K_3(\lambda, \mathbf{x}, \mathbf{y}') + R_{2B}^{-1}(\lambda, \mathbf{x}, \mathbf{y}')) \langle y_1' y_2' y_3' (1 - \mathbf{P}_1) P'(x) \rangle.$$

The term with the $K_3$-factor is regular by our previous discussion of $K_3$. The other one is analysed in the same way as $C_{x_3}^{-1} C_{x_2}^{-1} C_{x_1}^{-1} \langle x_1 x_2 x_3 (1 - \mathbf{P}_1) y_1 y_2 y_3 \rangle$. This time we do not worry about terms having a $\delta(x_i - y_i)$ since we will not have a right multiplication by $R_{03}^{-1}$ as it was the case for $G_3$ in defining $K_3$.

This completes also our proof that $L_3^*$ and $L_3$ are bounded by $O(\lambda)$ in momentum space.

Concerning $D_2(\lambda, x - y) = -\lambda^2 \langle P'(x)(1 - \mathbf{P}_0 - \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_3) P'(y) \rangle$, one can also easily see, using the above methods and the definition of $\mathbf{P}_n$, that a term like $\langle P'(x) \mathbf{P}_n P'(y) \rangle$ is locally regular in the sense we use this expression in this appendix.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