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A proof of spontaneous magnetization in the d-dimensional ising-, XY- and Heisenberg-models, $d \geq 2$*)

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Abstract. The existence of spontaneous magnetization has been proven for the Ising-, XY- and Heisenberg-models in $d \geq 2$ dimensions. The results are achieved by the fact that the partition functions are monotone increasing in the coupling constants. Further, it is shown that the two different definitions of the spontaneous magnetization are the same in these models. This last result is achieved by new cumulant inequalities.

1. The absence of an exact solution of the 3-dimensional Ising model has awaked the interest in the approximate studies of this model. Further there is no sound about the solution of the quantum XY- or Heisenberg-models in 2- or 3-dimensions. Recently progress has been made in proving the existence of the magnetizations of these models [1, 2].

Here we complete the ideas about the proofs that the magnetization is different from zero for $d \geq 2$ dimensions in the Ising-, quantum XY- and quantum Heisenberg-models, defined on the d-dimensional lattice Z^d. To do so, we consider at first the anisotropic Ising model. Its configuration energy is:

$$H_{\Lambda_d} = - \sum_{\{k \leqslant k_i \in \Lambda_d\}}^{i=0,\ldots,d} v_i(k, k_i)\sigma_k \sigma_{k_i} \quad (k = k_0), \tag{1}$$

where $k = \{k_1, \ldots, k_d\}$ and $k_i = \{k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_d\}$, $\Lambda_d \, c Z^d$. The coupling constants v_i are different in each space direction, i.e. they are anisotropic in space directions. In equation (1) σ_k takes values ± 1 and all of the constants $v_i(k, k_i)$ are positive. $v_0(k, k)$ are chosen such that

$$- \sum_{k \in \Lambda_d}^{i=0,\ldots,d} v_i(k, k_i)\sigma_k \sigma_{k_i} \geq 0 \tag{2}$$

and therefore $-H_{\Lambda_d} \geq 0$. To be able to say something about the bounds of the partition function, we give some mathematical preliminaries about monotone increasing functions, then to relate them to thermodynamic functions.

A real function $f(x)$ is convex in an interval I of the real line if for $0 \leq \lambda \leq 1$, $x_1, x_2 \in I$ one has

$$f(\lambda x_1 + [1 - \lambda]x_2) \leq \lambda f(x_1) + [1 - \lambda]f(x_2). \tag{3}$$

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λ is real. The second derivative of a convex function $f(x)$ is non-negative, wherever it exists:

$$f''(x) \geq 0. \tag{4}$$

Further it holds:

$$f(x) \geq f(x_0) + [x - x_0]f'(x_0) \tag{5}$$

for x and $x_0 \in I$, if f is differentiable at x_0. This inequality equation (5) states that the first two terms of the Taylor expansion of a convex function $f(x)$ give a lower bound to $f(x)$ [3]. The inequality equation (5) can be generalized to linear self-adjoint operators A and B on a Hilbert space if: the spectrums of the operators A and B are in the interval I; $f(x)$ is differentiable on $I_1 \subset I$ which contains the spectrum of B; the operator $f(A) - f(B) - (A - B)f'(B)$ is of trace class. For us this generalization is not necessary.

A monotone increasing-function $f(x)$ has the further property that the first derivative of $f(x)$ is positive definite:

$$f'(x_0) > 0. \tag{6}$$

Here the interest is in the singular behaviour of the free energy per site and of the susceptibility. The free energy is defined as

$$-\beta f_d = \lim_{\Lambda_d \to \infty} \frac{1}{\Lambda_d} \ln Z_{\Lambda_d}(\{v_i\}_1^d) = \lim_{\Lambda_d \to \infty} \frac{1}{\Lambda_d} \ln \operatorname{Tr} \{\exp(-\beta H_{\Lambda_d})\}. \tag{7}$$

From this definition and equations (1) and (2) it is clear that $-\beta f_d$ is a monotone increasing function in the coupling constants $\{v_i > 0\}$, $i = 1, \ldots, d$. This is, because $x - x_0$ of equation (5) corresponds to an increase in the coupling constant v_i and $-f'(x_0)$ of equations (6–7) to the energy expectation value. The latter is positive by Griffiths-inequalities [4]. $f''(x_0)$ corresponds to the specific heat expression of the model. From the above equations (1–7) it follows that the partition functions $Z_{\Lambda_1}(v_d)$, $Z_{\Lambda_{d-1}}(\{v_i\}_1^{d-1})$ and $Z_{\Lambda_d}(\{v_i\}_1^d)$ fulfil the inequalities

$$\begin{aligned} &\frac{1}{2\Lambda_1} \ln Z_{\Lambda_1}(2v_d) + \frac{1}{2\Lambda_{d-1}} \ln Z_{\Lambda_{d-1}}(\{2v_i\}_1^{d-1}) \\ &\qquad \geq \frac{1}{\Lambda_d} \ln Z_{\Lambda_d}(\{v_i\}_1^d) > \frac{1}{\Delta_{d-1}} \ln Z_{\Lambda_{d-1}}(\{v_i\}_1^{d-1}), \end{aligned} \tag{8}$$

where we used Cauchy–Schwartz inequality to derive the upper-bound. The above inequalities are the generalizations of the isotropic cases [2]. Specially for $d > 2$, the bounds can be given by the 1- and 2-dimensional ones, where we have the Ising- and Onsager-solutions of the 1- and 2-dimensional Ising-models [5, 6]. The 1-dimensional Ising-model possesses no finite magnetization, it does, however, exist in the 2-dimensional model and that it is different from zero. The mathematical tool, to prove this fact in 2 dimensions, is given by the Szegö–Kac-theorem [7]. In higher dimension than two we prove the existence of finite magnetization with help of the Szegö–Kac-theorem by constructing a lower bound to the $d > 2$ dimensional susceptibility

$$\chi_d(m_1 n | v_i) = \beta \langle \sigma_m \sigma_n \rangle_d(\{v_i\}) = \lim_{\Lambda_d \to \infty} \beta \langle \sigma_m \sigma_n \rangle_{d, \Lambda_d}(\{v_i\}) \tag{9}$$

where $m, n \in \Lambda_{d-1}$.

As we have seen above the free energy is a monotone increasing function in the coupling constants. This fact allows to construct lower bounds to the susceptibility equation (9). One can exploit the monotone increasing character for our purpose by introducing an extra term into the configuration energy by

$$H_{\Lambda_d}(\lambda) = H_{\Lambda_d} - \lambda\sigma_m\sigma_n \tag{10}$$

with $m, n \in \Lambda_{d-1}$ or $m, n \in \Lambda_1$. Then one has

$$\ln Z_{\Lambda_d}(\{v^i\}_1^d \mid \lambda) > \ln Z_{\Lambda_{d-1}}(\{v_i\}_1^{d-1} \mid \lambda) \tag{11}$$

where we used $Z_{\Lambda_{d-1}}^{\Lambda_1} > Z_{\Lambda_{d-1}}$. Because $\ln Z_{\Lambda_d}$ and $\ln Z_{\Lambda_{d-1}}$ is monotone increasing in $\{v_i\}$ and in λ the inequality is retained if one takes the derivatives after λ in (11). One has

$$\langle\sigma_m\sigma_n\rangle_{d,\Lambda_d}(\{v_i\}_1^d) > \langle\sigma_m\sigma_n\rangle_{d-1,\Lambda_{d-1}}(\{v_i\}_1^{d-1}) \tag{12a}$$

or

$$\langle\sigma_m\sigma_n\rangle_d(\{v_i\}_1^d) > \langle\sigma_m\sigma_n\rangle_{d-1}(\{v_i\}_1^{d-1}) \tag{12b}$$

in the thermodynamic limit at $\lambda \equiv 0$. The inequality (12) allows to prove finite magnetization for d > 2 dimensions by applying Szegö–Kac-theorem to the 2-dimensional lower bound:

$$\langle\sigma_m\sigma_n\rangle_d > \langle\sigma_m\sigma_n\rangle_2 \quad (d > 2), \tag{13}$$

and $m, n \in Z^1$.

2. For the d-dimensional XY- and Heisenberg-models the susceptibility can also be bounded from below by the 2-dimensional one. We denote the XY- and Heisenberg-model Hamiltonians by $H_{\Lambda_d}^{xy}$ and $H_{\Lambda_d}^{\mathscr{H}}$, which we assume negative definite, and add to them a term $-\lambda\sigma_m^x\sigma_n^x$ [10] to generate the corresponding susceptibilities. An appropriate choice for m and n can be in the subspace Λ_1 or in Λ_{d-1}, $\Lambda_1 \subset Z^1$ and $\Lambda_{d-1} \subset Z^{d-1}$. The Hamiltonians with the extra λ-term are: $H_{\Lambda_d}^{xy}(\lambda) = H_{\Lambda_d}^{xy} - \lambda\sigma_m^x\sigma_n^x$ and the corresponding Heisenberg–Hamiltonian. It has been shown that the partition functions belonging to these models fulfil certain inequalities [2], which we generalized to the anisotropic lattice here. The result is:

$$\begin{aligned}\ln Z_{\Lambda_d}(\{2v_i\}_1^d \mid \lambda) &\geq \tfrac{1}{2}\ln Z_{\Lambda_d}^{XY}(\{v_i\}_1^d \mid \lambda)\\ &\geq \ln Z_{\Lambda_d}(\{\tfrac{1}{2}v_i\}_1^d \mid \tfrac{\lambda}{2}) - \tfrac{1}{2}\ln Z_{\Lambda_d}(\{-v_i\}_1^d \mid 0)\end{aligned} \tag{14}$$

and

$$\begin{aligned}\tfrac{1}{2}\ln Z_{\Lambda_d}^{XY}(\{2v_i\}_1^d \mid \lambda) + \tfrac{1}{2}\ln Z_{\Lambda_d}(\{2v_i\}_1^d \mid \lambda) &\geq \tfrac{1}{2}\ln Z_{\Lambda_d}^{\mathscr{H}}(\{v_i\}_1^d \mid \lambda)\\ &\geq \ln Z_{\Lambda_d}(\{\tfrac{1}{2}v_i\}_1^d \mid \tfrac{\lambda}{2}) - \tfrac{1}{2}\ln Z_{\Lambda_d}^{XY}(\{-2v_i\}_1^d \mid 0),\end{aligned} \tag{15}$$

where the Golden–Thomson-theorem [8, 9] has been used. Z_{Λ_d}, $Z_{\Lambda_d}^{XY}$ and $Z_{\Lambda_d}^{\mathscr{H}}$ denote the Ising-, XY- and Heisenberg-partition functions on the d-dimensional, anisotropic lattice Z^d. Because it holds

$$-\ln Z_{\Lambda_d}(\{-v_i\}_1^d \mid 0) > 0 \tag{16}$$

and

$$-\ln Z_{\Lambda_d}^{XY}(\{-2v_i\}_1^d \mid 0) > 0, \tag{17}$$

the corresponding terms in the inequalities (14–15) can be neglected. Further using the upper bound (14) in the upper bound of (15), one gets the inequality

$$\tfrac{3}{2}\ln Z_{\Lambda_d}(\{4v_i\}_1^d \mid \lambda) \geq \tfrac{1}{2}\ln Z_{\Lambda_d}^{\mathscr{H}}(\{v_i\}_1^d \mid \lambda) \geq \ln Z_{\Lambda_d}(\{\tfrac{1}{2}v_i\}_1^d \mid \lambda). \quad (18)$$

The ln of all ferromagnetic partition functions in (14–15) and (18) are monotone increasing functions in the variables $v_i(>0)$ and $\lambda(>0)$. This fact allows us to differentiate the inequalities (14) and (18) with respect to λ. One gets

$$\langle\sigma_m^x\sigma_n^x\rangle_d(\{2v_i\}_1^d) \geq \tfrac{1}{2}\langle\sigma_m^x\sigma_n^x\rangle_{d,XY}(\{v_i\}_1^d) \geq \langle\sigma_m^x\sigma_n^x\rangle_d(\{\tfrac{1}{2}v_i\}_1^d), \quad (19)$$

$$\tfrac{3}{2}\langle\sigma_m^x\sigma_n^x\rangle_d(\{4v_i\}_1^d) \geq \tfrac{1}{2}\langle\sigma_m^x\sigma_n^x\rangle_{d,\mathscr{H}}(\{v_i\}_1^d) \geq \langle\sigma_m^x\sigma_n^x\rangle_d(\{\tfrac{1}{2}v_i\}_1^d), \quad (20)$$

where we have taken the thermodynamic limit and $m, n \in Z^1$.

Now it has been shown that the $d > 2$ dimensional Ising susceptibility can be bounded from below by those of the 2-dimensional one. Therefore we just proved the

Theorem: *The space anisotropic, quantum XY- and Heisenberg correlation functions, in the case of $d \geq 2$-dimensions, are bounded from above and below by the 2-dimensional ones and therefore the magnetization of the XY- and Heisenberg-models are different from zero below a critical parameter values $v_i^c(d \geq 2, v_i^c \sim T_c)$:*

$$\lim_{m-n\infty} < \sigma_m^x\sigma_n^x\rangle_{d,\alpha}(\{v_i\}_1^d) = M_{d,\alpha}^2(\{v_i\}_1^d) \geq M_2^2(\{v_i\}_1^2) > 0, \quad (21)$$

$\alpha = XY$ or $\mathscr{H}$.
$M_{a,xy}$ denotes the magnetization in the XY-model, $M_{d,\mathscr{H}}$ and M_2 those of the Heisenberg- and 2-dimensional Ising-model.

The isotropic models also show, of course, spontaneous magnetization, which is a special case of the above theorem.

3. In this section we will show that the two known definitions of the spontaneous magnetization coincide in the considered d-dimensional models.

Let us define the partition functions $Z_{\Lambda_d;\alpha}(\{v_i\}_1^d \mid \lambda)$ and $Z_{\Lambda_{d,1}\cup\Lambda_{d,2};\alpha}(\{v_i\}_1^d \mid \lambda)$ with the model Hamiltonians $H_{\Lambda_d;\alpha}(\lambda) = H_{\Lambda_d;\alpha} - \lambda\sigma_m^x\sigma_n^x$, where $H_{\Lambda_d;\alpha}$ stands for the d-dimensional, space anisotropic Ising-, quantum XY- or quantum Heisenberg-models ($\alpha = I, XY$ or $\mathscr{H}$). Further, we assume the following: $m \in \Lambda_{2,1}$, $n \in \Lambda_{2,2}$ and $H_{\Lambda_d;\alpha} < 0$.

Lemma 1. *It holds for any d-dimensional Ising-, XY- and Heisenberg-model and any finite $\Lambda_{d,1}, \Lambda_{d,2} \subset \Lambda_d \subset Z^d$ with condition $\Lambda_{d,1} \cap \Lambda_{d,2} = \phi$:*

$$\ln Z_{\Lambda_d;\alpha}(\{v_i\}_1^d \mid \lambda) > \ln Z_{\Lambda_{d,1}\cup\Lambda_{d,2};\alpha}(\{v_i\}_1^d \mid \lambda) \quad (22)$$

and

$$\langle\sigma_m^x\sigma_n^x\rangle_{d,\alpha} \geq \langle\sigma_m^x\rangle_{d,\alpha}\langle\sigma_n^x\rangle_{d,\alpha} \quad (23)$$

at $\lambda = 0$ in the thermodynamic limit.

The proof of the inequality (22) can be established along the ideas of ref. [2], combined with the Trotter product formula for instance. Afterwards, the inequality (23) follows from the inequality (22) by differentiation after λ at $\lambda = 0$. The above lemma uses again the convexity and the monotone increasing property of the partition function in the coupling constants.

Lemma 2. *Let $\Lambda_{d,1}$ and $\Lambda_{d,2}$ be finite sets with the property that their intersection is empty, $\Lambda_{d,1} \cap \Lambda_{d,2} = \phi$; $\Lambda_{d,1}, \Lambda_{d,2} \subset \Lambda_d \subset Z^d$. Then, it holds for any d-dimensional, space-anisotropic Ising-, XY- and Heisenberg-model:*

$$\langle \sigma_m^x \rangle_{d,\beta}(\{v_i\}_1^d) > \langle \sigma_m^x \rangle_d(\{yv_i\}_1^d) > \langle \sigma_m^x \rangle_{d-1}(\{yv_i\}_1^{d-1}) \tag{24}$$

and

$$\langle \sigma_m^x \rangle_{d-1}(\{4dv_i\}_1^{d-1}) > \langle \sigma_m^x \rangle_d(\{4v_i\}_1^d) > \langle \sigma_m^x \rangle_{d,\beta}(\{v_i\}_1^d) \tag{25}$$

where $0 < y \leq 1$ and $\beta = XY$ or $\mathscr{H}$.

In the Ising-expectation values we drop the Ising-index I.

To sketch the proof, we note that the inequalities

$$\begin{aligned} \ln Z_{\Lambda_{d,1} \cup \Lambda_{d,2};\beta}(\{v_i\}_1^d \mid \lambda) &> \ln Z_{\Lambda_{d,1} \cup \Lambda_{d,2}}(\{yv_i\}_1^d \mid \lambda) \\ &> \ln Z_{\Lambda_{d-1,1} \cup \Lambda_{d-1,2}}(\{yv_i\}_1^{d-1} \mid \lambda) \end{aligned} \tag{26}$$

and

$$\begin{aligned} \ln Z_{\Lambda_{d,1} \cup \Lambda_{d,2}}(\{4dv_i\}_1^{d-1} \mid \lambda) &> \ln Z_{\Lambda_{d,1} \cup \Lambda_{d,2}}(\{4v_i\}_1^d) \\ &> \ln Z_{\Lambda_{d,1} \cup \Lambda_{d,2};\beta}(\{v_i\}_1^d \mid \lambda) \end{aligned} \tag{27}$$

are fulfilled and the partition functions are defined with the Hamiltonians $H_{\Lambda_{d,\alpha}}(\lambda)$ as in Lemma 1. Here we have taken different coupling constants to satisfy the above inequalities. Taking the first λ-derivative of the inequalities (26) and (27) at $\lambda = 0$, the inequalities (24) and (25) follow immediately.

From the above two lemmas one has a consequence the following theorem:

Theorem. *The two point cumulants of the d-dimensional, space-anisotropic Ising-, XY- and Heisenberg-models fulfil the inequalities*

$$\langle \sigma_m^x \sigma_n^x \rangle_2^c(\{4dv_i\}_1^2) \geq \langle \sigma_m^x \sigma_n^x \rangle_{d,\beta}^c(\{v_i\}_1^d) \geq \langle \sigma_m^x \sigma_n^x \rangle_2^c(\{v_i\}_1^2) \tag{28}$$

and the equality

$$\lim_{m-n\to\infty} \langle \sigma_m^x \sigma_n^x \rangle_{d,\beta}(\{v_i\}_1^d) = \langle \sigma_k^x \rangle_{d,\beta}(\{v_i\}_1^d) \langle \sigma_l^x \rangle_{d,\beta}(\{v_i\}_1^d) \tag{29}$$

under the condition, $\{v_i\}_1^2 \geq \{v_j\}_3^d$. Further, $\beta = I$, XY or $\mathscr{H}$, $d \geq 2$ and $m, n \in Z^1$.

Proof. The Ising-, XY- and Heisenberg-partition functions fulfil the inequalities

$$\ln Z_{\Lambda_d}(\{4dv_i\}_1^2 \mid \lambda) \geq \ln Z_{\Lambda_d;\beta}(\{v_i\}_1^d \mid \lambda) \geq \ln Z_{\Lambda_2}(\{v_i\}_1^2 \mid \lambda)$$

from which follows:

$$\langle \sigma_m^x \sigma_n^x \rangle_2(\{4dv_i\}_1^2 \geq \langle \sigma_m^x \sigma_n^x \rangle_{d,\beta}(\{v_i\}_1^d) \geq \langle \sigma_m^x \sigma_n^x \rangle_2(\{v_i\}_1^2).$$

We dropped again the Ising-index I for the two dimensional case in the theorem and in its proof. When we combine these above inequalities to the Lemmas 1 and 2, the inequalities (28) follow by using the fact

$$\langle \sigma_m^x \rangle_2(\{4dv_i\}_1^2) \geq \langle \sigma_m^x \rangle_{d,\beta}(\{v_i\}_1^d).$$

This above relation can be derived on a similar way as we have derived the two above lemmas. If we take the limit $\lim_{m-n\to\infty}$ in (28) the upper and lower bounds are going to zero [7, 11] and therefore the equality (29) follows. This theorem completes the theorem of Section 2.

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