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Covariance group in the presence of external electromagnetic fields

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Abstract. In the frame of a research program concerned with the investigation by group theoretical methods of space-time symmetries of interacting systems we explicitly derive in this paper the general covariance group for a charged particle moving in an arbitrary external (classical) electromagnetic field. We obtain this group independently of any equation of motion, essentially on the simple basis of the invariance properties of the Maxwell equations under Poincaré transformations. As a consequence of these results, the group theoretical definition of an elementary particle can usefully be extended to the case where an external field is present, and useful information can be obtained on the characterization of covariant equations of motion.

0. Introduction

The group theoretical definition of elementary particles in terms of representations of the Poincaré group $IO(3, 1)$ is a well known and successful one [1]. In this frame, the Poincaré group plays the role of covariance group of special relativity, i.e. embodies the basic postulate of the theory that physical results should be left invariant under a space-time change of reference frame relating by definition two inertial systems. The elementary particles can then be characterized by the values of their spin and their squared mass and it is also possible, using group theoretical methods, to tackle successfully the description of the time evolution of the states representing such particles, i.e. to relate covariant equations of motion to representations of this group. It is the purpose of the present work to extend this well known and successful analysis to the case where an external (classical) e.m. field is present.

Usually such a field is introduced via a potential A in the free equation of motion by so-called minimal coupling, the momentum operators P_μ being replaced by $\Pi_\mu = P_\mu - (e/c)A_\mu(x)$ (e the charge of the particle). Except for the well understood high field limit and for the serious troubles encountered in the higher spin ($s \geqslant 1$) cases [2, 3], that we shall discuss later on [4], this recipe is a very satisfying and successful one.

From the point of view of the symmetries, however, the situation is not that clear. For example, as a consequence of the arbitrariness (gauge) in the choice of the potential, there corresponds to a given field not one equation of motion, but an infinite, physically indistinguishable, class of them. This implies that Poincaré covariance then only has to hold for this class as a whole and not for each element of that class separately. In addition, arbitrary gauge transformations are then covariant transformations for this class, too, and this gives rise to interesting but infinite dimensional groups which are quite difficult to handle [5, 6]. This is of course not new and a natural alternative would be to develop a formalism without introducing a potential, but based only on fields, as it is partly possible, e.g. for Dirac particles [7]. Such a formalism is however not exempt of difficulties and a generalization of it is not known.

In the present work we choose a way in between, using the advantages of these two formalisms, i.e. we use potentials, but we get rid of the arbitrariness mentioned above by fixing in some convenient way the gauge, for any given field. In this way it is possible to build up group theoretically a theory of elementary particles in interaction with an external field, by the construction first of the relevant covariance group, independently of any equation of motion, and by the analysis then of the representations of this group.

This paper will be organized as follows: in the first part we define more precisely what we mean by covariance. Then, after having made also more precise in part two the class of fields we consider, we define covariant transformations as acting on the potential space of a given field. This will be done, following our philosophy, with the help of the quite natural concept of compensating gauges together with a (fixed) chosen map $\pi: F \rightarrow A$ which applies any e.m. field F in an uniquely determined potential A. The result is shown to be essentially independent of the particular choice for π. In this way we construct covariance operator groups that may however, by construction, depend on the field we started with. We get rid of this dependence in the fourth part, by expliciting the general covariance group, valid for any field, and by making also explicit its relationship with the concept of covariant equation of motion in the presence of an external field. The representations of this group and their physical interpretation and consequences are the subject of a subsequent paper [4].

1. Covariance (definition)

Because the term covariance is quite a much used one (and unfortunately not always with the same meaning), let us first briefly define what we mean more precisely by covariance and covariance group. Let therefore α be an element of some parameter space X, $\psi \in \mathscr{H}(X)$ some (separable) Hilbert space of functions on X and

$$O(\alpha)\psi(\alpha) = 0, \qquad \forall \alpha \in X \tag{1.1}$$

some (scalar) wave equation on $\mathscr{H}(X)$ (e.g. an equation of motion). Let now g be an invertible map from X into itself with $g\alpha = \alpha'$. Then g is called a *covariant* transformation for O if and only if there exists a unitary (or antiunitary) operator V_g on $\mathscr{H}(X)$ such that

$$O'(g\alpha) \equiv V_g O(g\alpha) V_g^{-1} = O(\alpha) \tag{1.2}$$

implying of course that in the new frame, with $\psi'(\alpha) \equiv (V_g\psi)(\alpha)$ we have, for all ψ satisfying (1.1)

$$O'(\alpha)\psi'(\alpha) = 0. \tag{1.1$'$}$$

Note that in our case X will depend not only on space-time coordinates but also on a potential of the external field considered. The condition of covariance (1.2) is then as made more explicit in (4.11).

The above definition may be of course generalized to the case where $\psi(\alpha)$ is a n-components wave function and $O(\alpha)$ is in $n \times n$ matrix form: g will be then called a covariant transformation for O if and only if there exist an unitary (or antiunitary) operator V_g in $\mathscr{H}(X)$ and a non singular $n \times n$ matrix Λ_g which satisfy the condition

$$O'(g\alpha) \equiv V_g O(g\alpha) V_g^{-1} = \Lambda_g O(\alpha) \Lambda_g^{-1} \tag{1.2)'$$

implying the same equation (1.1)′ as before with similarly $\psi'(\alpha)$ given by $(V_g\psi)(\alpha)$.

One may verify that this definition implies that the elements g which are covariant transformations form a group G, the covariance group. Furthermore, if the set of all functions $\{(V_g\psi)\}$ in $\mathscr{H}(X)$ does not contain proper invariant subspaces, it follows, also from the definition and from Schur's Lemma, that the set $\{V_g \mid \forall g \in g\}$ satisfies the extra condition

$$V_g \cdot V_{g'} = \omega(g, g') V_{gg'} \tag{1.3}$$

where $\omega(g, g')$ is some complex number of modulus one, so that, taking into account the associativity properties of operators on Hilbert spaces and of matrices, the set $\{V_g\}$ forms an irreducible projective representation of G with carrier space $\mathscr{H}(X)$.

We define further g to be an *invariant* transformation of O, or a symmetry for O, if g is a covariant transformation with in addition

$$O'(g\alpha) = O(g\alpha). \tag{1.4}$$

The essential point to note here is that for G the Poincaré group and O the operator of an ordinary (free) equation of motion, the two concepts do coincide. This will however no longer be so in our case: covariance embodies the property of equivalence of reference systems (and there will thus be elements related to *every* Poincaré transformation) whereas invariance will be generated by the covariant transformations which in addition leave the external field invariant. As we shall see, in both cases it will be in general necessary to combine in a non-trivial way Poincaré and gauge transformations.

2. External e.m. fields

Because they will play quite an important role with respect to our space X, let us make first mathematically more precise what we consider as possible candidates for an external e.m. field: an external e.m. field will be a continuous differentiable map from the Minkovski space $M(4)$ into the space of the real anti-symmetric (time-pseudo-) tensors of rank two $F_{\mu\nu}$ with the following three properties

(i) It satisfies the Maxwell equations

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0$$
$$F_{\mu\nu}, \nu = \frac{4\pi}{c} j_\mu. \tag{2.1}$$

Because we make, from the physical point of view, no further restriction on the 4-current j_μ the second set of equations can and will be considered as a definition of this 4-current.

(ii) It transforms under an element $g = (a, \Lambda)$[1] of the Poincaré group $IO(3, 1)$ according to

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[1]) We adopt the notation $g = (a, \Lambda) \in IO(3, 1)$, with $(a, \Lambda)x = \Lambda x + a$, $x \in M(4)$, a a 4-translation, $\Lambda \in O(3, 1)$, an element of the homogeneous Lorentz group, $(\Lambda x)_\mu = \Lambda_\mu^\nu x_\nu$ for covariant and $(\Lambda x)^\mu = (\Lambda^{-1})_\nu^\mu x^\nu$ for contravariant vector components. In this notation the product in $IO(3, 1)$ reads then $(a, \Lambda)(a', \Lambda') = (a + \Lambda a', \Lambda\Lambda')$.

$$(gF)_{\mu\nu}(x) = \varepsilon(g)\Lambda_\mu^\rho\Lambda_\nu^\sigma F_{\rho\sigma}(\Lambda^{-1}(x-a)) \tag{2.2}$$

with $\varepsilon(g) = \text{sign}\,(\Lambda_0^0)$.

This law defines obviously, by the condition

$$g \in G_F \quad \text{if and only if} \quad (gF)_{\mu\nu}(x) = F_{\mu\nu}(x), \quad \forall x \in M(4) \tag{2.3}$$

a subgroup G_F of $IO(3, 1)$, the symmetry group of the field F.

(iii) It can be written as an integral

$$F_{\mu\nu}(x) = \int d^4k\hat{F}_{\mu\nu}(k)\exp(ikx) \tag{2.4}$$

with $k \in M^*(4)$, the dual of the Minkovski space, i.e. we consider any e.m. field for which the integral

$$\hat{F}_{\mu\nu}(k) = \left(\frac{1}{2\pi}\right)^4 \int d^4xF_{\mu\nu}(x)\exp(-ikx) \tag{2.5}$$

exists, in the sense of generalized functions.

3. Invariant and covariant transformations acting on e.m. potentials

(a) Invariant transformations

Let us now, for a given $F_{\mu\nu}(x)$, consider a potential $A_\mu(x)$ as a real 4-vector field over $M(4)$ satisfying

$$A_{\nu,\mu}(x) - A_{\mu,\nu}(x) = F_{\mu\nu}(x) \tag{3.1}$$

and transforming under the Poincaré group, correspondingly to (2.2), as

$$(gA)_\mu(x) = \varepsilon(g)\Lambda_\mu^\nu A_\nu(\Lambda^{-1}(x-a)). \tag{3.2}$$

It is clear that all $g \in IO(3, 1)$ satisfying the equation

$$(gA)_\mu(x) = A_\mu(x) \tag{3.3}$$

will satisfy the corresponding equation (2.3) for the e.m. field, too. But, as is well known, the converse is in general not true. It is also not always possible to choose a potential with the same symmetry as the field: if this is well the case e.g. for a crystal field, this is not so for a constant uniform field, as is easy to realize, or in the example of the field of a plane wave [8], for instance.

The symmetry of the field (i.e., of the physical system) has however to be restored in some way. This is possible, using the quite natural concept of compensating gauge: a symmetry of the field can, acting on a corresponding potential generate a new potential that can clearly only differ from the previous one by a gauge transformation. Combining thus gauge and space-time transformations one obtains for each symmetry of the field a coupled transformation leaving the potential invariant and hence an operator commuting with the equations of motion [9–12].

The so generated *invariance operator groups* are then, in general, not subgroups of the usual space-time transformation groups corresponding to the equation of motion considered. Invariant transformations, (relating *identical* physical systems) are however necessarily particular cases of covariant transformations (relating *equivalent* physical systems). It is in fact on this basis that we shall now extend the just sketched treatment for covariant transformations as well.

(b) Covariant transformations

Let us thus now consider the case where g runs over the whole Poincaré group, and construct in a first step a group of transformations on the space of potentials as follows:

Let F be an arbitrary but fixed external e.m. field and assume we have chosen some (convenient) fixed (linear) map π from the e.m. field tensors to the space of potentials

$$\pi : F \mapsto A. \tag{3.4}$$

In order to get rid of the arbitrariness introduced together with the potential we now want to construct a set $\{g^* \mid \forall g \in IO(3, 1)\}$ of transformations combining gauge and Poincaré transformations and acting on the space of the potentials in such a way that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\ \pi\ } & A \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g^*} \\ (gF) & \longrightarrow & \pi(gF) \end{array} \tag{3.5}$$

is commutative, i.e. such that

$$g^*(\pi F) = \pi(gF) \qquad \forall g \in IO(3, 1). \tag{3.6}$$

Because of the transformation laws (2.2) and (3.2) for fields and potentials both $\pi(gF)$ and $g(\pi F)$ correspond to the same field so that we have, in general,

$$(g(\pi F))(x) = (\pi(gF))(x) + \partial\chi_g(\pi(gF), x) \tag{3.7}$$

for some gauge function $\chi_g(\pi(gF), x)$ which may depend on x, on g, on the field F and on the chosen map π. This gauge function is then fixed by (3.7) uniquely up to a constant. We may now combine, analogously as in the invariance case, any Poincaré transformation with a gauge transformation in such a way that the transformed potential is kept in the same fixed gauge as defined by the map π. We thus define, for any $g \in IO(3, 1)$ a pair $\{\chi_g, g\} \stackrel{\text{def}}{=} g^*$, whose action on πF is given by

$$\{\chi_g, g\}(\pi F) \stackrel{\text{def}}{=} g(\pi F) - \partial\chi_g(\pi(gF), x), \tag{3.8)(i}$$

so that by construction we have, as required

$$\{\chi_g, g\}(\pi F) = \pi(gF). \tag{3.8)(ii}$$

The transformations defined by (3.8) do not however, in general, form a group, as it is not necessarily possible to choose the arbitrary constants in (3.7) in such a way that the gauge function associated with the product of two Poincaré transformations is the gauge function resulting from the product of the corresponding two pairs.

It is however possible to imbed them in a larger group $Q^*_{\pi F}$ combining constant gauge functions with Poincaré transformations, and which we now want to make more explicit.

Using first (3.2) and the fact that the operator ∂ transforms covariantly under the Poincaré group, the action of an element $g_1 \in IO(3, 1)$ on the gauge functions in (3.7) may be consistently defined as follows, $\forall g_1, g_2 \in IO(3, 1)$

$$\psi(g_1)\chi_{g_2}(\pi(g_2F), x) \stackrel{\text{def}}{=} \varepsilon(g_1)\chi_{g_2}(\pi(g_2F), g_1^{-1}x). \tag{3.9}$$

Using this relation and letting g_1 act on both sides of (3.7) one then obtains

$$\partial[\psi(g_1)\chi_{g_2}(\pi(g_2F), x)] - \partial\chi_{g_1g_2}(\pi(g_1g_2F), g_1^{-1}x) + \partial\chi_{g_1}(\pi(g_1g_2F), g_1^{-1}x) = 0. \tag{3.10}$$

The relation (3.10) can be integrated and one finds (after a shift in the field variable)

$$\chi_{g_1}(\pi F, x) + \psi(g_1)\chi_{g_2}(\pi(g_1^{-1}F), x) - \chi_{g_1 g_2}(\pi F, x) \in \mathscr{R}. \tag{3.11}$$

This expression does thus not depend on x but may of course depend on πF and on the space-time transformations g_1 and g_2. We shall denote this function in the sequel by $(f^*(g_1, g_2))(\pi F)$. Rewriting the second member of (3.11) as

$$\psi(g_1)\chi_{g_2}(\pi(g_1^{-1}F), x) \stackrel{\text{def}}{=} \zeta(g_1)\chi_{g_2}(\pi F, x) \tag{3.12}$$

one finally obtains for (3.11)

$$(f^*(g_1, g_2))(\pi F) = \chi_{g_1}(\pi F, x) + \zeta(g_1)\chi_{g_2}(\pi F, x) - \chi_{g_1 g_2}(\pi F, x). \tag{3.13}$$

Denoting now by $\Phi_{\pi F}$ the additive group of real functions of πF generated by the functions in (3.13), $\forall g_1, g_2 \in IO(3, 1)$, and with product

$$(\varphi_1 + \varphi_2)(\pi F) \stackrel{\text{def}}{=} \varphi_1(\pi F) + \varphi_2(\pi F), \ \forall \varphi_1, \varphi_2 \in \Phi_{\pi F}, \ \forall F \in O_F \tag{3.14}$$

where O_F denotes the orbit of F under the Poincaré group, one finds that the group $Q^*_{\pi F}$ which is *generated* by the transformations in (3.8) has the following structure:

Proposition 3.1. $Q^*_{\pi F}$ is an extension of $\Phi_{\pi F}$ by $IO(3, 1)$ with product rule, $\forall \varphi_1, \varphi_2 \in \Phi_{\pi F}$ and $\forall g_1, g_2 \in IO(3, 1)$

$$(\varphi_1, g_1)(\varphi_2, g_2) = (\varphi_1 + \zeta(g_1)\varphi_2 + f^*(g_1, g_2), g_1 g_2) \tag{3.15}$$

where $f^*(g_1, g_2)$ is given by (3.13) and $\zeta(g_1)$ is defined, from (3.12) and (3.13), by

$$(\zeta(g_1)\varphi)(\pi F) \stackrel{\text{def}}{=} \varepsilon(g_1)\varphi(\pi(g_1^{-1}F)) \tag{3.16}$$

the possible sign $\varepsilon(g_1)$ in (3.16) being as in (2.2) and (3.9). These properties can be summarized in the following exact sequence of groups:

$$0 \to \Phi_{\pi F} \to Q^*_{\pi F} \to IO(3, 1) \to 1, f^*, \zeta. \tag{3.17}$$

Proof: That $\Phi_{\pi F}$ is normal in $Q^*_{\pi F}$ and that $Q^*_{\pi F}/\Phi_{\pi F} \cong IO(3, 1)$ follow from the definitions. We just have to verify that $Q^*_{\pi F}$ is indeed the group generated by the transformations g^* in (3.8) and that the functions f^* in (3.13) do satisfy the factor system conditions.

For the former, defining the imbedding of g^* in $Q^*_{\pi F}$, $\forall g \in IO(3, 1)$, by the following map r

$$r \cdot g^* \equiv r\{\chi_g, g\} \stackrel{\text{def}}{=} (0, g) \in Q^*_{\pi F} \tag{3.18}$$

the verification is straightforward. Note that the map $g \mapsto r(g^*)$ is of course nothing else than a section for (3.17). Conversely the action of $Q^*_{\pi F}$ in (3.17) on the potential subspace defined by π, $\forall F \in O_F$, is given by the mapping σ defined by

$$(\sigma(\varphi, g))(\pi F) \stackrel{\text{def}}{=} \{\chi_g + \varphi, g\}(\pi F) = \pi(gF) \tag{3.19}$$

where we have used (3.7), (3.8) and the definition of $\Phi_{\pi F}$.

Finally, using the definitions (3.13) and (3.16), it is tedious but straightforward to verify that

$$f^*: IO(3, 1) \times IO(3, 1) \to \Phi_{\pi F} \tag{3.20}$$

satisfies, $\forall F \in O_F$ and $\forall g_1, g_2, g_3 \in IO(3, 1)$ the identity

$$f^*(g_1, g_2g_3) + \zeta(g_1)f^*(g_2, g_3) = f^*(g_1, g_2) + f^*(g_1g_2, g_3) \tag{3.21}$$

so that f^* is effectively a factor system.

We shall later on explicit the structure of this group. Before we do so, we first show that our definition is consistent, in the sense that $Q^*_{\pi F}$ does not, as an abstract group, depend on the particular choice of gauge we have made by fixing π, nor on the choice of a particular reference frame.

Proposition 3.2. The following groups are isomorphic:

$$Q^*_{\pi F} \cong Q^*_{\pi(gF)} \cong Q^*_{\pi F + \partial\xi(\pi F, x)} \tag{3.22}$$

$\forall g \in IO(3, 1)$ and for all differentiable (gauge) functions $\xi(\pi F, x)$ on space-time.

Proof: The first relation follows from the fact that a change of reference frame induces a conjugation in the r.h. side of (3.17) giving rise to an isomorphic extension. Further, let π' be a new choice of gauge (which may also be different for different $F \in O_F$), i.e. let

$$A'(x) = (\pi' F)(x) = (\pi F)(x) + \partial\xi(\pi F, x)$$

for some gauge functions $\xi(\pi F, x)$. We then have

$$\begin{aligned} (\pi(gF) - g(\pi F))(x) &= \partial\chi_g(\pi(gF), x) \\ (\pi'(gF) - g(\pi' F))(x) &= (\pi(gF) - g(\pi F))(x) \\ &\quad + \partial\xi(\pi(gF), x) - \psi(g)\partial\xi(\pi F, x) \end{aligned} \tag{3.23}$$

where we have used the linearity of the action of the Poincaré group on potentials. It follows from (3.23), ∂ transforming covariantly under $IO(3, 1)$, that

$$\chi_g(\pi'(gF), x) - \chi_g(\pi(gF), x) = (1 - \zeta(g))\xi(\pi(gF), x)$$

up to an arbitrary (inessential) constant function of πF.

Inserting this relation in (3.13) one then obtains

$$(f^*(g_1, g_2))(\pi F) - (f^*(g_1, g_2))(\pi' F) = 0, \quad \forall g_1 g_2 \in IO(3, 1)$$

so that both factor sets describe equivalent thus *a fortiori* isomorphic extensions. This completes the proof.

Let us now turn to the explicit calculation of the structure of our group. As a consequence of Proposition 3.2 we may choose some convenient gauge for the potential, i.e. some convenient map π. We therefore split the spectrum S of the field, i.e. the set of all k in (2.5) for which $\hat{F}_{\mu\nu}(k) \neq 0$ in three disconnected parts

$$\begin{aligned} &S = S^{(0)} \cup S^{(r)} \cup S^{(j)} \\ &S^{(\alpha)} \cap S^{(\beta)} = \varnothing, \qquad \alpha \neq \beta, \alpha, \beta \in 0, r, j \end{aligned} \tag{3.24}$$

with

$$\begin{aligned} S^{(0)} &= \{k \in S \mid k = 0\} \\ S^{(r)} &= \{k \in S \mid k^2 = 0, \quad k \neq 0\} \\ S^{(j)} &= \{k \in S \mid k^2 \neq 0\}. \end{aligned} \tag{3.25}$$

Note that this decomposition is Poincaré invariant. The field F splits correspondingly to (3.25) in three (independent) e.m. fields:

$$F_{\mu\nu}(x) = F_{\mu\nu}^{(0)} + F_{\mu\nu}^{(r)}(x) + F_{\mu\nu}^{(j)}(x) \tag{3.26}$$

where $F_{\mu\nu}^{(0)}$ will be assumed for the sake of this work to consist in a constant uniform field only, $F_{\mu\nu}^{(r)}$ is a radiation field, $F_{\mu\nu}^{(j)}$ a 'current' field. All these fields are obviously defined for $\alpha = 0, r, j$, by

$$F_{\mu\nu}^{(\alpha)}(x) = \int d^4k \hat{F}_{\mu\nu}^{(\alpha)}(k) \exp(ikx) \tag{3.27}$$

with

$$\hat{F}_{\mu\nu}^{(\alpha)}(k) = \begin{cases} \hat{F}_{\mu\nu}(k) & k \in S^{(\alpha)} \\ 0 & \text{else} \end{cases} \tag{3.28}$$

As each of these fields has to satisfy independently the Maxwell equations (2.1), one may choose for each of them a corresponding potential in a specific convenient gauge. We define thus a map π by its action on the various parts of (3.24)

$$(\pi F^{(\alpha)})(x) = A^{(\alpha)}(x)$$

with

$$A_\mu^{(0)}(x) \stackrel{\text{def}}{=} \tfrac{1}{2} x^\rho F_{\rho\mu}^{(0)}$$

$$A_\mu^{(r)}(x) \stackrel{\text{def}}{=} \int d^4k \left(\frac{\hat{F}_{0\mu}^{(r)}(k)}{ik_0} \right) \exp(ikx) \tag{3.29}$$

$$A_\mu^{(j)}(x) \stackrel{\text{def}}{=} \int d^4k \left(\frac{k^\rho \hat{F}_{\rho\mu}^{(j)}(k)}{k_\rho k^\rho} \right) \exp(ikx),$$

i.e. in the symmetric, radiation (or Coulomb), respectively Lorentz gauges. It can easily be verified, using the Maxwell equations (2.1) that each of these potentials satisfies (3.1) for the corresponding e.m. field. Furthermore, as π is linear, we shall then find (πF) for the whole field F by simple addition of the various parts in (3.23), i.e.

$$(\pi F)(x) = A^{(0)}(x) + A^{(r)}(x) + A^{(j)}(x).$$

We may now use this particular choice to calculate the compensating gauges defined by (3.7) and (3.8) and then, by (3.13), the representative factor set f^* defining the extension (3.17).

Using further the transformation law for the fields (2.2) and for the potentials (3.2), we get, after a long but straightforward calculation, that for the three parts of the field separately

$$\chi_g(\pi(gF^{(0)}), x) = (g(\pi F^{(0)}))_\mu(x) a^\mu + c_g = -\tfrac{1}{2}(gF^{(0)})_{\sigma\rho} a^\sigma x^\rho + c_g$$

$$\chi_g(\pi(gF^{(r)}), x) = \int^{x^0} (g(\pi F^{(r)}))_0(x)\, dx^0 + c_g' \tag{3.30}$$

$$\chi_g(\pi(gF^{(j)}), x) = c_g''$$

where $g = (a, \Lambda) \in IO(3, 1)$ as before and where c_g, c_g', c_g'' are free integration constants, which may depend on πF but may obviously be chosen to be identically zero (as giving rise to equivalent extensions (3.17)). Further we obtain, with (3.13) and (3.30)

$$(f^*(g_1, g_2))(\pi(gF^{(0)})) = \tfrac{1}{2}(gF^{(0)})_{\sigma\rho}(\Lambda_1 a_2)^\sigma (a_1)^\rho$$
$$(f^*(g_1, g_2))(\pi(gF^{(r)})) = 0 \qquad (3.31)$$
$$(f^*(g_1, g_2))(\pi(gF^{(j)})) = 0$$

Using finally the linearity of the map π and of the Poincaré transformations, the compensating gauges and the factor set of the extension (3.17) are obtained by simple addition of the different parts of (3.30) and (3.31) so that finally

$$\chi_g(\pi(gF), x) = -\tfrac{1}{2}(gF^{(0)})_{\sigma\rho}a^\sigma x^\rho + \int^{x^0} (g(\pi F^{(r)}))_0(x)\, dx^0$$
$$(f^*(g_1, g_2))(\pi(gF)) = \tfrac{1}{2}(gF^{(0)})_{\sigma\rho}(\Lambda_1 a_2)^\sigma (a_1)^\rho. \qquad (3.32)$$

This result solves incidentally the problem of the possible factor systems involved by symmetry groups of e.m. fields [9]: indeed $(f^*(g_1, g_2))(\pi(gF))$ becomes then a constant $f(g_1, g_2)$, (F remaining of course unchanged under transformations of the symmetry group) and χ_g is then also constant as a function of πF. We get directly from (3.32), for all $g \in G_F$ (see (2.3))

$$\chi_g(x) = -\tfrac{1}{2}F^{(0)}_{\sigma\rho}a^\sigma x^\rho + \int^{x^0} (g(\pi F^{(r)}))_0(x)\, dx^0$$
$$f(g_1, g_2) = \tfrac{1}{2}F^{(0)}_{\sigma\rho}(\Lambda_1 a_2)^\sigma (a_1)^\rho. \qquad (3.33)$$

These results look quite simple, given the large class of fields we have considered, but are in fact quite important. They show that there are so to say three kinds of fields: fields with sources with a Poincaré covariance; radiation fields, with a Poincaré isomorphic covariance (but with non-trivial compensating gauges) and fields with a non-zero contribution at the origin of the dual space, i.e. for the class we consider, fields carrying a constant uniform part, that give rise to non-trivial extensions in (3.17).

4. The general covariance group $\mathbb{M}$

In the previous section we have in a first step determined a covariance operator group by the analysis of its action on the potential subspace defined by a fixed cross-section π for an arbitrary but fixed external field and we have investigated some of its essential general structure properties. The group obtained in this way may however implicitly depend on the field we started with, as the function space $\Phi_{\pi F}$ is generated by the factor system f^* corresponding to this given field.

Using the obtained general explicit expression for the corresponding factor system (see (3.32)), we may now get rid of the above dependence, letting now F run over all possible fields and considering the most general function space generated by f^*. This function space will be denoted by $\mathbb{B}$.

It follows from (3.32), that only *linear* functions in the constant uniform part of the field do occur, so that this function space can be identified with the 6-dimensional dual vector space $T \wedge T$ of 4×4 antisymmetric contravariant real tensors. A basis for this function space can thus be given by the (antisymmetric) external tensor product of a basis of the Minkovski space with itself, i.e. by

$$E_{\mu\nu} = e_\mu \wedge e_\nu \qquad (4.1)$$

with $\mu, \nu = 0, 1, 2, 3$, and $(e_\mu)^\nu = \delta^\nu_\mu$. An arbitrary element B of $\mathbb{B}$ can then be expressed as

$$B = \sum_{\mu,\nu} B^{\mu\nu} E_{\mu\nu}, \quad B^{\mu\nu} \in \mathscr{R}, \forall \mu, \nu. \tag{4.2}$$

The group structure of $\mathbb{B}$ being generated because of (3.14) by the (additive) product of the real line, we have

$$(B_1 + B_2)^{\mu\nu} = B_1^{\mu\nu} + B_2^{\mu\nu}.$$

An element of the general covariance group $\mathbb{M}$ can thus be written as a pair $\langle B, g\rangle$, $B \in \mathbb{B}$, $g \in IO(3, 1)$ with product

$$\langle B, g\rangle\langle B', g'\rangle = \langle B + \zeta(g)B' + A(g, g'), gg'\rangle \tag{4.3}$$

where, as follows from (3.16), B' transforms under g as a contravariant tensor, i.e.

$$(\zeta(g)B')^{\sigma\rho} = (\Lambda^{-1})^{\sigma}_{\lambda}(\Lambda^{-1})^{\rho}_{\mu}(B')^{\lambda\mu} \tag{4.4}$$

and the general factor set $A(g, g')$ can be obtained from (3.15) and (3.32) as given by

$$A(g, g')^{\sigma\rho} = \tfrac{1}{2}[(\Lambda a')^{\sigma} \wedge a^{\rho}] = \tfrac{1}{4}[(\Lambda a')^{\sigma}a^{\rho} - (\Lambda a')^{\rho}a^{\sigma}]. \tag{4.5}$$

In other words, $\mathbb{M}$ appears as an extension of $\mathbb{B} \cong \mathscr{R}^6$ by $IO(3, 1)$, i.e. the following sequence of groups is exact

$$0 \to \mathbb{B} \to \mathbb{M} \to IO(3, 1) \to 1, A, \zeta. \tag{4.6}$$

The relation between $Q^*_{\pi F}$ and $\mathbb{M}$, for an arbitrary e.m. field F can then also be seen, using the preceding definitions and results, to be as illustrated in the following commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{B} & \longrightarrow & \mathbb{M} & \longrightarrow & IO(3,1) \to 1, A, \zeta \\ & & \downarrow \rho_F & & \downarrow & & \| \\ 0 & \longrightarrow & \Phi_{\pi F} & \longrightarrow & Q^*_{\pi F} & \longrightarrow & IO(3,1) \to 1, f^*, \zeta \end{array} \tag{4.7}$$

where ρ_F is an epimorphism whose kernel is given by the elements of $\mathbb{B}$ which as functions of πF, i.e. as mappings

$$b: \pi(O_F) \to \mathscr{R} \tag{4.8}$$

where $b(\pi F) \equiv B \cdot F^{(0)} = B^{\sigma\rho}F^{(0)}_{\sigma\rho} \in \mathscr{R}$, vanish identically on O_F.

Similarly one may obtain directly the action of $\mathbb{M}$ on the potential subspace defined by π for a given F by the following homomorphism Σ (compare with (3.19)):

$$\begin{aligned}(\Sigma\langle B, g\rangle)(\pi F) &\overset{\text{def}}{=} \{\chi_g + b, g\}(\pi F) \\ &= g(\pi F) - \partial[\chi_g(\pi(gF), x) + b(\pi(gF))] \\ &= \pi(gF)\end{aligned}$$

where we have used (3.8) and (4.8). The covariance operator groups $Q^*_{\pi F}$ appear thus, for each field, as representation operator groups of the general covariance group $\mathbb{M}$ as acting on the potential subspace defined by π and F.

For clarity we define here the *physical representations* of $\mathbb{M}$, as mappings V

$$V: \mathbb{M} \to U(\mathscr{H})$$

from $\mathbb{M}$ to the unitary/antiunitary operators of some separable Hilbert space $\mathscr{H}$ (to be specified later on) of (here scalar) functions $\psi(x, \pi F)$ depending on x and (via π), on the field F. The mapping V is explicitly given (for $g \in P^{\uparrow}$) by

$$V(\langle B, g\rangle)\psi(x, \pi F(x)) \overset{\text{def}}{=} \exp\left\{-i\frac{e}{c\hbar}[B\cdot gF^{(0)} + \chi_g(\pi(gF), x)]\right\} \times \psi(g^{-1}x, \pi F(g^{-1}x)) \quad (4.9)$$

with $\chi_g(\pi(gF), x)$ as found in (3.32). If the functions $\psi(x, \pi F)$ describe the states of a charged particle in the external field F, and thus obey some equation of motion, represented by a linear differential operator O on $\mathscr{H}$, we have

$$O(x, \pi F)\psi(x, \pi F) = 0, \qquad \forall x \in M(4). \quad (4.10)$$

From Section 1, and by definition of our group, O will be a covariant equation of motion if and only if it satisfies the condition

$$\begin{aligned} V(\langle B, g\rangle)O(x, \pi F)(V(\langle B, g\rangle))^{-1} &= O(g^{-1}x, \pi' F) \\ &= O'(x, \pi F) \equiv O(x, \pi gF). \end{aligned} \quad (4.11)$$

The equation (1.1)′ tells us now that if $\psi(x, \pi F)$ is a solution of (4.10),

$$V(\langle B, g\rangle)\psi(x, \pi F) \equiv \Phi(x, \pi gF)$$

is then a solution in the new frame thus satisfies the equation

$$O(x, \pi gF)\Phi(x, \pi gF) = 0. \quad (4.12)$$

Using these relations and the fact that, by (3.13), (3.32), and (4.5)

$$\begin{aligned} A(g, g')\cdot gg'F^{(0)} = f^*(g, g')(\pi(gg'F)) &\equiv \chi_g(\pi(gg'F), x) \\ &+ \psi(g)\chi_{g'}(\pi(g'F), x) - \chi_{gg'}(\pi((gg')F), x) \end{aligned} \quad (4.13)$$

one may verify, after a short calculation, that

$$V(\langle B, g\rangle\langle B', g'\rangle)\psi(x, \pi F) = V(\langle B, g\rangle)V(\langle B', g'\rangle)\psi(x, \pi F) \quad (4.14)$$

so that V is indeed an homomorphism and $\mathscr{H}$ is a representation space for the general covariance group $\mathbb{M}$. In analogy with the free particle case, the quantum system described by this space will be called an elementary particle in interaction with the field F, if it is irreducible.

As a general result, we have thus found that in the presence of an external e.m. field, the relevant covariance group is no longer the Poincaré group but contains the Poincaré group only as a factor group. Since our approach is in fact independent of any specific equation of motion it will be interesting to analyse more in detail the structure of this group $\mathbb{M}$ and of its representations. It will be possible in this way not only to obtain information on the covariant equations of motion in the presence of an external field but also to extend usefully the group theoretical definition of an elementary particle for this case where, as in a new world, an external field is present. This will be done in a subsequent paper [4].

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